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ON SUMS AND PRODUCTS OF INTEGERS

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Abstract. Erdös and Szemerédi proved that if A is a set of k positive integers, then there must be at least $ck^{1+\delta}$ integers that can be written as the sum or product of two elements of A, where c is a constant and $\delta > 0$. Nathanson proved that the result holds for $\delta = \frac{1}{31}$. In this paper it is proved that the result holds for $\delta = \frac{1}{5}$ and $c = \frac{1}{20}$.

1. Introduction

Let $h \geq 2$, and let A be a finite set of positive integers. Let

$$hA = \{a_1 + a_2 + \cdots + a_h | a_i \in A \text{ for } i = 1, \ldots, h\},$$
$$A^h = \{a_1 a_2 \cdots a_h | a_i \in A \text{ for } i = 1, \ldots, h\}.$$

We let

$$E_h(A) = hA \cup A^h.$$

It is not difficult to see that (see [3])

$$|E_h(A)| \leq \frac{2}{h!} k^h + O(k^{h-1}).$$

Erdös and Szemerédi [1, 2] conjectured that for every $\varepsilon > 0$,

$$|E_h(A)| \gg k^{h-\varepsilon}.$$

For $h = 2$, Nathanson and Tenenbaum [4] have proved that if $|A| = k$ and $|2A| \leq 3k - 4$, then

$$|A^2| \gg k^{2-\varepsilon}.$$

Erdös and Szemerédi [2] showed that there exists a real number $\delta > 0$ such that

$$|E_2(A)| \gg |A|^{1+\delta}.$$

Recently Nathanson [3] proved that the result holds for $\delta = \frac{1}{31}$. In this paper I show that the result holds for $\delta = \frac{1}{5}$.

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2. The main result

Lemma 1. *Let a_i $(i \in I)$ be nonnegative real numbers and $\beta > 1$. Then*

$$\sum_{i\in I} a_i^\beta \geq |I|^{1-\beta} \left(\sum_{i\in I} a_i\right)^\beta .$$

In particular,

$$\sum_{i\in I} a_i^2 \geq |I|^{-1} \left(\sum_{i\in I} a_i\right)^2 .$$

Proof. By the Hölder inequality we have

$$\sum_{i\in I} a_i \leq \left(\sum_{i\in I} 1^{\beta/(\beta-1)}\right)^{(\beta-1)/\beta} \left(\sum_{i\in I} a_i^\beta\right)^{1/\beta} .$$

Then the lemma follows immediately.

Lemma 2. *Let B be a nonempty, finite set of positive integers such that*

$$\max(B) \leq 2\min(B).$$

Then

$$|E_2(B)| \geq \frac{7}{5}\sqrt{2}\left(\frac{|B|}{5}\right)^{5/4} .$$

Proof. If $\min(B) \leq 4$, it is trivial, so we may assume that $\min(B) > 4$. Thus

$$B^2 \cap (2B) = \varnothing. \tag{1}$$

Let $|B| = k$ and

$$l = \left[\left(\frac{k}{5}\right)^{1/2}\right].$$

If $|B| < 12500$, then

$$|E_2(B)| \geq 2(2|B|-1) \geq 3|B| \geq 2\left(\frac{|B|}{5}\right)^{5/4},$$

so we may also assume that $|B| \geq 12500$. Thus $l \geq 49$. Let

$$\begin{gathered} B = \{b_1, b_2, \ldots, b_k\}, \qquad b_1 < b_2 < \cdots < b_k \leq 2b_1, \\ d_i = b_{i+l-1} - b_i, \qquad d_{i_0} = \min d_i, \\ B^* = \{b_{i_0}, b_{i_0+1}, \ldots, b_{i_0+l-1}\}, \\ B_i = \{b_{2il+1}, b_{2il+2}, \ldots, b_{2il+2l}\}, \\ E(B^*, B_i) = (B^* + B_i) \cup B^* B_i. \end{gathered}$$

If $b_1^*, b_2^* \in B^*$ and $j-i \ge 2l-1$, then

$$b_2^* + b_j - (b_1^* + b_i) = b_2^* - b_1^* + b_j - b_i > -d_{i_0} + d_i \ge 0,$$

$$\begin{aligned} b_2^* b_j - b_1^* b_i &= \left(b_1^* - \frac{1}{2} b_j\right)(b_j - b_i) + b_j \left(\frac{b_j - b_i}{2} + b_2^* - b_1^*\right) \\ &> b_j \left(\frac{d_i + d_{i+l-1}}{2} - d_{i_0}\right) \ge 0. \end{aligned}$$

Hence, if $j - i \ge 2$, then

$$\begin{aligned} (B^* + B_i) \cap (B^* + B_j) = \varnothing, \\ B^* B_i \cap B^* B_j = \varnothing. \end{aligned} \tag{2}$$

Let

$$\begin{aligned} B(i,u,v) = \{(b_1^*, b_2^*, b_3^*, b_4^*) \in (B^*)^4 : b_1^* \ne b_2^* \\ \text{and there exist } b_3', b_4' \in B_i \text{ such that} \\ b_1^* + b_3' = b_2^* + b_4' = u, b_3^* b_3' = b_4^* b_4' = v\}. \end{aligned}$$

Suppose that

$$B(i_1, u_1, v_1) \cap B(i_2, u_2, v_2) \ne \varnothing,$$

say

$$(b_1^*, b_2^*, b_3^*, b_4^*) \in B(i_1, u_1, v_1) \cap B(i_2, u_2, v_2).$$

Then

$$b_1^* + x = b_2^* + y, \qquad b_3^* x = b_4^* y \tag{3}$$

has at least one solution in integers x, y. It is easy to see that (3) has at most one solution. So (3) has a unique solution x_0, y_0. Thus

$$x_0 \in B_{i_1} \cap B_{i_2}, \quad u_1 = b_1^* + x_0 = u_2, \quad v_1 = b_3^* x_0 = v_2,$$

whence $i_1 = i_2, u_1 = u_2, v_1 = v_2$. Hence

$$\sum_{i,u,v} |B(i,u,v)| \le |B^*|^4 - |B^*|^3 = l^4 - l^3. \tag{4}$$

Let

$$\begin{aligned} B_i^{(v)} &= \{b : b \in B_i \text{ and there exists a } b^* \in B^* \text{ such that } b^* b = v\}, \\ \rho_{i,v}(u) &= |\{b^* : b^* \in B^* \text{ and there exists a } b' \in B_i^{(v)} \text{ such that } b^* + b' = u\}|. \end{aligned}$$

Then

$$\begin{aligned} |B(i,u,v)| \ge \rho_{i,v}(u)(\rho_{i,v}(u) - 1), \\ \sum_{u \in B^* + B_i^{(v)}} \rho_{i,v}(u) = |B^*|\,|B_i^{(v)}| = l|B_i^{(v)}|. \end{aligned} \tag{5}$$

By Lemma 1 and (5) we have

$$\begin{aligned}
\sum_{u,v}|B(i,u,v)| &\geq \sum_{v\in B^*B_i}\ \sum_{u\in B^*+B_i^{(v)}}|B(i,u,v)|\\
&\geq \sum_{v\in B^*B_i}\ \sum_{u\in B^*+B_i^{(v)}}\rho_{i,v}(u)(\rho_{i,v}(u)-1)\\
&\geq \sum_{v\in B^*B_i}\left(|B^*+B_i^{(v)}|^{-1}\left(\sum_{u\in B^*+B_i^{(v)}}\rho_{i,v}(u)\right)^2-\sum_{u\in B^*+B_i^{(v)}}\rho_{i,v}(u)\right)\\
&\geq \sum_{v\in B^*B_i}(|B^*+B_i^{(v)}|^{-1}l^2|B_i^{(v)}|^2-l|B_i^{(v)}|)\\
&\geq l^2|B^*+B_i|^{-1}\sum_{v\in B^*B_i}|B_i^{(v)}|^2-l\sum_{v\in B^*B_i}|B_i^{(v)}|\\
&\geq l^2|B^*+B_i|^{-1}|B^*B_i|^{-1}\left(\sum_{v\in B^*B_i}|B_i^{(v)}|\right)^2-l\sum_{v\in B^*B_i}|B_i^{(v)}|\\
&\geq 4l^6|B^*+B_i|^{-1}|B^*B_i|^{-1}-2l^3\\
&\geq 16l^6(|B^*+B_i|+|B^*B_i|)^{-2}-2l^3\\
&\geq 16l^6|E(B^*,B_i)|^{-2}-2l^3.
\end{aligned}$$

Let

$$I_1=\{i:|E(B^*,B_i)|<2l^{3/2}\}\quad\text{and}\quad I_2=\{i:|E(B^*,B_i)|\geq 2l^{3/2}\}.$$

If $I_1\neq\varnothing$, then by (4) and the inequality just above we have

$$\begin{aligned}
2l^3|I_1| &< \sum_{i\in I_1}(16l^6|E(B^*,B_i)|^{-2}-2l^3)\\
&\leq \sum_{i\in I_1}\sum_{u,v}|B(i,u,v)|\leq l^4-l^3.
\end{aligned}$$

So

$$\begin{aligned}
|I_1| &< \frac{1}{2}l-\frac{1}{2},\qquad |I_1|\leq\frac{1}{2}l-1,\\
|I_2| &\geq \left[\frac{k}{2l}\right]-\frac{1}{2}l+1\geq\frac{k}{2l}-\frac{1}{2}l.
\end{aligned}$$

By (1), (2) and $l\geq 49$ we have

$$\begin{aligned}
|E_2(B)| &\geq \left|\bigcup_{i\in I_2}E(B^*,B_i)\right|\geq\frac{1}{2}\sum_{i\in I_2}|E(B^*,B_i)|\geq\frac{1}{2}\cdot 2|I_2|l^{3/2}\\
&\geq \frac{1}{2}(k-l^2)(l+1)^{1/2}\left(\frac{l}{l+1}\right)^{1/2}\geq\frac{1}{2}\left(k-\frac{1}{5}k\right)\left(\frac{1}{5}k\right)^{1/4}\left(\frac{49}{50}\right)^{1/2}\\
&\geq \frac{7}{5}\sqrt{2}\left(\frac{k}{5}\right)^{5/4}.
\end{aligned}$$

This completes the proof of Lemma 2.

Theorem. *Let A be a nonempty, finite set of positive integers. Then*

$$|E_2(A)| \geq \frac{1}{20}|A|^{6/5}.$$

Proof. Let

$$|A| = k, \quad c_1 = \frac{7}{5}\sqrt{2}5^{-5/4}, \quad A_j = [2^{j-1}, 2^j) \cap A.$$

Since

$$\left|\bigcup_{j=1}^{\infty} A_j\right| = k,$$

without loss of generality, we may assume that

$$\left|\bigcup_{j=1}^{\infty} A_{2j-1}\right| \geq \frac{1}{2}k.$$

Let $j_1, j_2, \ldots, j_T$ be all positive odd integers with $A_{j_i} \neq \varnothing$. Let

$$\sum_{t \leq 0} |A_{j_t}| = 0.$$

Since

$$\sum_{t \leq T} |A_{j_t}| \geq \frac{1}{2}k,$$

there exists an integer t_0 such that

$$\sum_{t \leq t_0} |A_{j_t}| \geq \frac{1}{12}k, \qquad \sum_{t \leq t_0 - 1} |A_{j_t}| < \frac{1}{12}k.$$

Thus

$$\sum_{t \geq t_0} |A_{j_t}| = \sum_{t=1}^{T} |A_{j_t}| - \sum_{t \leq t_0-1} |A_{j_t}| \geq \frac{5}{12}k.$$

If $i < u$, then $\max(A_{j_i}) < \min(A_{j_u})$, whence

$$A_{j_i}^2 \cap A_{j_u}^2 = \varnothing, \qquad 2A_{j_i} \cap 2A_{j_u} = \varnothing.$$

Hence each integer n belongs to at most two of the sets $E_2(A_{j_i})$. Therefore, by Lemma 1,

(6)

$$E_2(A) \geq \left|\bigcup_{t=t_0}^{T} E_2(A_{j_t})\right| \geq \frac{1}{2}\sum_{t=t_0}^{T} |E_2(A_{j_t})| \geq \frac{1}{2}c_1 \sum_{t=t_0}^{T} |A_{j_t}|^{5/4}$$

$$\geq \frac{1}{2}c_1(T - t_0 + 1)^{-1/4} \left(\sum_{t \geq t_0} |A_{j_t}|\right)^{5/4} \geq \frac{1}{2}c_1(T - t_0 + 1)^{-1/4} \left(\frac{5}{12}k\right)^{5/4}.$$

Suppose that $i < u$ and

$$a_i' \in \bigcup_{t \le i} A_{j_t}, \qquad a_i \in A_{j_i},$$
$$a_u' \in \bigcup_{t \le u} A_{j_t}, \qquad a_u \in A_{j_u}.$$

Then

$$a_i' + a_i \le 2^{j_i+1} \le 2^{j_{i+1}-1} \le a_u < a_u' + a_u.$$

Hence

$$\left(\bigcup_{t \le i} A_{j_t} + A_{j_i} \right) \cap \left(\bigcup_{t \le u} A_{j_t} + A_{j_u} \right) = \varnothing, \quad \text{if } i \ne u.$$

Therefore

$$\begin{aligned} E_2(A) &\ge \left| \bigcup_{i=1}^{T} \left(\bigcup_{t \le i} A_{j_t} + A_{j_i} \right) \right| \ge \sum_{i=1}^{T} \left| \bigcup_{t \le i} A_{j_t} + A_{j_i} \right| \\ &\ge \sum_{i=1}^{T} \left| \bigcup_{t \le i} A_{j_t} \right| = \sum_{i=1}^{T} \sum_{t \le i} |A_{j_t}| \\ &\ge (T - t_0 + 1) \sum_{t \le t_0} |A_{j_t}| \ge \frac{1}{12}(T - t_0 + 1)k. \end{aligned} \tag{7}$$

Therefore, by (6) and (7) we have

$$\begin{aligned} E_2(A) &\ge \max\left\{ \frac{1}{2} c_1 (T - t_0 + 1)^{-1/4} \left(\frac{5}{12} k \right)^{5/4}, (T - t_0 + 1) \frac{1}{12} k \right\} \\ &\ge (6c_1)^{4/5} \frac{5}{144} k^{6/5} \\ &> \frac{1}{20} k^{6/5}. \end{aligned}$$

This completes the proof of the theorem.

Added in proof

One should also refer to Elekes, Gy., Acta Arith. **81** (1997), 365–367, for a very different method and Kevin Ford, The Ramanujan J. **2** (1998), 59–66.

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