

A QUASISYMMETRIC SURFACE WITH NO RECTIFIABLE CURVES

CHRISTOPHER J. BISHOP

(Communicated by Frederick W. Gehring)

ABSTRACT. There is a quasiconformal mapping f of \mathbb{R}^3 to itself such that the image of $\mathbb{R}^2 \times \{0\}$ contains no rectifiable curves.

1. INTRODUCTION

A mapping $f : X \rightarrow Y$ is called *quasiconformal* if there is an $H < \infty$ so that the linear dilatation

$$H(x) \equiv \lim_{r \rightarrow 0} \frac{\max_{|x-y|=r} |f(x) - f(y)|}{\min_{|x-y|=r} |f(x) - f(y)|} \leq H,$$

for every x . The purpose of this note is to prove

Theorem 1.1. *There is a quasiconformal mapping f of \mathbb{R}^3 to itself such that*

$$(1) \quad C^{-1} \leq \frac{|f(z) - f(w)|}{\varphi(|z - w|)} \leq C, \quad \forall z, w \in \mathbb{R}^2 \times \{0\},$$

for some $C < \infty$ and some increasing function φ such that

$$(2) \quad \lim_{t \rightarrow 0} \varphi(t)/t = \infty.$$

For $E \in \mathbb{R}^2 \times \{0\}$, if the image $f(E)$ can be covered by balls $\{B(f(x_j), \varphi(r_j))\}$, then E is covered by balls $B(x_j, Cr_j)$. Then (2) easily implies that if $f(E)$ has finite α -dimensional Hausdorff measure, then E has zero α -measure. In particular, any curve in $f(\mathbb{R}^2)$ must have infinite length, since otherwise its preimage would be a non-degenerate curve of zero length, a contradiction. Recall that a homeomorphism $f : X \rightarrow Y$ is called *quasisymmetric* if there is a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$ such that

$$(3) \quad |x - y| \leq t|x - z| \quad \text{implies} \quad |f(x) - f(y)| \leq \eta(t)|f(x) - f(z)|.$$

Quasiconformal maps of \mathbb{R}^d , $d \geq 2$, to itself are quasisymmetric (e.g., [4]), so Theorem 1.1 answers a question of S. Rohde (Question 18 of [2]) by implying

Corollary 1.2. *There is a quasisymmetric embedding of \mathbb{R}^2 into \mathbb{R}^3 such that the image $f(\mathbb{R}^2)$ contains no rectifiable curves.*

Received by the editors September 22, 1997.

1991 *Mathematics Subject Classification.* Primary 30C65.

Key words and phrases. Quasisymmetric maps, quasiconformal mappings, rectifiable curves, Jacobian.

The author was supported in part by NSF grant # DMS 95-00577.

In this case, one can avoid the theorem that quasiconformal implies quasisymmetric; our function φ will be constructed so that $\varphi(tr) \leq \varphi(t)\varphi(r)$, and this combined with (1) easily implies f is quasisymmetric on \mathbb{R}^2 .

In [3], J. Heinonen and P. Koskela define, for a quasiconformal map f on the upper half space \mathbb{R}_+^n ,

$$a_f(x) = \exp\{n^{-1}\text{vol}(B(x,r))^{-1} \int_{B(x,r)} \log J_f(y) dy\}$$

(J_f denotes the Jacobian of f and r is half the distance of $x = (x_1, \dots, x_n)$ to the boundary of the half space), and

$$E_\infty = \{x \in \partial\mathbb{R}_+^n : 0 < \liminf_{r \rightarrow 0} a_f(x + re_n) \leq \limsup_{r \rightarrow 0} a_f(x + re_n) = \infty\}.$$

Question 6 of [3] asks if there is an example where E_∞ has positive Lebesgue measure and whether it can be the whole boundary. In our example, the Jacobian at (x, y, t) is comparable to $(\varphi(t)/t)^3$, and thus E_∞ is the whole boundary.

It is unknown whether there is an embedding of \mathbb{R}^2 into \mathbb{R}^3 which satisfies (1) with $\varphi(t) = t^\alpha$, for some $\alpha < 1$ (it is easy to see that $\alpha < 2/3$ is impossible). Such maps are easy to find taking \mathbb{R} into \mathbb{R}^2 (just take the usual parameterization of the von Koch snowflake), and are known to exist for $\mathbb{R}^d \rightarrow \mathbb{R}^n$ if n is large enough with respect to d (see [1]).

After seeing my first proof of Corollary 1.2, S. Rohde pointed out the map I constructed on \mathbb{R}^2 extended to a quasiconformal map of \mathbb{R}^3 and hence gave Theorem 1.1. J. Heinonen then observed that this map solved the problem from [3]. I thank them for their comments and I thank the referees for numerous suggestions which improved the clarity of the paper.

2. THE PICTURE OF THE NON-RECTIFIABLE SURFACE

We will start by describing a picture of the surface $f(\mathbb{R}^2)$ which gives the main idea of the construction. We will then give a more careful construction of the map on all of \mathbb{R}^3 .

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth 1-periodic function with $\psi(0) = 0$ and such that the length of the graph Γ of ψ above $[0, 1]$ is 2. Define a surface $S \subset \mathbb{R}^3$ by $S = \Gamma \times [0, 1]$. Then S is a smooth surface of area 2 and is isometric to $[0, 2] \times [0, 1]$ (with respect to the intrinsic path metric).

Start with the unit square Q_0 in \mathbb{R}^2 . Define a map $f_0 : Q_0 \rightarrow \mathbb{R}^3$ by first stretching the square in the x direction by a factor of 2 (i.e., apply the map $(x, y) \rightarrow (2x, y)$ and then map the image rectangle isometrically onto the surface S ; see Figure 1).



FIGURE 1. The map F_1 : Stretching a square and folding it.

Now choose a large integer k_1 . The dyadic subsquares of Q_0 of size 2^{-k_1} are mapped by F_1 to pieces of S which look like small rectangles of size $2^{-k_1+1} \times 2^{-k_1}$, and since F_2 is smooth, the approximation is better and better the larger k_1 is.

Replace each such approximate rectangle by a scaled copy of $[0, 1] \times \Gamma$, now oriented to the “oscillations” of the surface are orthogonal to the first set. See Figure 2. The scaled copies will not quite meet along their edges, but we can make them do so by bending them with arbitrarily small angle (depending on k_1). Because we have stretched in the y direction, this time we see that small enough squares of size r in Q_0 are now mapped onto regions which approximate small squares of size $2r \times 2r$, within a small error.

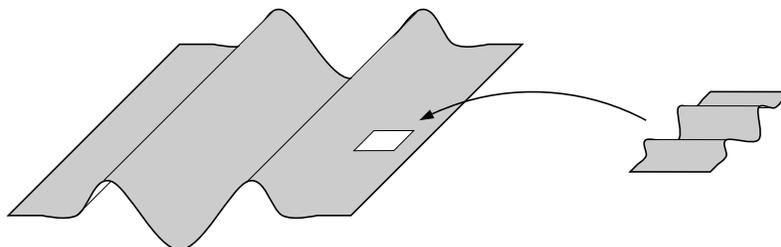


FIGURE 2. Folding in the orthogonal direction

We now continue by induction. After n steps, we have a smooth surface $f_n(Q_0)$ which maps small squares of size $r = 2^{-k_n}$ to pieces of the surface which approximate a $2^n r \times 2^n r$ square with small error. Next take each dyadic square in Q_0 of some size $\ll r$ and stretch by a factor of 2 in the x direction and map it onto a scaled copy of S , bending it to make the edges fit together. Then stretch in the y direction on an even smaller scale. If we choose the $\{k_n\}$ to increase quickly enough, then (1) will hold for $\varphi(t) = t2^{\alpha(t)}$, where $\alpha(t)$ is the minimal n so $2^{-k_n} < t$.

3. THE CONSTRUCTION

Now we give a more careful construction of the desired map on \mathbb{R}^3 . It will be the limit of a sequence of compositions $f_n = g_1 \circ \dots \circ g_n$ where the g_i 's are given by the following

Lemma 3.1. *There is a $C < \infty$, so that for any $0 < \delta, \epsilon, \eta \leq 1$, there is a $\nu = 2^{-m} > 0$ and a quasiconformal map $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ so that if $z = (x, y, t) \in \mathbb{R}^3$, then*

1. $g(z) = z$ if $|t| > \delta$.
2. The linear dilatation of g is $\leq 1 + C(\epsilon)$ in all of \mathbb{R}^3 , where $C(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.
3. The linear dilatation of g is $\leq 1 + \eta$ if $|t| < \nu$.
4. For $z, w \in \mathbb{R}^2 \times \{0\}$, we have

$$\left. \begin{aligned} (1 + \epsilon)(1 - \eta) \\ 1 - \eta \\ (1 + \frac{C\delta}{|z-w|})^{-1} \end{aligned} \right\} \leq \frac{|g(z) - g(w)|}{|z - w|} \leq \begin{cases} 1 + \epsilon + \eta, & |z - w| \leq \nu. \\ 1 + \epsilon + \eta, & \nu \leq |z - w| \leq \delta. \\ 1 + \frac{C\delta}{|z-w|}, & |z - w| \geq \delta. \end{cases}$$

First we check that Theorem 1.1 follows from the lemma. Choose a sequence $\{\epsilon_n\}$ in $(0, 1]$, so that $\sum_n \epsilon_n = \infty$. For example, $\epsilon_n = 1$ for all n will correspond to the picture described in the previous section and taking $\epsilon_n \rightarrow 0$, will give an asymptotically flat example. Let $\eta_n = 10^{-n}$. Let $\delta_1 = 1$. Now apply the lemma with $\epsilon_1, \delta_1, \eta_1$ to obtain a quasiconformal mapping g_1 , and a positive number ν_1 . In general, given ν_{n-1} , choose $\delta_n < \min(\nu_{n-1}/8, \delta_{n-1}/100)$, and apply the lemma to the triple $\epsilon_n, \delta_n, \eta_n$, to obtain the map g_n and the number ν_n . Let $f_n = g_1 \circ \dots \circ g_n$.

Then f_n is a finite composition of quasiconformal maps and hence is quasiconformal. It is the identity map on $\{z = (x, y, t) : |t| \geq 1\}$ and to estimate its dilatation at a point z with $|t| < 1$, choose k so that $\delta_{k+1} \leq |t| < \delta_k$. Then $g_{k+1} \circ \dots \circ g_n$ is the identity at z . The map g_k has linear dilatation at most $1 + C(\epsilon_k)$ everywhere, and hence at z . For each $1 \leq j < k$, the image of z under $g_j \circ \dots \circ g_k$ is in $\{z = (x, y, t) : |t| < \delta_j\}$ (since the inverse is the identity on the complement) and hence the dilatation is at most $1 + \eta_j$. Thus the linear dilatation of f_n at z is at most $(1 + \sup_{k \leq n} C(\epsilon_k)) \prod_{k \leq n} (1 + \eta_j) \leq C$. It is easy to check that $\{f_n\}$ is Cauchy with respect to uniform convergence on compacta, so $f = \lim_n f_n$ exists and is quasiconformal.

To prove (1) choose $z, w \in \mathbb{R}^2$ and choose k so that $2\delta_{k+1} \leq |z - w| \leq 2\delta_k$. For $j = n - 1, \dots, 1$, let $z_j = g_j \circ \dots \circ g_n(z)$ and similarly for w_j . For $k < j \leq n - 1$, induction and the fact that $\prod(1 - (\delta_j/\delta_{k+1})) > 1/2$ imply

$$|z_j - w_j| \geq (1 - \eta_j)|z_{j+1} - w_{j+1}| \geq \frac{1}{2}|z - w| \geq \delta_{k+1} \geq \delta_j.$$

Thus case 3 of part 4 of the lemma holds and

$$(1 + C \frac{\delta_j}{\delta_{k+1}})^{-1} \leq \frac{|z_{j-1} - w_{j-1}|}{|z_j - w_j|} \leq (1 + C \frac{\delta_j}{\delta_{k+1}}).$$

For $j = k$ we have (no matter which case holds),

$$(1 - \eta_k)|z_{k+1} - w_{k+1}| \leq |z_k - w_k| \leq 2(1 + \eta_k)|z_{k+1} - w_{k+1}|.$$

For $j = k - 1, |z_{k-1} - w_{k-1}| \leq 4|z - w| \leq 8\delta_k \leq \nu_{k-1}$ so case 1 holds in part 4 of the lemma. Furthermore, for $j = k - 2, \dots, 1, |z_{j-1} - w_{j-1}| \leq 4|z_j - w_j|$ and $\nu_j < \delta_j \leq \nu_j/4$. Thus $|z_j - w_j| \leq \nu_j$ implies $|z_{j-1} - w_{j-1}| \leq \nu_{j-1}$. Thus for $j < k$, case 1 of part 4 of Lemma 3.1 applies, and so

$$1 - \eta_j \leq \frac{|z_{j-1} - w_{j-1}|}{(1 + \epsilon_j)|z_j - w_j|} \leq 1 + \eta_j.$$

Putting all these together and telescoping we get,

$$\begin{aligned} \prod_{j=k+1}^n (1 + C \frac{\delta_j}{\delta_{k+1}})^{-1} \prod_{j=1}^k (1 - \eta_j) &\leq \frac{|z_1 - w_1|}{|z - w| \prod_{j=1}^k (1 + \epsilon_j)} \\ &\leq 2 \prod_{j=k+1}^n (1 + C \frac{\delta_j}{\delta_{k+1}}) \prod_{j=1}^k (1 + \eta_j). \end{aligned}$$

Since the products converge (independent of k and n) and $f(z) = z_1, f(w) = w_1$, we have

$$C^{-1} \leq \frac{|f(z) - f(w)|}{\varphi(|z - w|)} \leq C,$$

where $\varphi(t) = t \prod_{j=1}^{\alpha(t)} (1 + \epsilon_j)$, with $\alpha(t)$ defined by $\alpha(t) = \max\{k : t \leq 2\delta_k\}$. This is (1). Since $\sum \epsilon_n = \infty$, we also have (2).

Thus Lemma 3.1 proves Theorem 1.1. Now we prove the lemma. We will build g as a composition $g = g_1 \circ g_2 \circ g_3$. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth 1-periodic function with $\psi(0) = 0$ and such that the length of the graph Γ of ψ above $[0, 1]$ is $1 + \epsilon$. Define a surface $S \subset \mathbb{R}^3$ by $S = \Gamma \times \mathbb{R}$. Let $L(x) = \int_0^x \sqrt{1 + |\psi'(t)|^2} dt$ be the

arclength of the graph over $[0, x]$ and let J be the inverse, L^{-1} . Define a map $F : \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$ by

$$(4) \quad F : (x, y) \rightarrow (J((1 + \epsilon)x), y, \psi(J((1 + \epsilon)x))).$$

Now extend F to a function G defined on all of \mathbb{R}^3 by setting $G(x, y, t) = (x, y, t)$ if $|t| \geq 1$, G is quasiconformal with dilatation at most $1 + C(\epsilon)$, and the dilatation of G tends to 1 as $t \rightarrow 0$. For example, we can do this by making sure that $\partial_t G(x, y, 0)$ has norm 1 and is perpendicular to $S = F(\mathbb{R}^2)$, e.g., set

$$G(x, y, t) = F(x, y) + t \cdot N(x, y),$$

for small enough $t < t_0$, where

$$N(x, y) = \frac{(-\psi'(J((1 + \epsilon)x), 0, 1))}{\sqrt{|\psi'(J((1 + \epsilon)x))|^2 + 1}},$$

is the normal vector to $F(\mathbb{R}^2)$ at the point $F(x, y)$.

To define g_1 , let $g_1(x, y, t) = \frac{\delta}{2} G(\frac{2}{\delta}(x, y, t))$. Thus the tangent map near $\mathbb{R}^2 \times \{0\}$ looks like an affine stretch by a factor of $1 + \epsilon$ in the x direction, followed by a rotation in the (x, t) plane.

To define the map g_2 we do a similar thing, but at a much smaller scale and reversing the roles of x and y . Choose ν_1 so small that g_1 , restricted to the set $\{(x, 0, t) : t < \nu_1\}$, has linear dilatation $\leq 1 + \eta/10$. Now let $g_2(x, y, t) = \nu_1 G(\nu_1^{-1}(y, x, t))$. Choose ν_2 so that g_2 has linear dilatation $\leq 1 + \eta/10$ on $\{(0, y, t) : t < \nu_2\}$. Thus near \mathbb{R}^2 , the tangent map looks like an affine stretch in the y direction, followed by a rotation in the (y, t) plane.

Finally, define g_3 by

$$g_3(x, y, t) = \begin{cases} (x, y, t), & \text{if } |t| \geq \nu_2/2, \\ (x, y, \nu_2 h(t/\nu_2)), & \text{if } \nu_2/8 \leq |t| \leq \nu_2/2, \\ (x, y, (1 + \epsilon)t), & \text{if } t \geq \nu_2/8, \end{cases}$$

where h smoothly interpolates between $(1 + \epsilon)/8$ and $1/2$ on $[\frac{1}{8}, \frac{1}{2}]$ and satisfies $|h' - 1| \leq C(\epsilon)$. Thus near \mathbb{R}^2 , the map is an affine stretch in the t direction and g_3 is $1 + C(\epsilon)$ quasiconformal.

We claim that $g = g_1 \circ g_2 \circ g_3$ satisfies the lemma with $\nu = \nu_2/16$. Most of the conditions are clear from the construction. For example, it is clear that g is quasiconformal and is the identity on $\{(x, y, t) : |t| > \delta\}$. Moreover, it has dilatation bounded everywhere by $1 + C(\epsilon)$ (since each of the three component maps does). Finally, for points in $\{(x, y, t) : |t| < \nu\}$, by considering the tangent maps of the composition, it is easy to see that the map acts as a conformal dilation of size $1 + \epsilon$ with error at most η . This also proves the expansion estimates on the small and intermediate scales (i.e., $|z - w| \leq \delta$). For large scales (i.e., $|z - w| \geq \delta$), we just use the fact that $|G(z) - G(w)| = |z - w| + O(1)$ which implies $|g(z) - g(w)| = |z - w| + O(\delta)$. This completes the proof of the lemma.

Added in proof. G. David and T. Toro have improved Corollary 1.2 by showing that there is an embedding of \mathbb{R}^2 into \mathbb{R}^3 which satisfies (1) with $\varphi(t) = t^\alpha$ for some $\alpha < 1$. Their result appears in a preprint “Reifenberg flat metric spaces, snowballs and embeddings”.

REFERENCES

- [1] Assouad, P. (1983) Prolongements Lipschitziens dans \mathbb{R}^n , *Bull. Soc. Math. France* **111**, 429–448. MR **86f**:54050
- [2] Heinonen, J. and Semmes, S. (1997) Thirty-three *yes* or *no* questions about mappings, measures and metrics, *Conf. Geometry and Dynamics* **1**, 1–12. CMP 97:13
- [3] Heinonen, J. and Koskela, P. (1994) The boundary distortion of a quasiconformal mapping, *Pacific J. Math* **165**, 93–114. MR **95f**:30031
- [4] P. Tukia and J. Väisälä (1980) Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **5**, 97–114. MR **82g**:30038

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT STONY BROOK, STONY BROOK, NEW YORK 11794-3651

E-mail address: `bishop@math.sunysb.edu`