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GENERALIZED POWER SYMMETRIC STOCHASTIC MATRICES

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Abstract. We characterize stochastic matrices A which satisfy the equation $(A^p)^T = A^m$ where $p < m$ are positive integers.

1. Introduction

An $n \times n$ matrix is called stochastic if every entry is nonnegative and each row sum is one. The transpose of the matrix A will be denoted by A^T. A matrix A is doubly stochastic if both A and A^T are stochastic.

Sinkhorn [5] characterized stochastic matrices A which satisfy the condition $A^T = A^p$, where $p > 1$ is a positive integer. Such matrices were called power symmetric in [5]. In this paper we consider a generalization. Call a square matrix A generalized power symmetric if $(A^p)^T = A^m$, where $p < m$ are positive integers. We give a characterization of generalized power symmetric stochastic matrices, thereby generalizing Sinkhorn's result. The proof makes nontrivial use of the machinery of generalized inverses. In view of the fact that $(A^n)_{i,j}$ represents the probability of an event to change from the state i to the state j in n units of time, the reader may find it interesting to interpret the condition $(A^p)^T = A^m$ on the matrix $A = (A_{i,j})$.

The paper is organized as follows. In the next section, we introduce some definitions and prove several preliminary results. The main results are proved in Section 3.

2. Preliminary results

If A is an $m \times n$ matrix, then an $n \times m$ matrix G is called a generalized inverse of A if $AGA = A$. If A is a square matrix, then G is the group inverse of A if $AGA = A, GAG = G$ and $AG = GA$. We refer to ([1], [2], [3]) for the background concerning generalized inverses. It is well known that A admits group inverse if and only if $rank(A) = rank(A^2)$, in which case the group inverse, denoted by $A^{\#}$, is unique.

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If A is an $n \times n$ matrix, then the index of A, denoted by $index(A)$, is the least positive integer k such that $rank(A^k) = rank(A^{k+1})$. Thus A has group inverse if and only if $index(A) = 1$.

If A is an $m \times n$ real matrix, then the $n \times m$ matrix G is called the Moore-Penrose inverse of A if it satisfies $AGA = A, GAG = G, (AG)^T = AG, (GA)^T = GA$. The Moore-Penrose inverse of A, which always exists and is unique, is denoted by $A^\dagger$. A real matrix A is said to be an EP matrix if the column spaces of A and A^T are identical. We refer to [1] or [3] for elementary properties of EP matrices.

Lemma 1. *Let A be a real $n \times n$ matrix and suppose $(A^p)^T = A^m$, where $p < m$ are positive integers. Then the following assertions are true:*

(i) $rank(A^p) = rank(A^k), \ k \geq p$,
(ii) $index(A) \leq p$,
(iii) $index(A^p) = 1$,
(iv) A^p *is an* EP *matrix,*
(v) $(A^p)^\# = (A^p)^\dagger$.

Proof. (i) Clearly, $rank(A^p) \geq rank(A^k) \geq rank(A^m)$ for $p \leq k \leq m$. Since $(A^p)^T = A^m$, we have

$$rank(A^p) \ = \ rank(A^p)^T \ = \ rank(A^m)$$

and thus $rank(A^p) = rank(A^k), \ p \leq k \leq m$. It follows that $rank(A^p) = rank(A^k), \ k \geq p$.

(ii) By (i), $rank(A^p) = rank(A^{p+1})$ and hence $index(A) \leq p$.
(iii) By (i), $rank(A^p) = rank(A^{2p})$ and hence $index(A^p) = 1$.
(iv) Since $rank(A^p) = rank(A^m)$, the column space of A^p is the same as that of A^m. Also, since $(A^p)^T = A^m$, the column space of A^m is the same as the row space of A^p, written as a set of column vectors. Therefore, the column spaces of A^p and $(A^p)^T$ are identical and A^p is an EP matrix.
(v) It is known (see, for example, [3], p. 129) that for an EP matrix the group inverse exists and coincides with the Moore-Penrose inverse.

Lemma 2. *Let A be an $n \times n$ matrix such that $(A^p)^T = A^m$, where $p < m$ are positive integers. Let $\gamma = m - p$. Then $A^p = A^p A^{i\gamma}(A^{i\gamma})^T$ for any integer $i \geq 0$.*

Proof. We have

$$\begin{aligned} A^p &= (A^{p+\gamma})^T \\ &= (A^p)^T (A^\gamma)^T \\ &= A^{p+\gamma}(A^\gamma)^T \\ &= A^\gamma A^p (A^\gamma)^T \\ &= A^\gamma(A^\gamma A^p (A^\gamma)^T)(A^\gamma)^T \\ &= A^{2\gamma} A^p (A^{2\gamma})^T. \end{aligned} \tag{1}$$

Repeating the argument, we get

$$A^p = A^{i\gamma} A^p (A^{i\gamma})^T = A^p A^{i\gamma}(A^{i\gamma})^T$$

for any $i \geq 0$ and the proof is complete.

Lemma 3. *Let A be a nonnegative $n \times n$ matrix and suppose $(A^p)^T = A^m$, where $p < m$ are positive integers. Then $(A^p)^\#$ exists and is nonnegative.*

Proof. By Lemma 1 (iii), $index(A^p) = 1$ and therefore $(A^p)^\#$ exists. Let $\gamma = m - p$. Setting $i = p$ in Lemma 2, we get

$$A^p = A^p A^{p\gamma}(A^{p\gamma})^T = A^p A^{p\gamma} A^{(p+\gamma)\gamma}$$

in view of $(A^p)^T = A^m$. Thus, since $\gamma \geq 1$,

$$A^p = A^{2p} A^{p(\gamma-1)} A^{(p+\gamma)\gamma}$$

where A^o is taken to be the identity matrix. Let $B = A^{p(\gamma-1)} A^{(p+\gamma)\gamma}$. Then it follows from the previous equation that $A^p B A^p = A^p$. Also, since B is a power of A, $A^p B = B A^p$. Thus $(A^p)^\# = B A^p B$ is also a power of A and hence is nonnegative.

Lemma 4. *Let A be a stochastic $n \times n$ matrix and suppose $(A^p)^T = A^m$, where $p < m$ are positive integers. Then there exists a permutation matrix P such that PAP^T is a direct sum of matrices of the following two types I and II:*

I. *C_{11} where $C_{11}^v = xx^T, x > 0$ for some positive integer v.*

II. $$\begin{bmatrix} 0 & C_{12} & 0 & \cdots & 0 \\ 0 & 0 & C_{23} & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \cdots & & & C_{d-1,d} \\ C_{d1} & 0 & \cdots & & 0 \end{bmatrix}$$ *where there exists a positive integer u such that $(C_{12}\ C_{23}\ \cdots\ C_{d1})^u = x_1 x_1^T,\ \cdots, (C_{d1}\ C_{12}\ \cdots\ C_{d-1,d})^u = x_d x_d^T$ for some positive vectors $x_1, \cdots, x_d$.*

Proof. Let $C = A^p(A^p)^\#$. Since $(A^p)^\# = (A^p)^\dagger$ by Lemma 1, C is symmetric. By Lemma 3, $(A^p)^\#$ is nonnegative and hence C is nonnegative. Also, C is clearly idempotent. As observed in the proof of Lemma 3, $(A^p)^\#$ is a power of A and hence C is also a power of A.

The nonnegative roots of symmetric, nonnegative, idempotent matrices have been characterized in [4], Theorem 2. Applying the result to C, we get the form of A asserted in the present result. We remark that according to Theorem 2 of [4], a type III summand is also possible along with the two types mentioned above. However, a matrix of this type is necessarily nilpotent and since A is stochastic, such a summand is not possible.

Remark. Using the notation of Lemma 4, it can be verified that the type II summand given there has the property that its (du)th power is

$$\begin{bmatrix} x_1 x_1^T & 0 & \cdots & 0 \\ 0 & x_2 x_2^T & \cdots & 0 \\ \cdot & & & \\ \cdot & & & \\ 0 & 0 & \cdots & x_d x_d^T \end{bmatrix}$$

This observation will be used in the sequel.

We now introduce some notation. Denote by $J_{m \times n}$ the $m \times n$ matrix with each entry equal to one. If $n_1, \cdots, n_d$ are positive integers adding up to n, then

$\hat{J}_{(n_1,\,\cdots\,,n_d)}$ will denote the $n \times n$ matrix

$$\begin{bmatrix} 0 & \frac{1}{n_2}\ J_{n_1 \times n_2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{n_3}\ J_{n_2 \times n_3} & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & \cdots & & \frac{1}{n_d}\ J_{n_{d-1} \times n_d} \\ \frac{1}{n_1}\ J_{n_d \times n_1} & 0 & \cdots & & 0 \end{bmatrix}$$

where the zero blocks along the diagonal are square, of order $n_1, n_2, \ \cdots\ , n_d$, respectively. We remark that if $d = 1$, then $\hat{J}_{(n_1)} = \frac{1}{n_1}\ J_{n_1 \times n_1}$.

From now onwards, we make the convention that when we deal with integers $n_1, \ \cdots\ , n_d$, the subscripts of n should be interpreted modulo d. Thus, for example, $n_{d+1} = n_1, n_{-2} = n_{d-2}$ and so on.

Lemma 5. *Let $n_1, \ \cdots\ , n_d$ be positive integers summing to n and let $p < m$ be positive integers. Let $p = \mu$ mod $d, m = \mu'$ mod d where $0 \le \mu \le d-1, 0 \le \mu' \le d-1$. Let $S = \hat{J}_{(n_1,\,\cdots\,,n_d)}$. Then the following conditions are equivalent:*

- (i) $(S^p)^T = S^m$;
- (ii) *(a) d divides both p and m, or (b) d divides neither p nor $m, \mu + \mu' = d$ and $n_i = n_{i+\mu'}, \quad 1 \le i \le d$;*
- (iii) *(a) d divides both p and m, or (b) d divides neither p nor $m, \mu + \mu' = d$ and $n_i = n_{i+\delta}, \quad 1 \le i \le d$, where $\delta = (\mu, \mu')$ is the g.c.d. of μ and μ'.*

Proof. We first observe that

$$S^d = \begin{bmatrix} \frac{1}{n_1}\ J_{n_1 \times n_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{n_2}\ J_{n_2 \times n_2} & \cdots & 0 \\ \cdot & & & \\ \cdot & & & \\ 0 & 0 & \cdots & \frac{1}{n_d}\ J_{n_d \times n_d} \end{bmatrix}$$

$S^d S = S$ and so $S^d S^i = S^i$, for all $i > 0$.

Note (i) implies d divides p if and only if d divides m. We first prove (i)⇔(ii):

Assume d divides neither p nor m. Let $(S^m)_i$ denote the (i,j)-block in the partitioning of S^μ, in conformity with the partitioning of $S, 1 \le i, j \le d$. Similarly $(S^{\mu'})_{ij}$ will denote the (i,j)-block in $S^{\mu'}$. A straightforward multiplication involving partitioned matrices shows that

$$(S^\mu)_{ij} = \begin{cases} \frac{1}{n_{\mu+i}}\ J_{n_i \times n_{\mu+i}}, & \text{if } j = (\mu + i) \ mod\ d, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly,

$$(S^{\mu'})_{ij} = \begin{cases} \frac{1}{n_{\mu'+i}}\ J_{n_i \times n_{\mu'+i}}, & \text{if } j = (\mu' + i) \ mod\ d. \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$(S^\mu)^T_{ij} = \begin{cases} \frac{1}{n_{\mu+j}}\ J_{n_{\mu+j} \times n_j}, & \text{if } i = (\mu + j) \ mod\ d, \\ 0, & \text{otherwise.} \end{cases}$$

or equivalently,

$$(S^\mu)^T_{ij} = \begin{cases} \frac{1}{n_{\mu+j}}\ J_{n_{\mu+j} \times n_j}, & \text{if } j = (i - \mu) \ mod\ d, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $(S^\mu)^T = S^{\mu'}$ holds if and only if $\mu' + i = (i - \mu)\ mod\ d$ and $n_i = n_{\mu'+i}$. Since $(S^p)^T = S^m$ holds if and only if $(S^\mu)^T = S^{\mu\prime}$, it follows that under the condition d does not divide p and d does not divide m, (i) holds if and only if $\mu + \mu\prime = d$ and $n_i = n_{i+\mu\prime}, 1 \leq i \leq d$.

Furthermore, because $d \mid p$ and $d \mid m$ trivially imply $(S^p)^T = S^m$, the proof of (i)$\Leftrightarrow$(ii) is completed.

If (ii) holds, then $n_i = n_{\mu'+i} = n_{d-\mu+i}, \quad 1 \leq i \leq d$. Hence

$$n_i = n_{\alpha\mu' - \beta\mu + \beta d + i} = n_{\alpha\mu' - \beta\mu + i}\ ,$$

where α, β are positive integers, $1 \leq i \leq d$. Choosing α, β such that $\delta = \alpha\mu' - \beta\mu$, we get $n_i = n_{\delta+i},\ 1 \leq i \leq d$. This proves (iii).

Conversely, if (iii) holds, then $n_i = n_{\mu'+i}$ since δ divides μ', establishing (ii). This completes the proof.

3. The main results

We are now ready to give a characterization of generalized power symmetric stochastic matrices. In the next result we consider matrices of index one.

Theorem 1. *Let A be a stochastic $n \times n$ matrix with $index(A) = 1$ and let $p < m$ be positive integers. Then $(A^p)^T = A^m$ if and only if there exists a permutation matrix P such that PAP^T is a direct sum of matrices of the following two types I and II:*

I. $\frac{1}{k}\ J_{k\times k}$ for some positive integer k;

II. $d \times d$ block partitioned matrices of the form $\hat{J}_{(n_1,\ \cdots\ ,n_d)}$ satisfying (a) $d \mid p$ and $d \mid m$, or (b) d divides neither p nor m such that if $p = \mu (mod\ d), m = \mu\prime (mod\ d)$ where $0 \leq \mu, \mu\prime \leq d-1$ and $\delta = (\mu, \mu\prime)$, then $\mu + \mu\prime = d, n_i = n_{i+\delta}$.

Proof. First suppose that $(A^p)^T = A^m$. By Lemma 4 we conclude that there exists a permutation matrix P such that PAP^T is a direct sum of matrices of types I, II given in Lemma 4.

Let $S = C_{11}$ be a summand of type I so that $C_{11}^v = xx^T, x > 0$ for some positive integer v. Since $index(C_{11}) = 1$, it follows that $rank(C_{11}) = rank(C_{11}^v) = 1$. Suppose $C_{11} = \tilde{x}\tilde{y}^T$ for some $\tilde{x} > 0, \tilde{y} > 0$. Now using $(C_{11}^p)^T = C_{11}^m$, it can be shown that C_{11} is symmetric and hence doubly stochastic. It is well known that a positive, rank one doubly stochastic matrix must be $\frac{1}{k}\ J_{k\times k}$ for some k.

Now suppose S is a summand of type II and let

$$S = \begin{bmatrix} 0 & C_{12} & 0 & \cdots & 0 \\ 0 & 0 & C_{23} & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & \cdots & & C_{d-1,d} \\ C_{d1} & 0 & \cdots & & 0 \end{bmatrix}$$

where the zero blocks on the diagonal are square of order $n_1,\ \cdots, n_d$, respectively. As observed in the remark following Lemma 4, there exists a positive integer u such that S^{du} is a direct sum of d positive, symmetric, idempotent, rank one matrices.

We first claim that $rank(C_{12}) = rank(C_{23}) = \ \cdots\ = rank(C_{d1}) = 1$. For otherwise, $rank(S) > d$. However, $rank(S^{du}) = d$ and since $index(S) = 1, rank(S) = rank(S^{du})$, giving a contradiction. Therefore, the claim is proved. We let $C_{12} =$

$x_1y_2^T, C_{23} = x_2y_3^T, \cdots, C_{d1} = x_dy_1^T$, where x_i, y_i are positive vectors. Since $C_{12}, C_{23}, \cdots, C_{d1}$ have rank one, we may choose $x_i = J_{n_i\times 1},\ 1 \leq i \leq d$.

In view of the description of S^{du} given earlier, we may write

$$(C_{12}\ C_{23}\ \cdots\ C_{d1})^u = xx^T$$

for some positive vector x. Thus

$$(x_1y_2^T\ x_2y_3^T\ \cdots\ x_dy_1^T)^u = xx^T$$

and hence $\lambda x_1y_1^T = xx^T$ for some $\lambda > 0$. It follows that $y_1 = \gamma_1 J_{n_1\times 1}$ for some $\gamma_1 > 0$. Similarly we conclude that $y_i = \gamma_i\ J_{n_i\times 1}, \gamma_i > 0,\ 1 \leq i \leq d$. Now, interpreting subscripts modulo d as usual, we have

$$\begin{aligned} C_{i,i+1} &= x_iy_{i+1}^T \\ &= J_{n_i\times 1}\ (\gamma_{i+1}\ J_{n_{i+1}\times 1})^T \\ &= \gamma_{i+1}\ J_{n_i\times n_{i+1}} \\ &= \frac{1}{n_{i+1}}\ J_{n_i\times n_{i+1}}\ , \end{aligned} \tag{2}$$

since $C_{i,i+1}$ has row sums one, $1 \leq i \leq d$. The remaining assertions in (ii) follow from Lemma 5.

Conversely, it is clear that if S is a summand of type I or II, then $(S^p)^T = S^m$. This completes the proof.

Corollary 1. *Let A be an $n\times n$ stochastic matrix of index one and suppose $(A^p)^T = A^m$ where $p < m$ are positive integers with $(p, m) = 1$. Then there exists a permutation matrix P such that PAP^T is a direct sum of matrices of the following types I and II:*

I. $\frac{1}{k}\ J_{k\times k}$ *for some positive integer k;*
II. $\hat{J}_{(\ell,\ \cdots\ ,\ell)}$ *for some positive integer ℓ, occurring d times and d divides $p + m$.*

Proof. By Theorem 1 there exists a permutation matrix P such that PAP^T is a direct sum of matrices of types I, II given in Theorem 1. Consider a summand of type II, say $\hat{J}_{(n_1,\ \cdots\ ,n_d)}$. Let $\mu = p\ mod\ d, \mu' = m\ mod\ d$. Then by Theorem 1, $\mu + \mu' = d$ and $n_i = n_{i+\delta}$, where δ is the *g.c.d.* of μ, μ'. Since $(p, m) = 1$ and $\mu + \mu' = d$, it follows that $\delta = 1$. Thus $n_1 = n_2 = \cdots = n_d = \ell$, say. Thus the summand equals $\hat{J}_{(\ell,\ \cdots\ ,\ell)}$. Also, $\mu + \mu' = d$ implies that $p + m = 0\ mod\ d$ and thus d divides $p + m$.

We now derive the main result in [5].

Corollary 2. *Let A be an $n \times n$ stochastic matrix and suppose $A^T = A^m$ for some positive integer $m > 1$. Then there exists a permutation matrix P such that PAP^T is a direct sum of matrices of the following types:*

(1) $\frac{1}{k}\ J_{k\times k}$ *for some positive integer k;*
(2) $\hat{J}_{(\ell,\ \cdots\ ,\ell)}$ *for some positive integer ℓ, where ℓ occurs d times and d divides $m + 1$.*

Proof. By Lemma 1, *index*$(A) = 1$. Since the *g.c.d.* of $1, m$ is 1, by Corollary 1 there exists a permutation matrix P such that PAP^T is a direct sum of matrices of types given in Corollary 1. The rest of the proof follows easily.

Our next objective is to describe a stochastic matrix A, not necessarily of index one, which satisfies $(A^p)^T = A^m$ for positive integers $p < m$.

If A is an $n \times n$ matrix, then recall that A can be expressed as $A = C_A + N_A$ where C_A, the core part of A, is of index one, N_A is nilpotent and $C_A N_A = N_A C_A = 0$. This is referred to as the core-nilpotent decomposition of A. We refer to [2] for basic properties of this decomposition.

Lemma 6. *Let A be a stochastic $n \times n$ matrix such that $(A^p)^T = A^m$, where $p < m$ are positive integers. Let $A = C_A + N_A$ be the core-nilpotent decomposition. Then C_A is nonnegative and stochastic.*

Proof. By Lemma 1, $index(A) \le p$. Then $A^p = C_A^p$. Since $index(C_A) = 1$, we can write $C_A = C_A^p X$ for some matrix X. Now

$$\begin{aligned} A^p(A^p)^{\#}C_A &= A^p(A^p)^{\#}\ C_A^p X \\ &= A^p(A^p)^{\#}\ A^p X \\ &= A^p X \\ &= C_A^p X \\ &= C_A \end{aligned} \tag{3}$$

and then

$$\begin{aligned} A^p(A^p)^{\#}A &= A^p(A^p)^{\#}(C_A + N_A) \\ &= C_A + A^p(A^p)^{\#}\ N_A \\ &= C_A + (A^p)^{\#}\ C_A^p N_A \\ &= C_A. \end{aligned} \tag{4}$$

Since $(A^p)^{\#}$ is a power of A, as observed in the proof of Lemma 3, it follows that C_A is also a power of A. Thus C_A is nonnegative and stochastic.

Theorem 2. *Let A be an $n \times n$ stochastic matrix. Then $(A^p)^T = A^m$, where $p < m$ are positive integers if and only if there exists a permutation matrix P such that PAP^T is a direct sum of matrices $C_{ii} + N_{ii}, 1 \le i \le k$, where*

1. *C_{ii} are stochastic matrices of index $1, N_{ii}$ are nilpotent matrices of index $\le p$ with sum of entries of each row as zero, $C_{ii}N_{ii} = 0 = N_{ii}C_{ii}, 1 \le i \le k$, and*
2. *Each C_{ii} is a direct sum of matrices of types (I) and (II) as described in Theorem 1.*

Proof. ‘Only if’ part: By Lemma 1, $index(A) \le p$ and hence $A^p = C_A^p, A^m = C_A^m$. Thus $(C_A^p)^T = C_A^m$. Since $index(C_A) = 1$ and by Lemma 6 C_A is stochastic, Theorem 1 yields that there exists a permutation matrix P such that PC_AP^T is a direct sum of matrices of types I, II as given in Theorem 1.

Let

$$PAP^T = [A_{ij}], PC_AP^T = [(C_A)_{ij}], \quad PN_AP^T = [(N_A)_{ij}]$$

be compatible partitions. Note that $(C_A)_{ij} = 0$ if $i \ne j$. Since $A_{ij} = (C_A)_{ij} + (N_A)_{ij}$, we get that $(N_A)_{ij} \ge 0$ for $i \ne j$. However, since $C_AN_A = 0$, we have $(C_A)_{ii}(N_A)_{ij} = 0, i \ne j$. It follows from the description of $(C_A)_{ii}$, in Theorem 1 and the remark following Lemma 4 that for some positive integer $s, (C_A)_{ii}^s$ has positive diagonal entries. This, along with $(C_A)_{ii}(N_A)_{ij} = 0$, and $(N_A)_{ij} \geq 0$ for $i \neq j$, yields $(N_A)_{ij} = 0$. Thus $A_{ij} = 0, i \neq j$. Setting $C_{ii} = (C_A)_{ii}, N_{ii} = (N_A)_{ii}$ for all i, the result follows. The 'if part' is straightforward.

We conclude with an example. Let

$$A = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix}$$

Then A is stochastic and $(A^2)^T = A^4$. The core-nilpotent decomposition is given by $A = C_A + N_A$, where

$$C_A = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \quad , \quad N_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{-1}{6} & 0 & 0 \end{bmatrix}$$

Thus C_A is stochastic and is of type II as described in Theorem 2. This example also shows that N_A need not be nonnegative.

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