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NAKAI'S CONJECTURE FOR VARIETIES SMOOTHED BY NORMALIZATION

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Abstract. The notion of D-simplicity is used to give a short proof that varieties whose normalization is smooth satisfy Ishibashi's extension of Nakai's conjecture to arbitrary characteristic. This gives a new proof of Nakai's conjecture for curves and Stanley-Reisner rings.

Introduction

Nakai's conjecture concerns a very natural question: can differential operators detect singularities on algebraic varieties? On a smooth complex variety, it is well known that the ring of differential operators is generated by derivations. Nakai asked whether the converse holds: if the ring of differential operators is generated by derivations, is the variety smooth? In this paper, the notion of D-simplicity is used to give a short proof that varieties whose normalization is smooth satisfy Ishibashi's extension [2] of Nakai's conjecture to arbitrary characteristic. For example, any variety whose irreducible components are smooth satisfies Nakai's conjecture, giving a proof that Nakai's conjecture holds for Stanley-Reisner rings. Furthermore, this gives a simple new characteristic-independent proof of Nakai's conjecture for curves. The argument is quite short and rederives characteristic-dependent results of Mount and Villamayor (characteristic zero) [4] and Ishibashi (prime characteristic) [2].

Definitions and notation

Let $X = \mathrm{Spec}(R)$ be an affine algebraic variety defined over a field k of characteristic zero. When R is regular (that is, when X is smooth), the ring of differential operators $D(R/k)$ (see *EGA* [1] or [3]) equals $der(R/k)$, the R-subalgebra generated by the derivations (see McConnell and Robson [3, Corollary 15.5.6]). Nakai conjectured that this condition characterizes nonsingularity: R is regular if and only if $der(R/k) = D(R/k)$. Ishibashi [2] extended Nakai's conjecture to varieties defined over an arbitrary perfect field.

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A Hasse-Schmidt derivation $\Delta = \{\delta_n\}_{n=0}^{\infty} \subseteq End_k(R)$ is a collection of k-linear endomorphisms of R such that $\delta_0 = id_R$ and

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(a)\delta_j(b).$$

For example, if R has characteristic zero and d is a derivation, $\delta_n = \frac{1}{n!}d^n$ determines a Hasse-Schmidt derivation. Let $HS(R/k)$ be the R-algebra generated by the components δ_n of Hasse-Schmidt derivations on R. $HS(R/k)$ is a subalgebra of $D(R/k)$ and if R has characteristic zero, $der(R/k) = HS(R/k)$. In characteristic zero Grothendieck [1, 16.11.2 and 17.12.4] showed that when R is smooth over k, $HS(R/k) = D(R/k)$; a proof in arbitrary characteristic can be found in Traves [9]. Ishibashi's extension[1] of Nakai's conjecture is that R is smooth over k if and only if $HS(R/k) = D(R/k)$. The main result of this paper is that varieties whose normalization is smooth satisfy this extension of Nakai's conjecture.

Preliminary results

Let R be a reduced algebra of finite type over a perfect field k. For a reduced ring R, the normalization R' of R is the integral closure of R in its total ring of quotients, $L = S^{-1}R$, where S is the multiplicative set of nonzerodivisors in R. The conductor of R' into R is

$$C = \{c \in R : \; cR' \subseteq R\}.$$

The conductor is an ideal of both R and R'.

Lemma 1. *(1) The conductor is $HS(R/k)$-stable.*
(2) Powers of $HS(R/k)$-stable ideals are $HS(R/k)$-stable.

Proof. Part (1) is well-known: see Seidenberg [6, Corollary on page 169 and section 5]. To establish (2) it suffices to show that for $a_1, \ldots, a_s$ in the $HS(R/k)$-stable ideal I and $\Delta = \{\delta_n\}$ a Hasse-Schmidt derivation, $\delta_n(a_1 \cdots a_s) \in I^s$. Now

$$\delta_n(a_1 \cdots a_s) = \sum_{i+j=n} \delta_i(a_1)\delta_j(a_2 \cdots a_s)$$

and the claim follows by induction on s. □

Lemma 2. *The conductor is not contained in any minimal prime of R.*

Proof. The conductor C equals $Ann_R(\frac{R'}{R})$. If $C \subseteq P$ with P a minimal prime of R, then, since $Supp(\frac{R'}{R}) = \mathbb{V}(Ann(\frac{R'}{R}))$, $P \in Supp(\frac{R'}{R})$. Localizing the exact sequence of R-modules

$$0 \to R \to R' \to \frac{R'}{R} \to 0$$

at the minimal prime P gives an exact sequence

$$0 \to R_P \to R'_P \to (\frac{R'}{R})_P \to 0.$$

But since P is a minimal prime of a reduced ring, R_P is a field and the normalization map $R_P \to R'_P$ is an isomorphism. This forces $(\frac{R'}{R})_P = 0$, a contradiction. So C is not contained in any minimal prime of R. □

[1]Actually, Ishibashi requires that k be algebraically closed and conjectures R regular, but it is clear that smoothness is the relevant notion

Lemma 3. *If R is a domain that is smooth over k, then R is $D(R/k)$-simple.*

Proof. Since k is perfect, R is regular. In the characteristic zero case, it is well known that R is $D(R/k)$-simple (see McConnell and Robson [3, Theorem 15.3.8 and Corollary 15.5.6]). For the prime characteristic case, note that since R is regular, R is strongly F-regular and since R is an algebra of finite type over a perfect field, R is F-finite. Smith [7, Theorem 2.2] has shown that strongly F-regular F-finite domains are $D(R/k)$-simple. □

1. Nakai's conjecture

Theorem 4. *Let R be a reduced k-algebra of finite type and let R' be its integral closure in its total ring of quotients. If R' is a product of D-simple rings and $HS(R/k) = D(R/k)$, then R is normal.*

Proof. Note that R' is isomorphic to $R_1 \times \cdots \times R_t$, where R_i is the normalization of $\frac{R}{P_i}$ with the $\{P_i\}$ ranging over the minimal primes of R. By Lemma 1 (1), C is $HS(R/k)$-stable and by Lemma 1 (2), C^2 is also $HS(R/k)$-stable. Thus, C^2 is $D(R/k)$-stable.

Assume that $C^2 \neq C$. Since C is not contained in any of the minimal primes P_i of R, there are elements $c_i \in C \setminus P_i$ with some $c_j \notin C^2$. To see this, take $x \in C \setminus C^2$ and note that $x \neq 0 \Rightarrow x \notin \bigcap P_i \Rightarrow x$ is not in some P_j. Set $c_j = x$ and pick the other $c_i \in C \setminus P_i$. Let $c = (c_1, \ldots, c_t) \in R'$, after identifying R' with the product in the first paragraph. Then $c \in C \setminus C^2$ and c is nonzero in each component. By the D-simplicity of each of the R_i, there is an operator $\theta = (\theta_1, \ldots, \theta_t) \in D(R_1) \times \cdots \times D(R_t)$, such that $\theta(c^2) = 1$. If each $\theta_i \in D(R_i)$ is an operator of order $\leq n_i$ then $\theta = (\theta_1, \ldots, \theta_t)$ maps R' to itself and θ is a differential operator of order $\leq n = \max(n_i)$. Thus, $\theta \in D(R')$ and $c\theta \in D(R)$. Now $(c\theta)(c^2) = c \notin C^2$, contradicting the fact that C^2 is $D(R/k)$-stable. This forces $C^2 = C$.

In fact, $C = R$. Indeed, if C is contained in a maximal ideal m, then $C_m = C_m^2 \subset mC_m \subset C_m$, so $mC_m = C_m$. Now Nakayama's lemma forces $C_m = 0$. But then C must consist of zero divisors, contradicting Lemma 2. So $C = R$ and R is normal. □

Theorem 5. *If $HS(R/k) = D(R/k)$ and the normalization R' of R is smooth over k, then R is smooth over k.*

Proof. Since the normalization R' of R is a product of smooth domains, R' is a product of D-simple rings (by Lemma 3). The result now follows from Theorem 4. □

This theorem says that the ring of differential operators of a singular complex variety whose normalization is smooth is not generated by derivations. Thus, $D(R/\mathbb{C})$ is complicated even for very mild singularities (those that can be resolved by normalization). We expect that $D(R/\mathbb{C})$ will become more complicated as the singularities become worse. This provides further evidence for Nakai's conjecture.

The theorem shows that Nakai's conjecture holds for reduced varieties smoothed by normalization; for example, curves.

Corollary 6. *Let R be a reduced k-algebra of finite type, where k is a perfect field. If R is 1-dimensional and $HS(R/k) = D(R/k)$, then R is smooth over k. In* *particular, if R is a domain of characteristic 0 and $der(R/k) = D(R/k)$, then R is regular.*

Theorem 5 can also be used to recover a result due to Schreiner [5] (also, see Traves [8]): Nakai's conjecture holds for Stanley-Reisner rings (that is, for subvarieties of $\mathbb{A}_k^N$ which are the union of coordinate subspaces). More generally, the theorem implies that Nakai's conjecture holds for varieties all of whose components are smooth, as remarked by Lazarsfeld.

Corollary 7. *Let R be a reduced k-algebra of finite type, where k is a perfect field. If $HS(R/k) = D(R/k)$ and if $\frac{R}{P}$ is smooth over k for each minimal prime P of R, then R is smooth over k. In particular, if R is a Stanley-Reisner ring and $HS(R/k) = D(R/k)$, then R is a polynomial ring.*

Both Corollary 6 and Corollary 7 follow immediately from Theorem 5: just observe that the normalization of $X = \mathrm{Spec}(R)$ is isomorphic to the disjoint union of the normalization of the components of X, each of which is smooth over k.

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