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FOURIER MULTIPLIERS ON WEIGHTED L^p-SPACES

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Dedicated to Professor Leonard Y. H. Yap on the ocassion of his sixtieth birthday

Abstract. In his 1986 paper in the Rev. Mat. Iberoamericana, A. Carbery proved that a singular integral operator is of weak type (p,p) on $L^p(\mathbb{R}^n)$ if its lacunary pieces satisfy a certain regularity condition. In this paper we prove that Carbery's result is sharp in a certain sense. We also obtain a weighted analogue of Carbery's result. Some applications of our results are also given.

1. Introduction

We introduce some notation to be used in this paper. Let $\mathcal{S}(\mathbb{R}^n)$ be the class of Schwartz functions on $\mathbb{R}^n$ and let ϕ be a non-negative $C^\infty(\mathbb{R}^n)$ function supported in $\{|x| \le 1\}$ with $\int \phi = 1$. For $j \in \mathbb{Z}$ define the operator P_j on $\mathcal{S}(\mathbb{R}^n)$ by

$$P_j f = \phi_{2^j} * f,$$

where $\phi_{2^j}(x) = 2^{-jn}\phi(2^{-j}x), \quad x \in \mathbb{R}^n$. Now let ψ be a non-negative $C^\infty(\mathbb{R}^n)$ function supported in $\{1 \le |\xi| \le 4\}$ such that $\sum_{j\in\mathbb{Z}} \psi(2^j\xi) = 1$ for $\xi \ne 0$. For $j \in \mathbb{Z}$, define the operator Q_j on $\mathcal{S}(\mathbb{R}^n)$ by

$$(Q_j f)^\wedge(\xi) = \psi(2^j\xi)\hat{f}(\xi).$$

Let η be a non-negative $C^\infty(\mathbb{R}^n)$ function supported in $\{1 \le |x| \le 4\}$ such that $\sum_{j\in\mathbb{Z}} \eta(2^j x) = 1$ for $x \ne 0$. Let T and T_j be singular integral operators with kernels $K(x,y)$ and $K(x,y)\eta(2^{-j}(x-y))$, respectively.

Recently, Carbery proved the following theorem in [C, Theorem 1].

Theorem C. *Let T be a singular integral operator bounded on $L^2(\mathbb{R}^n)$ such that*

$$\sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \| \sum_{\ell\ge 0} Q_{j+k}T_{j+\ell}(I-P_j)\|_{\mathcal{M}_p} < \infty,$$

where $\|\cdot\|_{\mathcal{M}_p}$ is the operator norm on $L^p(\mathbb{R}^n)$ and I is the identity operator on $L^2(\mathbb{R}^n)$. Then T is of weak type (p,p) on $L^p(\mathbb{R}^n)$.

We show in Theorem 4.1 that Theorem C is sharp. We also obtain in Theorem 2.1 a weighted analogue of Theorem C. Theorem 2.1 is then applied to obtain two multiplier results on power-weighted L^p-spaces; see Theorems 3.1 and 3.4.

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2. Weighted analogue of Theorem C

Let w be a non-negative locally integrable function on $\mathbb{R}^n$. For $E \subset \mathbb{R}^n$ define $w(E) = \int_E w(x)dx$. For $1 \leq p < \infty$ let $L^p_w(\mathbb{R}^n)$ be the space of all measurable functions f on $\mathbb{R}^n$ such that $\|f\|_{p,w} < \infty$, where

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}.$$

We shall simply write $L^p_\alpha(\mathbb{R}^n)$ and $\|\cdot\|_{p,\alpha}$ if $w(x) = |x|^\alpha$.

The weight w is said to satisfy the Muckenhoupt A_p condition if there is a constant C such that

$$\left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C, \qquad 1 < p < \infty;$$

$$\frac{1}{|B|} \int_B w(x)dx \leq C \mathop{ess\,inf}_{x \in B} w(x), \qquad p = 1,$$

for all balls $B \subset \mathbb{R}^n$, where $|B|$ denotes the Lebesgue measure of B.

The main result in this section is the following weighted version of Theorem C.

Theorem 2.1. *Let $w \in A_p$, $1 < p < 2$, and let T be a singular integral operator bounded on $L^2_w(\mathbb{R}^n)$ such that*

$$\sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} \| \sum_{\ell \geq 0} Q_{j+k} T_{j+\ell} (I - P_j) \|_{\mathcal{M}_{p,w}} < \infty, \tag{2.1}$$

where $\|\cdot\|_{\mathcal{M}_{p,w}}$ is the operator norm on $L^p_w(\mathbb{R}^n)$. Then T is of weak type (p,p) on $L^p_w(\mathbb{R}^n)$.

The proof of Theorem 2.1 depends on the following weighted version of Calderón-Zygmund decomposition whose proof is standard and is therefore omitted.

Theorem 2.2. *Let $w \in A_p$, $1 \leq p < \infty$, and let $f \in \mathcal{S}(\mathbb{R}^n)$. For $\alpha > 0$, there exist a sequence of mutually disjoint balls $\{B_i\}$ and measurable functions g and b_i, $i \in \mathbb{N}$, such that*

(i) $f = g + \sum_{i=1}^{\infty} b_i$;

(ii) $b_i = f\chi_{B_i}$ *and* $\|b_i\|_1 \leq 2^n \alpha |B_i|$;

(iii) $\sum_{i=1}^{\infty} \|b_i\|^p_{p,w} \leq \|f\|^p_{p,w}, \quad 1 \leq p < \infty$;

(iv) $\sum_{i=1}^{\infty} w(B_i) \leq C_p \alpha^{-p} \|f\|^p_{p,w}, \quad 1 \leq p < \infty$;

(v) $\|g\|_\infty \leq \alpha$;

(vi) $\|g\|^2_{2,w} \leq C_p \alpha^{2-p} \|f\|^p_{p,w}, \quad 1 \leq p < 2.$

Proof of Theorem 2.1. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Fix $\alpha > 0$ and apply Theorem 2.2 to write $f = g + \sum_i b_i$ with $\|g\|_\infty \leq \alpha$ and each b_i supported in a ball B_i of radius $2^{j(i)}$. Let

$$G = g + \sum_i b_i * \phi_{2^{j(i)}}$$

and

$$h = \sum_i (b_i - b_i * \phi_{2^{j(i)}}),$$

where ϕ is a non-negative radial $C^\infty(\mathbb{R}^n)$ function supported in $\{|x| \le 1\}$ with $\int \phi = 1$ and $\phi_{2^{j(i)}}(x) = 2^{-nj(i)}\phi(2^{-j(i)}x)$. Then we have

$$\begin{aligned}\{x : |Tf(x)| > \alpha\} &\subseteq \{x : |TG(x)| > \alpha/2\} \cup \{x : |Th(x)| > \alpha/2\}\\ &:= E_\alpha \cup F_\alpha.\end{aligned}$$

It follows from hypothesis (2.1) and Carbery's arguments in [C, Theorem 1] that $w(F_\alpha) \le C\alpha^{-p}\|f\|_{p,w}^p$. To estimate $w(E_\alpha)$, we note that $w(E_\alpha) \le C\alpha^{-2}\|G\|_{2,w}^2$ because T is bounded on $L_w^2(\mathbb{R}^n)$. Since $1 < p < 2$, Theorem 2.2(vi) implies that

$$\|g\|_{2,w}^2 \le C\alpha^{2-p}\|f\|_{p,w}^p. \tag{2.2}$$

Note that

$$\begin{aligned}\|b_i * \phi_{2^{j(i)}}\|_\infty &\le \|b_i\|_1 \|\phi_{2^{j(i)}}\|_\infty\\ &\le C2^n\alpha|B_i|2^{-nj(i)}\\ &\le C\alpha\end{aligned}$$

for all $i \in \mathbb{N}$. Since $1 < p < 2$ we have

$$\begin{aligned}\sum_i \|b_i * \phi_{2^{j(i)}}\|_{2,w}^2 &\le C\alpha^{2-p}\sum_i \|b_i * \phi_{2^{j(i)}}\|_{p,w}^p\\ &\le C\alpha^{2-p}\sum_i \|b_i\|_{p,w}^p\\ &\le C\alpha^{2-p}\|f\|_{p,w}^p,\end{aligned} \tag{2.3}$$

where the second inequality follows from [ST, Theorem 6, p. 162] and the last inequality is by Theorem 2.2 (iii). It now follows from (2.2) and (2.3) that

$$\|G\|_{2,w}^2 \le C\alpha^{2-p}\|f\|_{p,w}^p.$$

Hence we also have

$$w(E_\alpha) \le C\alpha^{-p}\|f\|_{p,w}^p.$$

Consequently we have $w\{x : |Tf(x)| > \alpha\} \le C\alpha^{-p}\|f\|_{p,w}^p$; that is to say, T is of weak type (p,p) on $L_w^p(\mathbb{R}^n)$.

Theorem 2.1 has the following simple corollary.

Corollary 2.3. *Let* $\{\alpha(k)\}_{-\infty}^0$ *be a sequence of positive real numbers satisfying* $\sum_{k\le 0}|k|\alpha(k) < \infty$. *Let* $w \in A_p$, $1 \le p < 2$, *and let* $m \in L^\infty(\mathbb{R}^n)$ *be such that* m *is a multiplier on* $L_w^2(\mathbb{R}^n)$. *Suppose that for all* $j > i$, *we have*

$$\|m_i * (\hat{\eta})_{2^{-j}}\|_{\mathcal{M}_{p,w}} \le \alpha(i-j),$$

where $m_i(\xi) = m(\xi)\psi(2^i\xi)$. *Then* m *is a multiplier of weak type* (p,p) *on* $L_w^p(\mathbb{R}^n)$.

Proof. Let T be the convolution operator defined on $\mathcal{S}(R^n)$ by $(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$. For $j \in \mathbb{Z}$ define the operator T_j on $\mathcal{S}(R^n)$ by

$$(T_jf)^\wedge(\xi) = (m * (\hat{\eta})_{2^j}(\xi))\hat{f}(\xi).$$

Using Theorem 2.1 and [ST, Theorem 6, p.162], we can prove the corollary in the same manner as Theorem 3 is proved in [C]. Details of the proof are therefore omitted.

3. Multipliers on power-weighted $L^p(\mathbb{R}^n)$

Corollary 2.3 indicates the amount of regularity needed for each m_i so that m is a multiplier of weak type (p, p) on $L^p_w(R^n)$. Our next theorem shows that for certain power weights, such a regularity condition is implied by m_i satisfying a certain Lipschitz condition. We say a distribution f is in the Lipschitz space $\Lambda^{\beta}_{r,s}(\mathbb{R}^n)$ for $\beta > 0,\ 1 \le r, s \le \infty,$ if $\|f\|_{\Lambda^{\beta}_{r,s}} < \infty$, where

$$\|f\|_{\Lambda^{\beta}_{r,s}} = \left\{ \sum_{j\in\mathbb{Z}} 2^{-sj\beta} \|f * (\hat{\eta})_{2^j}\|_r^s \right\}^{1/s},$$

with the usual modification if $s = \infty$.

Theorem 3.1. *Let $\beta > 0$ and let $m \in L^{\infty}(\mathbb{R}^n)$ be such that $m_i \in \Lambda^{\beta}_{\infty,2}(\mathbb{R}^n)$ and*

$$\sup_{i\in\mathbb{Z}} 2^{-i\beta} \|m_i\|_{\Lambda^{\beta}_{\infty,2}} < \infty, \tag{3.1}$$

where m_i is as in Corollary 2.3.

- (i) *Let $1 < q < 2$ and let $n(2-q)/2q < \beta < n/2$. Then m is a multiplier on $L^r_{\alpha}(R^n)$ for all $q < r \le 2$ and $|\alpha| \le 2\beta(r-q)/(2-q)$.*
- (ii) *If $n/2 \le \beta$, then m is a multiplier on $L^r_{\alpha}(R^n)$ for all $1 < r \le 2$ and $|\alpha| < n(r-1)$.*

We need the following two lemmas in the proof of Theorem 3.1.

Lemma 3.2. *Let $1 < q < 2$ and let $n(2-q)/2q < \beta < n/2$. Let $m \in L^{\infty}(\mathbb{R}^n)$ be such that $m_i \in \Lambda^{\beta}_{\infty,\infty}(\mathbb{R}^n)$ for all $i \in \mathbb{Z}$ and*

$$\sup_{i\in\mathbb{Z}} 2^{-i\beta} \|m_i\|_{\Lambda^{\beta}_{\infty,\infty}} < \infty, \tag{3.2}$$

where m_i is as in Corollary 2.3. Then m is a multiplier on $L^q(\mathbb{R}^n)$.

Proof. Recall that $\sum_{k\in\mathbb{Z}} \psi(2^k\xi) = 1$ for $\xi \ne 0$ and write

$$\begin{aligned} m_i * (\hat{\eta})_{2^{-j}}(\xi) &= \sum_{k\in\mathbb{Z}} m_i * (\hat{\eta})_{2^{-j}}(\xi)\psi(2^k\xi) \\ &= \sum_{k<i-1} + \sum_{k=i-1}^{i+1} + \sum_{k>i+1} m_i * (\hat{\eta})_{2^{-j}}(\xi)\psi(2^k\xi) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Let τ_{ijk} be defined by $\tau_{ijk}(\xi) = (m_i * (\hat{\eta})_{2^{-j}}(\xi))\psi(2^k\xi)$. Then $\mathrm{II} = \sum_{k=i-1}^{i+1} \tau_{ijk}$.

Choose p such that $1 < p < q$ and $\beta > n(2-p)/2p$. We shall estimate $\|\tau_{ijk}\|_{\mathcal{M}_p}$ for $k = i$ by interpolation between $\|\tau_{iji}\|_{\mathcal{M}_2}$ and $\|\tau_{iji}\|_{\mathcal{M}_1}$. Clearly, we have

$$\begin{aligned} \|\tau_{iji}\|_{\mathcal{M}_2} &= \|\tau_{iji}\|_{\infty} \\ &\le \|m_i * (\hat{\eta})_{2^{-j}}\|_{\infty} \|\psi(2^i\cdot)\|_{\infty} \\ &\le C2^{(i-j)\beta}, \end{aligned} \tag{3.3}$$

where the last inequality follows from (3.2) and $\|\psi\|_\infty \le 1$. Note that $\|\tau_{iji}\|_{\mathcal{M}_1} = \|(\tau_{iji})^\vee\|_1$. Let $(\tau_{iji})^\vee = (\tau_{iji})^\vee\chi_A + (\tau_{iji})^\vee\chi_B$, where $A = \{|x| \le 2^{j+3}\}$, $B = \{|x| > 2^{j+3}\}$. Then

$$\begin{aligned}\|(\tau_{iji})^\vee\chi_A\|_1 &\le \|(\tau_{iji})^\vee\|_2\|\chi_A\|_2\\ &\le \|m_i * (\hat{\eta})_{2^{-j}}\|_\infty\|\psi(2^i\cdot)\|_2\|\chi_A\|_2\\ &\le C2^{(i-j)\beta}2^{-ni/2}2^{n(j+3)/2} \le C2^{(i-j)(\beta-n/2)},\end{aligned}$$

where the penultimate inequality follows from (3.2) and $\|\psi(2^i\cdot)\|_2 = 2^{-in/2}\|\psi\|_2$. To estimate $\|(\tau_{iji})^\vee\chi_B\|_1$ we write $B_\ell = \{2^\ell < |x| \le 2^{\ell+1}\}$ for $\ell \ge j+3$. Then

$$\|(\tau_{iji})^\vee\chi_B\|_1 = \sum_{\ell > j+2}\int_{B_\ell}|(\tau_{iji})^\vee(x)|dx.$$

Now for $x \in B_\ell$ and $t \in \mathbb{N}$, there exists a constant C_t such that

$$\begin{aligned}|(\tau_{iji})^\vee(x)| &\le 2^{-ni}\int_{\{1\le|2^{-j}y|\le 4\}}|(m_i)^\vee(y)\eta(2^{-j}y)||(\psi)^\vee(2^{-i}(x-y))|dy\\ &\le C_t2^{-2ni+nj+(i-\ell)t},\end{aligned}$$

where the last inequality follows from $\|(m_i)^\vee\|_\infty \le \|m_i\|_1 \le C2^{-ni}$. Now choose $t = 3n$ and we have

$$\|(\tau_{iji})^\vee\chi_B\|_1 \le C\sum_{\ell > j+2}2^{ni+nj-2n\ell} \le C2^{(i-j)n}.$$

Since $i < j$ and $\beta < n/2$, we have $\|(\tau_{iji})^\vee\|_1 \le C2^{(i-j)(\beta-n/2)}$. It follows that

$$\|\tau_{iji}\|_{\mathcal{M}_1} \le C2^{(i-j)(\beta-n/2)}. \tag{3.4}$$

Interpolating between (3.3) and (3.4) yields $\|\tau_{iji}\|_{\mathcal{M}_p} \le C2^{(i-j)(\beta-n(2-p)/2p)}$ for $1 < p < 2$.

Similar estimates of $\|\tau_{ijk}\|_{\mathcal{M}_p}$ for $k = i-1, i+1$ give

$$\|\,\mathrm{II}\|_{\mathcal{M}_p} \le C2^{(i-j)(\beta-n(2-p)/2p)}.$$

Routine calculations as in [C, p.395] show that for $|\gamma| \le n$ and $t > 2n$, there exists a constant C_t such that $\|\,\mathrm{I}\|_{\mathcal{M}_p} \le C_t\|m_i\|_\infty 2^{(j-i)(n+|\gamma|-t)}$ and $\|\,\mathrm{III}\|_{\mathcal{M}_p} \le C_t\|m_i\|_\infty 2^{(j-i)(n+|\gamma|-t)}$.

Consequently for $j > i$ and $t = 3n$, we have

$$\begin{aligned}\|m_i * (\hat{\eta})_{2^{-j}}\|_{\mathcal{M}_p} &\le \|\,\mathrm{I}\|_{\mathcal{M}_p} + \|\,\mathrm{II}\|_{\mathcal{M}_p} + \|\,\mathrm{III}\|_{\mathcal{M}_p}\\ &\le C(2^{(i-j)(\beta-n(2-p)/2p)} + 2^{(i-j)n}).\end{aligned}$$

It follows from Corollary 2.3 that m is a multiplier of weak type (p,p) on $L^p(\mathbb{R}^n)$. Since $1 < p < q < 2$ and m is a multiplier on $L^2(\mathbb{R}^n)$, we have m a multiplier on $L^q(\mathbb{R}^n)$.

Lemma 3.3. *Let $0 < \beta < n/2$. Let $m \in L^\infty(\mathbb{R}^n)$ be such that $m_i \in \Lambda^\beta_{n/\beta,2}(\mathbb{R}^n)$ and*

$$\sup_{i\in\mathbb{Z}}\|m_i\|_{\Lambda^\beta_{n/\beta,2}} < \infty,$$

where m_i is as in Corollary 2.3. Then m is a multiplier on $L^2_{2\beta}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Since $\|(m_i\hat{f})^{\vee}\|_{2,2\beta} \sim \|m_i\hat{f}\|_{\Lambda^{\beta}_{2,2}}$, it follows from Herz [He, Lemma 1.5*] that

$$\begin{aligned}\|m_i\hat{f}\|_{\Lambda^{\beta}_{2,2}} &\le C\{\|m_i\|_\infty + \|m_i\|_{\Lambda^{\beta}_{n/\beta,2}}\}\|\hat{f}\|_{\Lambda^{\beta}_{2,2}} \\ &\le C\|f\|_{2,2\beta}.\end{aligned}$$

Thus we have $\sup_{i\in\mathbb{Z}} \|m_i\|_{\mathcal{M}_{2,2\beta}} < \infty$. Hence m is a multiplier on $L^2_{2\beta}(\mathbb{R}^n)$.

We now turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. (i). Note that supp $m_i \subset \{2^{-i} \le |\xi| \le 2^{-i+2}\}$. Now Theorem 6.3.1 of [BL] and hypothesis (3.1) of Theorem 3.1 imply that m satisfies the hypotheses of both Lemmas 3.2 and 3.3. Thus m is both a multiplier on $L^q(\mathbb{R}^n)$, $1 < q < 2$, and a multiplier on $L^2_{2\beta}(\mathbb{R}^n)$. The result now follows from the Stein-Weiss interpolation of L^p spaces with change of measures; see [SW, Theorem 2.11].

(ii). The proof of Lemma 3.2 can be easily modified to show that m is a multiplier on $L^p(\mathbb{R}^n)$ for all $1 < p < 2$. Now $\Lambda^{\beta}_{n/\beta,2}(\mathbb{R}^n) \subset \Lambda^{\alpha}_{n/\alpha,2}(\mathbb{R}^n)$ for all $0 < \alpha < n/2$. It follows from Lemma 3.3 that m is a multiplier on $L^2_{2\alpha}(\mathbb{R}^n)$ for all $|\alpha| < n/2$. The result again follows from the Stein-Weiss interpolation with change of measures.

As an application of Theorem 3.1 we consider the following multiplier discussed in (3.6) of Baernstein and Sawyer [BS, p.22].

Let a and b be positive real numbers and let s be an integer larger than b/a. Let m be a strongly singular multiplier such that

$$\begin{cases}|D^{\beta}m(\xi)| \le C|\xi|^{-b}|\xi|^{(a-1)|\beta|}, & 0 \le |\beta| \le s,\\ m(\xi) = 0, & |\xi| \le 1,\end{cases} \tag{3.5}$$

where $\beta = (\beta_1, ..., \beta_n)$, $\quad D^{\beta} = D_1^{\beta_1}...D_n^{\beta_n}$ and $D_j = \frac{\partial}{\partial \xi_j}$.

The prototypical example is the function $m_{a,b}$ defined by $m_{a,b}(\xi) = \Theta(\xi)|\xi|^{-b}e^{i|\xi|^a}$, where $\Theta \in C^{\infty}(\mathbb{R}^n)$, $\Theta = 0$ on $|\xi| < 1$, $\Theta = 1$ on $|\xi| \ge 2$.

The next theorem shows that m is a multiplier on certain power-weighted $L^p(\mathbb{R}^n)$.

Theorem 3.4. *Let* $m \in L^{\infty}(\mathbb{R}^n)$ *satisfy* (3.5) *above. Then we have*

(i) *if* $1 < q < 2$ *and* $n(2-q)/2q < b/a < n/2$*, then* m *is a multiplier on* $L^r_{\alpha}(\mathbb{R}^n)$ *for all* $q < r \le 2$ *and* $|\alpha| \le 2b(r-q)/a(2-q)$*;*

(ii) *if* $n/2 \le b/a$*, then* m *is a multiplier on* $L^r_{\alpha}(\mathbb{R}^n)$ *for all* $1 < r \le 2$ *and* $|\alpha| < n(r-1)$.

Proof. Let s be an integer larger than b/a. Since $m_i(\xi) = m(\xi)\psi(2^i\xi)$ and $m = 0$ for $|\xi| \le 1$, we have $m_i = 0$ for $i \ge 2$. For $i < 2$ our hypothesis (3.5) implies that

$$\|D^{\beta}m_i\|_\infty \le C2^{i\{b-(a-1)|\beta|\}}, \quad 0 \le |\beta| \le s,$$

where C is independent of i. Assume that b/a is not an integer and let $b/a = \nu + \sigma$, where ν is a non-negative integer and $0 < \sigma < 1$. For $1 \le j \le n$ we have

$$\begin{aligned}\left\|\Delta_h \frac{\partial^{\nu} m_i}{\partial \xi_j^{\nu}}\right\|_\infty &\le C \;\; \min\{|h| \max_{|\beta|=\nu+1}\|D^{\beta}m_i\|_\infty, \;\; \|\frac{\partial^{\nu} m_i}{\partial \xi_j^{\nu}}\|_\infty\} \\ &\le C \;\; \min\{|h|2^{i\{b-(a-1)(\nu+1)\}}, \;\; 2^{i\{b-(a-1)\nu\}}\},\end{aligned}$$

where $\Delta_h f(x) = f(x+h) - f(x)$. Let $\omega_\infty(t, \frac{\partial^\nu m_i}{\partial \xi_j^\nu}) = \sup_{|h|<t} \|\Delta_h \frac{\partial^\nu m_i}{\partial \xi_j^\nu}\|_\infty$. Then we have

$$\begin{aligned}
&\sum_{j=1}^{n} \left(\int_0^\infty (t^{-\sigma} \omega_\infty(t, \frac{\partial^\nu m_i}{\partial \xi_j^\nu}))^2 \, dt/t \right)^{\frac{1}{2}} \\
&\leq \sum_{j=1}^{n} \Bigg\{ \left(\int_0^{2^{i(a-1)}} (t^{-\sigma} \omega_\infty(t, \frac{\partial^\nu m_i}{\partial \xi_j^\nu}))^2 \, dt/t \right)^{\frac{1}{2}} \\
&\qquad\qquad + \left(\int_{2^{i(a-1)}}^\infty (t^{-\sigma} \omega_\infty(t, \frac{\partial^\nu m_i}{\partial \xi_j^\nu}))^2 \, dt/t \right)^{\frac{1}{2}} \Bigg\} \\
&\leq C 2^{ib/a}.
\end{aligned}$$

It follows from [BL, Theorem 6.3.1] that $m_i \in \Lambda_{\infty,2}^{b/a}$ and $\|m_i\|_{\Lambda_{\infty,2}^{b/a}} \leq C 2^{ib/a}$.

If b/a is an integer, we write $b/a = \nu + 1$, where $\nu \geq 0$. Then for $1 \leq j \leq n$ we have

$$\left\| \Delta_h^2 \frac{\partial^\nu m_i}{\partial \xi_j^\nu} \right\|_\infty \leq C \ \min\{|h|^2 2^{i\{b-(a-1)(\nu+2)\}}, \ 2^{i\{b-(a-1)\nu\}}\},$$

where $\Delta_h^2 f = \Delta(\Delta_h f)$.

Routine calculation as in the case where b/a is a non-integer then shows that $m_i \in \Lambda_{\infty,2}^{b/a}$ and $\|m_i\|_{\Lambda_{\infty,2}^{b/a}} \leq C 2^{ib/a}$. The theorem now follows from Theorem 3.1.

4. Sharpness of Theorem C

In this section we prove that Theorem C is sharp in the following sense.

Theorem 4.1. *Let $1 < p < 2$. There exists a convolution operator T bounded on $L^2(\mathbb{R}^n)$ so that*

(i) $\sum\limits_{k \in \mathbb{Z}} \sup\limits_{j \in \mathbb{Z}} \| \sum_{\ell \geq 0} Q_{j+k} T_{j+\ell} (I - P_j) \|_{\mathcal{M}_p} < \infty$;

(ii) *T is bounded on $L^p(\mathbb{R}^n)$;*

(iii) *T is not of weak type (r, r) on $L^r(\mathbb{R}^n)$ for $1 < r < p$.*

Proof. Let $1 < r < p < 2$. Choose q such that $r < q < p$ and choose α such that $0 < \alpha < 1$, $q < 2/(2-\alpha) < p$. Following Onneweer [O, p.56] we construct a sequence $(P_k)_{k \geq 0}$ of Rudin-Shapiro-like polynomials on the n-dimensional torus $\mathbb{T}^n$ such that $\|P_k\|_\infty \leq 2^{n(k+1)/2}$ and $|\hat{P}_k(j)| = 1$ for $j = (j_1, j_2, \cdots, j_n) \in \mathbb{Z}^n$, where $0 \leq j_i < 2^k$ for $i = 1, \cdots, n$. Note that $\hat{P}_k$ is supported on $\mathbb{Z}^n$. Now for $k \geq 0$ define γ_k on $\mathbb{Z}^n$ by $\gamma_k = 2^{n(\alpha-1)k/2} \hat{P}_k$. Let $\tilde{k}$ denote the center of the cube $[2^{k+2}, 2^{k+3}]^n$ and define Φ_k on $\mathbb{Z}^n$ by $\Phi_k(j) = \gamma_k(j - \tilde{k})$. Write $\Phi = \sum\limits_{k \geq 0} \Phi_k$ and it can be shown as in Figà-Talamanca and Gaudry [FG, Theorem B] that Φ is a multiplier on $L^p(\mathbb{T}^n)$ but not a multiplier on $L^q(\mathbb{T}^n)$.

Let m be the function defined on $\mathbb{R}^n$ by $m(\xi) = \sum_{j \in \mathbb{Z}^n} S(\xi - j) \Phi(j)$, where $S(\xi) := (\ \max\{1 - |\xi|^2, 0\})^n$ for $\xi \in \mathbb{R}^n$. Now define the convolution operator T on $\mathcal{S}(R^n)$ by $(Tf)^\wedge = m\hat{f}$. Then T is bounded on $L^2(\mathbb{R}^n)$. Since $n(1-\alpha)/2 >$

$n(2-p)/2p$, it follows from the proofs of Theorem 3 of [C] and Lemma 3.2 that T will satisfy (i) if we have

$$\sup_{k\in\mathbb{Z}} 2^{-nk(1-\alpha)/2}\|m_k\|_{\Lambda^{n(1-\alpha)/2}_{\infty,\infty}} < \infty, \tag{4.1}$$

where $m_k(\xi) = m(\xi)\psi(2^k\xi)$. Note that $\operatorname{supp} m_k \subset \{2^{-k} \le |\xi| \le 2^{-k+2}\}$ and so $m_k = 0$ for $k \ge 0$. Thus we only need to prove (4.1) for $k < 0$. Let $\xi \in \mathbb{R}^n$ and let $A_\xi = \{j \in \mathbb{Z}^n : |\xi - j| \le 1\}$. Then A_ξ is a finite set with at most 2^n elements. Let $\beta = n(1-\alpha)/2$ and we have $|\Phi(j)| \le 2^{(k+2)\beta}$ for $j \in A_\xi$, $\xi \in \operatorname{supp} m_k$. Furthermore, $k < 0$ implies that there exists a constant C so that $\|D^\mu S(\cdot - j)\psi(2^k\cdot)\|_\infty \le C$ for all multi-indices μ with $0 \le |\mu| \le n$ and all $j \in A_\xi$, $\xi \in \operatorname{supp} m_k$. Consequently we have

$$|D^\mu m_k(\xi)| \le \sum_{j\in A_\xi} |D^\mu S(\xi - j)\Phi(j)\psi(2^k\xi)| \le C2^{k\beta}.$$

If β is not an integer, we write $\beta = \nu + \sigma$ with $0 < \sigma < 1$. Then ν is an integer less than $n/2$. For $i = 1, 2, \cdots, n$ we have

$$\left\| \Delta_h \frac{\partial^\nu m_k}{\partial \xi_i^\nu} \right\|_\infty \le C|h|^\sigma 2^{k\beta}.$$

Thus we have $\|m_k\|_{\Lambda^\beta_{\infty,\infty}} \le C2^{k\beta}$ if β is not an integer. If β is an integer, then routine calculation as in the proof of Theorem 3.4 shows that $\|m_k\|_{\Lambda^\beta_{\infty,\infty}} \le C2^{k\beta}$. Consequently, the sequence $\{m_k\}_{-\infty}^{\infty}$ satisfies (4.1) and we have T satisfying (i).

Since $\beta = n(1-\alpha)/2 > n(2-p)/2p$, (4.1) and Lemma 3.2 imply that m is a multiplier on $L^p(\mathbb{R}^n)$. Hence T satisfies (ii). Lastly, if T were of weak type (r, r) on $L^r(\mathbb{R}^n)$ for $1 < r < p$, then T would be bounded on $L^q(\mathbb{R}^n)$ since $r < q < p$. Hence m would be a multiplier on $L^q(\mathbb{R}^n)$. By deLeeuw's theorem [L, Proposition 3.3] the restriction Φ of m to $\mathbb{Z}^n$ would then be a multiplier on $L^q(\mathbb{T}^n)$, but this contradicts our earlier observation that Φ is not a multiplier on $L^q(\mathbb{T}^n)$. Thus T satisfies (iii).

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