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SOME REMARKS ON METRIC SPACES WHOSE PRODUCT WITH EVERY LINDELÖF SPACE IS LINDELÖF

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ABSTRACT. Let us assume that Martin's Axiom holds. We prove that if X is a metrizable space whose product with every Lindelöf space is Lindelöf, then for every metric d on X, consistent with the topology of $X, (X,d)$ is a countable union of totally bounded subsets.

E. Michael proved that under the Continuum Hypothesis, abbreviated (CH), if X is a metric space whose product with every Lindelöf space is Lindelöf, then X is σ-compact. It is an open question whether (CH) can be relaxed to Martin's Axiom, abbreviated (MA). The aim of this note is to prove the result mentioned in the abstract.

The topological terminology follows [E], and we have adopted set-theoretical terminology from [K2]. In this note, N stands for the set of natural numbers and P for the space of the irrational numbers. The irrational numbers will be viewed as sequences of natural numbers. The symbol $\mathfrak{c}$ will denote the initial ordinal number of the cardinality continuum. Other ordinal numbers will be denoted by Greek letters. If $x = (x(n))_{n=1}^{\infty}$ and $y = (y(n))_{n=1}^{\infty}$ are irrational numbers, we say that $x <_* y$, $x \leq_* y$ or $x =_* y$ if for all but finitely many n $x(n) < y(n)$, $x(n) \leq y(n)$ or $x(n) = y(n)$ respectively. If A is a subset of a metric space (X,d), then the symbol $\operatorname{diam}(A) < \epsilon$ means that the distance between any two points of A is less than ϵ.

Theorem (MA). *If X is a metrizable space whose product with every Lindelöf space is Lindelöf, then the following conditions hold:*

a) For every metric d on X, consistent with the topology of $X, (X,d)$ is a countable union of totally bounded subsets.

b) Every closed cover of X of cardinality less than continuum has a countable subcover.

c) If X is an analytic space then X is σ-compact.

Proof of a). Let us first assume that (X,d) is a zero-dimensional space. Note that since X is separable, $X = Z \cup A$, where A is σ-compact and Z is closed and not locally compact at any point. Without loss of generality we can assume that $X = Z$.

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There is a sequence $(\mathcal{U}_i)_{i=1}^{\infty}$, where $\mathcal{U}_i = \{U_p : p \in N^i\}$, of open covers of X such that

1) $\mathcal{U}_i$ consists of pairwise disjoint non-empty sets,
2) if $U_p \in \mathcal{U}_i$ then $\operatorname{diam}(U_p) < \frac{1}{i}$,
3) if $p \in N^i$ and $q \in N^{i+1}$ and q extends p, then $U_q \subset U_p$.

Let us define a continuous function g of X to P by the following formulas: $g(x) = p_x$, where $p_x \in P$ is such that $\{x\} = \bigcap_{n=1}^{\infty} U_{p_x|n}$, $p_x = (p_x(n))_{n=1}^{\infty}$ and $p_x|n = (p_x(1), \dots, p_x(n))$. It is obvious that g is well defined and that it is a continuous function.

Let $\{x_\alpha : \alpha < \mathfrak{c}\}$ be a $\mathfrak{c}$-scale in P, which exists under (MA) (see [K2], exercise 8, p. 87).

Claim. There is $\alpha < \mathfrak{c}$ such that

$$g(X) \subset \{p \in P : p \leq_* x_\alpha\} = P_\alpha.$$

Assume that the claim holds. Observe that if $\{p \in P : p =_* x_\alpha\} = \{p_n : n \in N\}$ and $Z_n = \{p \in P : p \leq p_n\}$, then $g^{-1}(Z_n) = X_n$ is totally bounded for $n \in N$. Indeed if $\epsilon > 0$, $\frac{1}{i} < \epsilon$, and $T = \{t \in N^i : t \leq p_n|i \text{ and } U_t \cap X_n \neq \emptyset\}$, then let x_t be a point of $U_t \cap X_n$ for $t \in T$. Observe that the union of balls with centers at x_t, for $t \in T$, and radius ϵ covers X_n; and this means that X_n is totally bounded.

Proof of Claim. Suppose that

4) $g(X) \setminus P_\alpha \neq \emptyset$ for $\alpha < \mathfrak{c}$.

Let Y be a metric compactification of P and put $P_{\mathfrak{c}} = Y$. The topology on $P_{\mathfrak{c}}$ is generated by the sets of the form $P_0 \cap H$ or $(P_{\alpha_2} \setminus P_{\alpha_1}) \cap H$, where $0 \leq \alpha_1 < \alpha_2 \leq \mathfrak{c}$ and H is open in Y. In [A1] it was proved that $P_{\mathfrak{c}}$ is Lindelöf. Observe that $K = \{(y, y) : y \in g(X)\}$ is a closed subset of $g(X) \times P_{\mathfrak{c}}$; from 4) it follows that the family $\{(g(X) \times P_\alpha) \cap K : \alpha < \mathfrak{c}\}$ covers K and does not have a countable subcover, contradicting the Lindelöf property of $g(X) \times P_{\mathfrak{c}}$.

Let X be an arbitrary metric space whose product with every Lindelöf space is Lindelöf. There exist a zero-dimensional metric and separable space Y and a perfect continuous function f of Y onto X. Since a perfect preimage of a Lindelöf space is Lindelöf, we conclude that for every Lindelöf space Z the product $Y \times Z$ is Lindelöf. Let d_1 and d be metrics on Y and X respectively. Let us define a new metric d_2 on Y by

5) $d_2(y_1, y_2) = d_1(y_1, y_2) + d(f(y_1), f(y_2))$.

Then

$$(Y, d_2) = \bigcup_{n=1}^{\infty} Y_n,$$

where Y_n is totally bounded. As $d_2(y_1, y_2) \geq d(f(y_1), f(y_2))$, we conclude that $f(Y_n)$ is totally bounded in (X, d). □

Proof of b). Let $\{X_\alpha : \alpha < \lambda\}$ be a closed cover of X with $\lambda < \mathfrak{c}$ which does not have a countable subcover. Let (Y, τ') be a metric compactification of X and τ a new topology on Y generated by $\{X_\alpha : \alpha < \lambda\}$ and τ'. Put $Z = (Y, \tau)$. By (MA) and the fact that $Z \setminus X$ is a Lindelöf subset of Z, we infer that Z is Lindelöf. The product $X \times Z$ is not Lindelöf, because $K = \{(x, x) : x \in X\}$ is a closed subset of $X \times Z$ and $\{(X \times X_\alpha) \cap K : \alpha < \lambda\}$ is an open cover of K without a countable subcover. □

Proof of c). By the Hurewicz theorem, (see [K1], theorem 21.18), if X is not σ-compact then X contains a closed subset homeomorphic to the irrationals. By the example from [A1] there is a Lindelöf space Y such that $X \times Y$ is not Lindelöf. $\square$

Question 1 (MA)**.** Let X be a metrizable separable space such that for every metric d on X, consistent with the topology of $X, (X,d)$ is a countable union of totally bounded subsets. Is it true that X is a union of less than continuum compact sets?

We do not know the answer to this question even for (CH). In this case we ask if X is σ-compact.

Remark 2. Observe that the following conditions are equivalent for a metrizable separable space X :

(i) For every metric d on X, consistent with the topology of $X, (X,d)$ is a countable union of totally bounded subsets,

(ii) For every complete metric space Y containing X there is a σ-compact set Z such that $X \subset Z \subset Y$.

Proof. (i)→(ii). If $X \subset (Y,d)$, where (Y,d) is a complete metric space, then by (i) $(X, d|X \times X) = \bigcup_{n=1}^{\infty} X_n$, where X_n is totally bounded, and $\overline{X}_n^Y$ is compact as a complete totally bounded metric space.

(ii)→(i). There is a complete separable metric space (Y,d') such that $(X,d) \subset (Y,d')$ and $d'|X \times X = d$ (see [E], Theorem 4.3.14). By (ii) there is Z such that $X \subset Z = \bigcup_{n=1}^{\infty} Z_n \subset Y$, where Z_n is compact. Then $X_n = Z_n \cap X$ is totally bounded.

Another proof of this implication follows from the fact that (ii) implies that there is $a \in P$ such that $g(X) \subset \{p \in P : p \leq a\}$, where g is taken from the proof of part a) of the theorem. $\square$

Remark 3. One can also prove the theorem using Remark 2, but in this case the reasoning is not so constructive as in the original proof (cf. [A2], Remark 16).

Question 4. Is (MA) essential in the theorem?

Question 5. Is it consistent with ZFC to assume that there is a metric space (X,d) which is not a countable union of totally bounded subsets and whose product with every Lindelöf space is Lindelöf?

Question 5 is related to a well known problem:

Question 6. Is it consistent with ZFC to assume that for every Lindelöf space Y the product $Y \times P$ is Lindelöf?

Let us finish with the following:

Proposition (MA). *If X is a metrizable space whose product with every Lindelöf space is Lindelöf, then the following condition holds:*

There is an increasing sequence $\{A_\alpha : \alpha < \lambda\}$ of σ-compact subsets in X, where $\lambda < \mathfrak{c}$, which cannot be extended, which means that there is no σ-compact set Z in X containing $\bigcup\{A_\alpha : \alpha < \lambda$ as a proper subset.

Moreover, if in addition X is not σ-compact, then there is a compact set C in X such that $\{\alpha < \lambda : A'_\alpha \cap C \neq \emptyset\}$ is cofinal with λ, where $A'_\alpha = A_\alpha \setminus \bigcup\{A_\beta : \beta < \alpha\}$.

Proof. Suppose not, and let $\{C_\alpha : \alpha < \mathfrak{c}\}$ be the family of all compact subsets of X. Put $A_0 = C_0$. If A_α has been defined for $\alpha < \beta$, then by the assumption there is a σ-compact set Z such that the union of $\{A_\alpha : \alpha < \beta\}$ is a proper subset of Z. Put $A_\beta = Z \cup C_\beta \cup \{x_\beta\}$, where x_β is an arbitrary point of $X \setminus \bigcup\{A_\alpha : \alpha < \beta\}$.

From the construction it follows that

$$(*) \qquad \bigcup\{C_{\alpha'} : \alpha' < \alpha\} \subset \bigcup\{A_{\alpha'} : \alpha' < \alpha\} \text{ for } \alpha < \mathfrak{c}.$$

Let (Y', τ_1) be a metrizable compactification of X. Then put $A_{\mathfrak{c}} = Y'$ and $Y = (Y', \tau_2)$, where τ_2 is generated by $\tau_1 \cup \{A_\alpha \setminus A_{\alpha'} : \alpha' < \alpha \leq \mathfrak{c}\} \cup \{A_0\}$.

We shall show that Y is a Lindelöf space and $X \times Y$ is not. Observe that $Y \setminus X$ is a Lindelöf space as a subspace of (Y', τ_1). Let U be an open subset of Y containing $Y \setminus X$. Then there is a σ-compact set C in X such that $Y \setminus U \subset C$, and by $(*)$ there is α such that $C \subset A_\alpha$. To finish the proof of the Lindelöf property of Y it is enough to show that A_α is a Lindelöf subspace of Y for all α. It is obvious that A_0 is Lindelöf. Suppose that A_β is Lindelöf for $\beta < \alpha$. Note that $A'_\alpha = A_\alpha \setminus \bigcup\{A_\beta : \beta < \alpha\}$ is Lindelöf as a subspace of X. Let U be an open set in Y containing A'_α. Then there is a σ-compact set D in X such that $A_\alpha \setminus U \subset D \subset \bigcup\{A_\beta : \beta < \alpha\}$. Since (MA) implies that every closed cover of a metrizable compact set of cardinalinality less than continuum has a countable subcover (see [A1], Lemma 2), by the inductive assumption we conclude that A_α is a Lindelöf space.

The product $X \times Y$ is not Lindelöf, because $K = \{(x, x) : x \in X\}$ is a closed subset of $X \times Y$ and the family $\{K \cap (X \times A_\alpha) : \alpha < \mathfrak{c}\}$ is an open cover of K without countable subcover.

Suppose that $\{A_\alpha : \alpha < \lambda\}$ is an increasing sequence of σ-compact subsets in X, where $\lambda < \mathfrak{c}$, and there is no a compact subset C in X such that $\{\alpha < \lambda : A'_\alpha \cap C \neq \emptyset\}$ is cofinal with λ, where $A'_\alpha = A_\alpha \setminus \bigcup\{A_\beta : \beta < \alpha\}$. Then, using a similar reasoning as in the first part of the proof, we conclude that $T = \bigcup\{A_\alpha : \alpha < \lambda\} \cup Y \setminus X$ is a Lindelöf subspace of Y and $X \times T$ is not Lindelöf. □

Remark 7. One can strengthen the second part of the proposition by showing that if Z is a σ-compact set then there is a compact set C in X such that $\{\alpha < \lambda : (C \setminus Z) \cap A'_\alpha \neq \emptyset\}$ is cofinal with λ.

Question 8 (MA)**.** Let λ' be an ordinal number less than continuum. Is it possible to relax the phrase "σ-compact", in the formulation of the proposition, to "union of λ' compact sets"?

Added in proof. J. Chaber pointed out to me that W. Hurewicz (*On sequences of continuous functions*, Fund. Math. 9(1927), 193-204) proved that a metric space M is a Hurewicz space if and only if every sequence (f_n) of continuous real valued functions on M is bounded, i.e. there is a sequence (r_n) of real numbers such that for all but finitely many n the range of f_n is bounded by r_n.

Recently R. Pol has answered Question 1 in the negative way.

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