

IRREDUCIBLE PLANE CURVES WITH THE ALBANESE DIMENSION 2

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ABSTRACT. Let B be a plane curve given by an equation $F(X_0, X_1, X_2) = 0$, and let B_a be the affine plane curve given by $f(x, y) = F(1, x, y) = 0$. Let S_n denote a cyclic covering of \mathbf{P}^2 determined by $z^n = f(x, y)$. The number $\max_{n \in \mathbf{N}} (\dim \operatorname{Im}(S_n \rightarrow \operatorname{Alb}(S_n)))$ is called the Albanese dimension of B_a . In this article, we shall give examples of B_a with the Albanese dimension 2.

INTRODUCTION

Let B be a plane curve in \mathbf{P}^2 given by an equation $F(X_0, X_1, X_2) = 0$, X_0, X_1, X_2 being homogeneous coordinates. Let B_a be the affine plane curve given by $f(x, y) = F(1, x, y) = 0$. Define an n -cyclic extension, K , of the rational function field of \mathbf{P}^2 , $\mathbf{C}(\mathbf{P}^2) = \mathbf{C}(x, y)$, by:

$$K = \mathbf{C}(x, y)(\zeta), \quad \zeta^n = f(x, y).$$

Let S'_n be the K -normalization of \mathbf{P}^2 ; and we denote its smooth model by S_n . S_n is an n -fold cyclic covering of \mathbf{P}^2 branched along B and possibly the line at infinity, $X_0 = 0$. It is called a cyclic multiple plane, of which investigation has been done by many mathematicians (for example, [BdF], [C], [CC], [L], [S], [Z1] and [Z2]). Their main interest has been to study the first Betti number, $b_1(S_n)$, the Alexander polynomial of B , and S_n itself. In this note we focus our attention on the image of the Albanese mapping $\alpha_n : S_n \rightarrow \operatorname{Alb}(S_n)$. In [Ku], Kulikov defined the Albanese dimension of B_a as follows:

Definition 0.1. The number

$$a(B_a) := \max_{n \in \mathbf{N}} \dim \alpha_n(S_n)$$

is called the Albanese dimension of B_a .

A priori, $a(B_a) = 0, 1$, or 2 . The purpose of this note is to give examples of irreducible affine plane curves, B_a , with $a(B_a) = 2$.

Note that although Kulikov gives a sufficient condition for $a(B_a)$ to be 2 (Theorem 1.1 below) in [Ku], he does not give any example of an irreducible affine plane curve enjoying the condition. Now we state our result.

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Theorem 0.2. (i) Let E be a smooth plane cubic curve and let E^\vee be its dual curve. Put $B = E^\vee$ and consider the 6-fold cyclic covering S_6 as above. Then $\alpha_6(S_6) = 2$.

(ii) Let B be a plane sextic curve defined by the equation

$$F(X_0, X_1, X_2) = X_0^3 X_1^3 - 3X_0 X_1 X_2^4 + 24X_0^4 X_1 X_2 + 2X_2^6 + 40X_0^3 X_2^3 - 16X_0^6 = 0,$$

and consider the 6-fold cyclic covering S_6 as above. Then (a) B is irreducible and its normalization is a rational curve, and (b) $\alpha_6(S_6) = 2$.

Corollary 0.3. Let B_a be the affine plane curve obtained from B in Theorem 0.2. Then $a(B_a) = 2$.

§1. KULIKOV'S THEOREM

We keep the same notation as in the Introduction.

Theorem 1.1 (Theorem 1 (iii), [Ku]). Let p, q be integers with $p, q \geq 2$ and $\gcd(p, q) = 1$. If $F(X_0, X_1, X_2)$ possess two decompositions

$$\begin{aligned} F(X_0, X_1, X_2) &= G_1(X_0, X_1, X_2)^p + H_1(X_0, X_1, X_2)^q \\ &= G_2(X_0, X_1, X_2)^p + H_2(X_0, X_1, X_2)^q \end{aligned}$$

such that two pencils of plane curves, $\{\lambda_0 G_1^p + \lambda_1 H_1^q = 0\}_{[\lambda_0:\lambda_1] \in \mathbf{P}^1}$ and $\{\lambda_0 G_2^p + \lambda_1 H_2^q = 0\}_{[\lambda_0:\lambda_1] \in \mathbf{P}^1}$, are different, then $\dim \alpha_{pq}(S_{pq}) = 2$. In particular, $a(B_a) = 2$.

For a proof, see [Ku].

The decomposition as above is called a (p, q) torus decomposition; and we call B a (p, q) torus curve if B has a (p, q) torus decomposition.

§2. PROOF OF THEOREM 0.2 (i)

Let E be a smooth cubic curve in \mathbf{P}^2 given by $F_E(X_0, X_1, X_2) = 0$, and let $\iota : E \rightarrow E^\vee$ be the canonical map given by

$$p \in E \mapsto [\partial F_E / \partial X_0(p) : \partial F_E / \partial X_1(p) : \partial F_E / \partial X_2(p)] \in E^\vee.$$

Our approach to prove Theorem 0.2 (i) is based on Theorem 1.1. In fact, we show that E^\vee has 12 different $(2, 3)$ torus decompositions. To this end, the following proposition is crucial:

Proposition 2.1. Suppose that there exists a conic, C , which meets E^\vee at 6 distinct cusps. Then E^\vee has a $(2, 3)$ torus decomposition, and C is defined by the quadratic form appearing in the $(2, 3)$ torus decomposition.

Proof. Let D be the unique adjoint of degree 3 to E^\vee . Then obviously $3C$ and $2D$ cut on E^\vee the same divisor.

Thus our problem is reduced to finding a conic enjoying the condition in Proposition 2.1. Although the following lemma may be well-known to experts, we give its proof for completeness. The proof below is due to F. Catanese.

Proposition 2.2. Let p_i ($i = 1, \dots, 9$) be 9 inflection points on E , and put $\check{p}_i = \iota(p_i)$ for each i . If three distinct inflection points p_i, p_j, p_k are on a line, then there exists a conic through the six cusps \check{p}_l ($l \in \{1, \dots, 9\} \setminus \{i, j, k\}$).

Remark 2.3. Any two torus decompositions in Proposition 2.2 satisfy the condition in Theorem 1.1 since their base points are different.

Proof. If E is projectively equivalent to the Fermat cubic curve, we can check the statement by direct computation. Hence, in what follows, we always assume that E is not projectively equivalent to the Fermat cubic curve.

By relabeling the 9 inflection points if necessary, we may assume that p_1, p_2, p_3 are on a line. Let $\hat{\mathbf{P}}^2 \rightarrow \mathbf{P}^2$ be a composition of blowing-ups at $\check{p}_1, \dots, \check{p}_9$. We denote by μ^*E^\vee and $\mu^{-1}E^\vee$ the total transform and the proper transform of E^\vee , respectively. We also denote by $E_i = \mu^{-1}(\check{p}_i)$, $i = 1, \dots, 9$, the 9 exceptional curves of the first kind. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_{\hat{\mathbf{P}}^2}(-\mu^{-1}E^\vee + L) \rightarrow \mathcal{O}_{\hat{\mathbf{P}}^2}(L) \rightarrow \mathcal{O}_{\mu^{-1}E^\vee}(L|_{\mu^{-1}E^\vee}) \rightarrow 0,$$

where $L := 2\mu^*l - \sum_{i=4}^9 E_i$, l being a line in \mathbf{P}^2 . By the definition of ι and Abel's theorem on an elliptic curve, we have $\mathcal{O}_{\mu^{-1}E^\vee}(L|_{\mu^{-1}E^\vee}) \cong \mathcal{O}_{\mu^{-1}E^\vee}$. Hence, in order to prove the statement, by the cohomology exact sequence obtained from the above exact sequence, it is enough to show the following claim:

Claim 2.4. $h^1(\hat{\mathbf{P}}^2, \mathcal{O}_{\hat{\mathbf{P}}^2}(-\mu^{-1}E^\vee + L)) = 0$.

Proof of Claim 2.4. By the Riemann-Roch theorem and the Serre duality, we have

$$\begin{aligned} & \chi\left(\hat{\mathbf{P}}^2, \mathcal{O}_{\hat{\mathbf{P}}^2}(-\mu^{-1}E^\vee + L)\right) \\ &= \chi\left(\hat{\mathbf{P}}^2, \mathcal{O}_{\hat{\mathbf{P}}^2}(\mu^*E^\vee - E_1 - E_2 - E_3)\right) \\ &= \frac{1}{2}(\mu^*l - E_1 - E_2 - E_3)(\mu^*l - E_1 - E_2 - E_3 - K_{\hat{\mathbf{P}}^2}) + 1 \\ &= 0. \end{aligned}$$

As $h^0\left(\hat{\mathbf{P}}^2, \mathcal{O}_{\hat{\mathbf{P}}^2}(-\mu^{-1}E^\vee + L)\right) = 0$, Claim 2.4 easily follows from the following lemma:

Lemma 2.5. *If $h^0(\hat{\mathbf{P}}^2, \mathcal{O}(\mu^*l - E_1 - E_2 - E_3)) \neq 0$, then E is projectively equivalent to the Fermat cubic curve.*

Proof. Suppose that three cusps \check{p}_1, \check{p}_2 and \check{p}_3 on E^\vee are on a line. This means that three tangent lines at p_1, p_2 and p_3 of E meet at one point, which we denote by Q . Let $\pi_Q : \mathbf{P}^2 \setminus Q \rightarrow \mathbf{P}^1$ denote the projection from Q to \mathbf{P}^1 . Then the above fact implies that $\pi_Q|_E : E \rightarrow \mathbf{P}^1$ is a triple covering having the ramification index 3 at each ramification point. Hence by Corollary 3.2 in [T], $\pi_Q|_E$ is cyclic. Thus E has an automorphism of degree 3 with 3 fixed points. Hence $E \cong \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega$, $\omega = \exp(2\pi\sqrt{-1}/3)$, i.e., $j(E) = 0$. This means that E is projectively equivalent to the Fermat cubic curve. This contradicts our assumption.

By Remark 2.3, any two of the twelve (2, 3) torus decompositions as above satisfy the condition in Theorem 1.1. Hence we have Theorem 0.2 (i).

§3. PROOF OF THEOREM 0.2 (ii)

We first show that B has two torus decompositions. The key idea to find them is an elementary fact on an elliptic curve as follows:

Fact 3.1. Let E be an elliptic curve over a field k ($\text{char}(k) \neq 2, 3$) given by the Weierstrass equation $y^2 = x^3 + ax + b$ ($a, b \in k$). Let (u, v) be a k -rational point on E which is a 3-torsion point in the Mordell-Weil group, $MW(E)$, i.e., an inflection point. Then the right hand side of the Weierstrass equation of E has decomposition

$$x^3 + ax + b = (x - u)^3 + (\alpha x + \beta)^2,$$

where the line $y = \alpha x + \beta$ is the tangent line of E at (u, v) .

Let $F(X_0, X_1, X_2)$ be the equation in Theorem 0.2 (ii) and put $f(x, t) = F(1, x, t)$. Consider a smooth elliptic surface \mathcal{E} given by the Weierstrass equation

$$y^2 = f(x, t).$$

This elliptic surface is nothing but the surface X_{3333} in Table 5.3 in [MP]. The Mordell-Weil group, $MW(X_{3333})$, of X_{3333} has 8 non-trivial 3-torsion points (see Table 5.3, [MP]). We choose two of them,

$$(-3t^2, 4\sqrt{-1}(t^3 + 1)) \quad \text{and} \quad ((t - 2)^2, 4\sqrt{3}(t^2 - t + 1)).$$

By applying Fact 3.1 to these points, we have:

Lemma 3.2. B has two torus decompositions

$$(3.2.1) \quad (X_0X_1 + 3X_2^2)^3 - (3X_0X_1X_2 + 5X_2^3 - 4X_0^3)^2$$

and

$$(3.2.2) \quad (X_0X_1 - (X_2 - 2X_0)^2)^3 + 3(X_0X_1X_2 - 2X_0^2X_1 - X_2^3 + 2X_0X_2^2 - 8X_0^2X_2 + 4X_0^3)^2.$$

We show that (3.2.1) and (3.2.2) satisfy the condition in Theorem 1.1.

Lemma 3.3. *The two pencils*

$$(3.3.1) \quad \{\lambda_0(X_0X_1 + 3X_2^2)^3 + \lambda_1(3X_0X_1X_2 + 5X_2^3 - 4X_0^3)^2 = 0\}_{[\lambda_0:\lambda_1] \in \mathbf{P}^1}$$

and

$$(3.3.2) \quad \{\lambda_0(X_0X_1 - (X_2 - 2X_0)^2)^3 + \lambda_1(X_0X_1X_2 - X_0^2X_1 - X_2^3 + 2X_0X_2^2 - 8X_0^2X_2 + 4X_0^3)^2 = 0\}_{[\lambda_0:\lambda_1] \in \mathbf{P}^1}$$

are different.

Proof. Straightforward computations show that the intersection points of $X_0X_1 + 3X_2^2 = 0$ and $3X_0X_1X_2 + 5X_2^3 - 4X_0^3 = 0$ are $[1 : -3 : -1]$, $[1 : -3\omega^2 : -\omega]$, $[1 : -3\omega : -\omega^2]$ ($\omega = \exp(2\pi\sqrt{-1}/3)$), and $[0 : 1 : 0]$, where $[0 : 1 : 0]$ is an intersection point with multiplicity 3. Similarly the intersection points of $X_0X_1 - (X_2 - 2X_0)^2 = 0$ and $X_0X_1X_2 - X_0^2X_1 - X_2^3 + 2X_0X_2^2 - 8X_0^2X_2 + 4X_0^3 = 0$ are $[1 : -3\omega^2 : -\omega]$, $[1 : -3\omega : -\omega^2]$ ($\omega = \exp(2\pi\sqrt{-1}/3)$), and $[0 : 1 : 0]$, where $[0 : 1 : 0]$ is an intersection point with multiplicity 4. Thus the base points of (3.3.1) and (3.3.2) are different. Hence the two pencils are different.

Now the remaining statements follow from the next two lemmas.

Lemma 3.4. B is irreducible.

Proof. Put $f(x, t) = F(1, x, t)$. It is enough to show that f is irreducible. Let \mathcal{E} be the elliptic surface as above. Then, by the definition of the group law on \mathcal{E} , $f(x, t)$ is reducible if and only if $MW(\mathcal{E})$ has a non-trivial 2-torsion. In our case, $\mathcal{E} = X_{3333}$. Hence $MW(\mathcal{E}) \cong \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ by Table 5.3 in [MP]. Therefore $f(x, t)$ is irreducible.

Lemma 3.5. *The normalization of B is rational.*

Proof. By direct computations, we can see that B has singular points at $[1 : -3 : -1]$, $[1 : -3\omega^2 : -\omega]$, $[1 : -3\omega : -\omega^2]$ ($\omega = \exp(2\pi\sqrt{-1}/3)$), and $[0 : 1 : 0]$; and the singularities at the first three points are cusps and the singularity at $[0 : 1 : 0]$ is a triple point with the following property:

Let $\nu : \hat{\mathbf{P}}^2 \rightarrow \mathbf{P}^2$ be a blowing-up at $[0 : 1 : 0]$ and let $\nu^{-1}B$ be the proper transform of B . Then the singularity of $\nu^{-1}B$ lying over $[0 : 1 : 0]$ is a d_7 singularity.

This shows that the normalization of B is rational.

§4. FURTHER EXAMPLES

From the two examples in Theorem 0.2, we have infinitely many examples of irreducible affine plane curves with the Albanese dimension 2. In fact we have the following theorem:

Theorem 4.1. *Let B be as in Theorem 0.2, and let B_a be the affine plane curve given in the Introduction. Let $f(x, y) = 0$ be the defining equation of B_a , and let $g(t)$ be a polynomial. Then the Albanese dimension of an affine plane curve, $B_{g(t)}$, defined by the affine equation $f(x, g(t)) = 0$ is 2.*

Remark 4.2. In Theorem 4.1, if we choose $g(t)$ in such a way that $f(x, g(t))$ is irreducible, then $B_{g(t)}$ is an irreducible affine plane curve with $a(B_{g(t)}) = 2$.

Proof. Put $K = \mathbf{C}(x, t)(\zeta)$, $\zeta^6 = f(x, g(t))$. Let $S_6(g(t))$ be a smooth model of the K -normalization of \mathbf{P}^2 , and let S_6 be as before. Then $S_6(g(t))$ is a 6-fold cyclic covering of \mathbf{P}^2 branched along $B_{g(t)}$ and possibly the line at infinity. As the rational function field of S_6 is a subfield of that of $S_6(g(t))$, there exists a dominant rational map, $\varphi : S_6(g(t)) \cdots \rightarrow S_6$. Let $q : \hat{S}_6(g(t)) \rightarrow S_6(g(t))$ be the resolution of indeterminacy of φ . Then we have a commutative diagram

$$\begin{array}{ccc} \hat{S}_6(g(t)) & \xrightarrow{\alpha_{g(t)} \circ q} & Alb(S_6(g(t))) \\ \downarrow & & \downarrow \\ S_6 & \xrightarrow{\alpha_6} & Alb(S_6). \end{array}$$

This shows $\dim \alpha_{g(t)} \circ q(\hat{S}_{g(t)}) = 2$. Hence $a(B_{g(t)}) = 2$.

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