

UNIFORM DISTRIBUTION MODULO ONE ON SUBSEQUENCES

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(Communicated by David E. Rohrlich)

ABSTRACT. Let \mathcal{P} be a set of primes with a divergent series of reciprocals and let $\mathcal{K} = \mathcal{K}(\mathcal{P})$ denote the set of squarefree integers greater than one that are divisible only by primes in \mathcal{P} . G. Myerson and A. D. Pollington proved that $(u_n)_{n \geq 1} \subset [0, 1)$ is uniformly distributed (mod 1) whenever the subsequence $(u_{kn})_{n \geq 1}$ is uniformly distributed (mod 1) for every k in \mathcal{K} . We show that in fact $(u_n)_{n \geq 1}$ is uniformly distributed (mod 1) whenever the subsequence $(u_{pn})_{n \geq 1}$ is uniformly distributed (mod 1) for every $p \in \mathcal{P}$.

1. INTRODUCTION

In [4], G. Myerson and A. D. Pollington proved several intriguing results about uniform distribution modulo one, including the following theorem.

Theorem A (Myerson-Pollington). *Let \mathcal{P} be a set of primes such that $\sum\{1/p : p \in \mathcal{P}\}$ diverges. Let $\mathcal{K} = \mathcal{K}(\mathcal{P})$ denote the set of squarefree integers greater than one divisible only by primes in \mathcal{P} . If $(u_{kn})_{n \geq 1}$ is uniformly distributed (mod 1) for every k in \mathcal{K} , then $(u_n)_{n \geq 1}$ is uniformly distributed (mod 1).*

The set \mathcal{K} in Theorem A functions as a kind of “test set,” used to probe the distribution of the sequence (u_n) . The following definition formalizes this notion.

Definition. A set S of integers greater than 1 is called a *u.d. test set* if the sequence $(u_n)_{n \geq 1} \subset [0, 1)$ is uniformly distributed whenever the subsequence $(u_{sn})_{n \geq 1}$ is uniformly distributed for all $s \in S$.

Note that if S is a set of positive integers greater than 1 and S contains a u.d. test set, then *a fortiori* S is a u.d. test set.

It is natural to look for other u.d. test sets besides the sets $\mathcal{K}(\mathcal{P})$ and their supersets. Our main result, Theorem 1 below, shows that the set of primes \mathcal{P} in Theorem A is itself a u.d. test set. Note that Theorem A is a consequence of Theorem 1, because in the former result $\mathcal{P} \subset \mathcal{K}$.

Theorem 1. *Let \mathcal{P} be a set of primes. Then \mathcal{P} is a u.d. test set if and only if $\sum\{1/p : p \in \mathcal{P}\}$ diverges.*

Myerson and Pollington proved Theorem A by using the Weyl criterion for uniform distribution and properties of the Möbius μ -function. The path that we take

Received by the editors October 21, 1997.

1991 *Mathematics Subject Classification.* Primary 11K06; Secondary 11B05.

to Theorem 1 is quite different. We introduce a second type of integer set, called a *density test set*, whose definition makes no reference to uniform distribution. In Theorem 2, stated below, the property of being a u.d. test set is shown to be equivalent to that of being a density test set. After proving Theorem 2, we provide a sufficient condition for being a density test set. Finally, we prove that a set of primes is a density test set if and only if the series of reciprocals of its elements diverges.

We require some notation to state the definition of a density test set.

Notation. Let $A \subseteq \mathbb{N}$. We let $\mathbf{d}A = \lim_{N \rightarrow \infty} \frac{1}{N} \#\{n \leq N : n \in A\}$ denote the natural density of A , provided the limit exists. For each $s \in \mathbb{N}$, we define $A_s = \{n \in \mathbb{N} : sn \in A\}$. One may think of $\mathbf{d}A_s$ (if it exists) as being the density of the set $\{n \in A : s|n\}$ relative to the set $\{s, 2s, 3s, \dots\}$.

Definition. Let S be a set of integers greater than 1. We say that S is a *density test set* if whenever a set A of positive integers has the property that there exists $\alpha \in [0, 1]$ such that $\mathbf{d}A_s = \alpha$ for each $s \in S$, then $\mathbf{d}A = \alpha$.

Theorem 2. S is a u.d. test set if and only if S is a density test set.

2. PROOF OF THEOREM 2

We need a well-known result from the theory of uniform distribution. A proof is given in [3, Theorem I.4.1].

Lemma 3. Let $(a_n)_{n \geq 1}$ be a given sequence of distinct integers. Then the sequence $(a_n x)_{n \geq 1}$ is uniformly distributed (mod 1) for almost all real numbers x .

Proof of Theorem 2. Assume that S is a density test set. Let $(u_n)_{n \geq 1} \subset [0, 1)$, and suppose that $(u_{sn})_{n \geq 1}$ is uniformly distributed for each $s \in S$. Let $x \in (0, 1]$ be fixed, and let $A = A(x) = \{n : 0 \leq u_n < x\}$. For each $s \in S$ we have

$$\frac{1}{N} \#\{n \leq N : n \in A_s\} = \frac{1}{N} \#\{n \leq N : 0 \leq u_{sn} < x\} \rightarrow x \quad \text{as } N \rightarrow \infty,$$

that is, $\mathbf{d}A_s = x$. Since S is a density test set, we may conclude that $\mathbf{d}A = x$. Then

$$\frac{1}{N} \#\{n \leq N : 0 \leq u_n < x\} = \frac{1}{N} \#\{n \leq N : n \in A\} \rightarrow x \quad \text{as } N \rightarrow \infty.$$

It follows that (u_n) is uniformly distributed. Thus, S is a u.d. test set.

Next, we assume that S is a u.d. test set. Let A be a set of positive integers, and suppose that there is an $\alpha \in [0, 1]$ such that $\mathbf{d}A_s = \alpha$ for each $s \in S$. It is convenient to consider the cases $0 < \alpha < 1$, $\alpha = 1$, and $\alpha = 0$ separately.

We assume first that $0 < \alpha < 1$. Let $A'_s = \mathbb{N} \setminus A_s$ for each $s \in S$. The sets A_s and A'_s have positive density, and hence are infinite. For each $s \in S$, let Γ_s denote the set of all real $x \in [0, 1]$ for which the sequence $(snx : n \in A_s)$ is uniformly distributed (mod 1), and let Γ'_s denote the set of all real $x \in [0, 1]$ for which $(snx : n \in A'_s)$ is uniformly distributed (mod 1). Let $\Gamma = (\bigcap_{s \in S} \Gamma_s) \cap (\bigcap_{s \in S} \Gamma'_s)$. By Lemma 3, each of the sets $[0, 1] \setminus \Gamma_s$ and $[0, 1] \setminus \Gamma'_s$ has measure 0, whence $[0, 1] \setminus \Gamma$ has measure 0. In particular, Γ is not empty. Select any $\gamma \in \Gamma$. Put $w_n = \{n\gamma\}$ for $n \geq 1$, that is, w_n is the fractional part of $n\gamma$. By construction, the sequences $(w_{sn} : n \in A_s)$ and $(w_{sn} : n \in A'_s)$ are uniformly distributed for each $s \in S$. We now define a sequence

$(u_n)_{n \geq 1} \subset [0, 1)$ by

$$u_n = \begin{cases} \alpha w_n, & \text{if } n \in A, \\ \alpha + (1 - \alpha)w_n, & \text{otherwise.} \end{cases}$$

Let $s \in S$. We claim that $(u_{sn})_{n \geq 1}$ is uniformly distributed. Take $0 < x \leq 1$. If $x \leq \alpha$, then

$$\begin{aligned} \frac{1}{N} \#\{n \leq N : 0 \leq u_{sn} < x\} &= \frac{1}{N} \#\{n \leq N : sn \in A, 0 \leq \alpha w_{sn} < x\} \\ &= \frac{1}{N} \#\{n \leq N : n \in A_s, 0 \leq w_{sn} < x/\alpha\} \\ &\rightarrow \mathbf{d}A_s \cdot \frac{x}{\alpha} = x \quad \text{as } N \rightarrow \infty, \end{aligned}$$

since $(w_{sn} : n \in A_s)$ is uniformly distributed. On the other hand, if $x > \alpha$, then

$$\begin{aligned} \frac{1}{N} \#\{n \leq N : 0 \leq u_{sn} < x\} &= \frac{1}{N} \#\{n \leq N : sn \in A\} \\ &\quad + \frac{1}{N} \#\{n \leq N : sn \notin A, 0 \leq \alpha + (1 - \alpha)w_{sn} < x\} \\ &= \frac{1}{N} \#\{n \leq N : n \in A_s\} \\ &\quad + \frac{1}{N} \#\{n \leq N : n \in A'_s, 0 \leq w_{sn} < (x - \alpha)/(1 - \alpha)\} \\ &\rightarrow \mathbf{d}A_s + \mathbf{d}A'_s \cdot \frac{x - \alpha}{1 - \alpha} = x \quad \text{as } N \rightarrow \infty \end{aligned}$$

because $(w_{sn} : n \in A'_s)$ is uniformly distributed. Hence, $(u_{sn})_{n \geq 1}$ is uniformly distributed, as claimed. We deduce that $(u_n)_{n \geq 1}$ is uniformly distributed, by virtue of S being a u.d. test set. Noting that $A = \{n : 0 \leq u_n < \alpha\}$, we see that $\mathbf{d}A$ exists and equals α .

Suppose now that $\alpha = 1$. Define the sets Γ_s for $s \in S$ as above, and let $\Gamma = \bigcap_{s \in S} \Gamma_s$. Select $\gamma \in \Gamma$, and let $w_n = \{n\gamma\}$ for $n \geq 1$. We define a sequence $(u_n)_{n \geq 1} \subset [0, 1)$ by

$$u_n = \begin{cases} w_n, & \text{if } n \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Arguing as in the last paragraph, we show that the subsequences $(u_{sn})_{n \geq 1}$ are uniformly distributed for each $s \in S$. Then (u_n) is itself uniformly distributed because S is a u.d. test set. In particular, $1 - \mathbf{d}A = \mathbf{d}\{n : u_n = 0\} = 0$, and hence, $\mathbf{d}A = \alpha$.

Finally, we assume that $\alpha = 0$. Define the sets Γ'_s for $s \in S$ as before, and let $\Gamma = \bigcap_{s \in S} \Gamma'_s$. Select $\gamma \in \Gamma$, and let $w_n = \{n\gamma\}$ for $n \geq 1$. We define $(u_n)_{n \geq 1} \subset [0, 1)$ by

$$u_n = \begin{cases} 0, & \text{if } n \in A, \\ w_n, & \text{otherwise.} \end{cases}$$

As above, we show that the subsequences $(u_{sn})_{n \geq 1}$ are uniformly distributed for each $s \in S$, which implies that (u_n) is itself uniformly distributed because S is a u.d. test set. Consequently, $\mathbf{d}A = \mathbf{d}\{n : u_n = 0\} = 0 = \alpha$.

With all cases checked, we conclude that S is a density test set. □

3. PROOF OF THEOREM 1

We will deduce the principal implication in Theorem 1 from the following proposition.

Proposition 4. *Let S be a set of integers greater than 1 such that $\sum\{1/s : s \in S\} = \infty$, and let $f(y) := \sum\{1/s : s \in S, s \leq y\}$. Let $[s, t]$ denote the least common multiple of positive integers s and t . Suppose that*

$$(1) \quad \sum_{\substack{s,t \in S \\ s,t \leq y}} \frac{1}{[s,t]} \sim f(y)^2 \quad (y \rightarrow \infty).$$

Then S is a density test set.

Proof. Let $\tau_y(n, S)$ denote the number of $s \in S$ such that $s \leq y$ and $s|n$. In the course of the proof, we use Turán’s variance method (see [1, Chapter 4], for example) to show that $\tau_y(n, S)$ is close to $f(y)$ for “most” n .

Let A be a set of positive integers, and suppose that there exists $\alpha \in [0, 1]$ such that $\mathbf{d}A_s = \alpha$ for each $s \in S$.

Let $1 \leq y \leq x$. Observe that

$$\begin{aligned} \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} \tau_y(n, S) &= \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} \sum_{\substack{s \leq y \\ s \in S \\ s|n}} 1 = \sum_{\substack{s \leq y \\ s \in S}} \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A \\ s|n}} 1 = \sum_{\substack{s \leq y \\ s \in S}} \left(\frac{\alpha}{s} + o_s(1) \right) \\ &= \alpha f(y) + o_y(1) \quad (x \rightarrow \infty). \end{aligned}$$

It follows that

$$\frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} 1 + \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} \left(\frac{\tau_y(n, S) - f(y)}{f(y)} \right) = \alpha + o_y(1) \quad (x \rightarrow \infty),$$

so that

$$(2) \quad \limsup_{x \rightarrow \infty} \left| \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} 1 - \alpha \right| \leq \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \left| \frac{\tau_y(n, S) - f(y)}{f(y)} \right|$$

for all y large enough that $f(y) > 0$.

We now further restrict y so that $1 \leq y \leq \sqrt{x}$. We have

$$\sum_{n \leq x} \tau_y(n, S)^2 = \sum_{n \leq x} \sum_{\substack{[s,t]|n \\ s,t \in S \\ s,t \leq y}} 1 = \sum_{\substack{[s,t] \leq x \\ s,t \in S \\ s,t \leq y}} \left[\frac{x}{[s,t]} \right] = x \sum_{\substack{s,t \in S \\ s,t \leq y}} \frac{1}{[s,t]} + O(x).$$

A similar but simpler argument shows that

$$\sum_{n \leq x} \tau_y(n, S) = x f(y) + O(y).$$

We combine our last two results to obtain

$$\begin{aligned} \sum_{n \leq x} (\tau_y(n, S) - f(y))^2 &= \sum_{n \leq x} \tau_y(n, S)^2 - 2f(y) \sum_{n \leq x} \tau_y(n, S) + [x]f(y)^2 \\ &= x \left(\sum_{\substack{s,t \in S \\ s,t \leq y}} \frac{1}{[s,t]} - f(y)^2 \right) + O(x). \end{aligned}$$

Using the hypothesis (1) we deduce that

$$(3) \quad \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \left(\frac{\tau_y(n, S) - f(y)}{f(y)} \right)^2 = o(1) \quad (y \rightarrow \infty).$$

From (2), the Cauchy-Schwarz inequality, and (3), we have

$$\begin{aligned} \limsup_{x \rightarrow \infty} \left| \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} 1 - \alpha \right| &\leq \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \left| \frac{\tau_y(n, S) - f(y)}{f(y)} \right| \cdot 1 \\ &\leq \limsup_{x \rightarrow \infty} \frac{1}{x} \left\{ \sum_{n \leq x} \left| \frac{\tau_y(n, S) - f(y)}{f(y)} \right|^2 \right\}^{1/2} \cdot x^{1/2} \\ &= o(1) \quad (y \rightarrow \infty). \end{aligned}$$

We let $y \rightarrow \infty$ to conclude that $\mathbf{d}A = \alpha$. Thus, S is a density test set. □

While Proposition 4 provides us with a sufficient condition for being a density test set, the next lemma offers a necessary condition. To state the result concisely, we use some terminology from the theory of sets of multiples. For $S \subseteq \mathbb{N}$, let $\mathcal{M}(S) = \{ms : m \geq 1, s \in S\}$ be the set of multiples of numbers in S . A set $S \subseteq \mathbb{N} \setminus \{1\}$ is said to be *Behrend* if $\mathcal{M}(S)$ has natural density 1. The interested reader can find a detailed discussion of Behrend sets in [2].

Lemma 5. *If S is a density test set, then S is Behrend.*

Proof. Let S be a density test set, and set $A = \mathcal{M}(S)$. Since $\mathbf{d}A_s = 1$ for each $s \in S$, we have $\mathbf{d}A = 1$. Thus, S is Behrend. □

The proof of Theorem 1 uses a basic property of Behrend sets, which we state in Lemma 6. A proof is given in [2, Corollary 0.10].

Lemma 6. *The series of reciprocals of elements from a Behrend set diverges.*

Proof of Theorem 1. In view of Theorem 2, it suffices to show that \mathcal{P} is a density test set if and only if $\sum\{1/p : p \in \mathcal{P}\}$ diverges.

If \mathcal{P} is a density test set, then $\sum\{1/p : p \in \mathcal{P}\}$ diverges by Lemmas 5 and 6.

Now assume that $\sum\{1/p : p \in \mathcal{P}\} = \infty$. Let $f(y) = \sum\{1/p : p \in \mathcal{P}, p \leq y\}$. We have

$$\sum_{\substack{p, q \in \mathcal{P} \\ p, q \leq y}} \frac{1}{[p, q]} = f(y)^2 + f(y) + O(1) \sim f(y)^2 \quad (y \rightarrow \infty).$$

Then \mathcal{P} is a density test set by Proposition 4. □

4. CONCLUDING REMARKS

A problem that is complementary to the one considered in this paper is the following. If $(u_n) \subset [0, 1)$ is uniformly distributed, need the subsequence $(u_{kn})_{n \geq 1}$ be uniformly distributed for some $k > 1$? Myerson and Pollington [4] have shown that the answer is a resounding “no.” They constructed a uniformly distributed sequence (u_n) such that the subsequence $(u_{kn+j})_{n \geq 1}$ is not uniformly distributed for *all* integers $k \geq 2$ and $j \geq 0$.

It would be of interest to find a simple characterization of density test sets (or equivalently, of u.d. test sets). One might conjecture that a set S is a density test

set if and only if S is Behrend. Indeed, Theorem 1 and Lemmas 5 and 6 establish this equivalence when S consists only of primes. We can prove that the equivalence also holds when the elements of S have at most two prime factors, but at this point we are unable to settle the general case.

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