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CRYSTAL BASES FOR $U_q(sl(2,1))$

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Abstract. A construction of the crystal bases for the quantized enveloping algebra of $sl(2,1)$ is discussed.

1. Introduction

The crystal bases introduced by Kashiwara for the quantized universal enveloping algebras of the symmetrizable Kac-Moody Lie algebras have many remarkable properties (see [7, 8]). Since the same idea of Drinfeld [1] and Jimbo [4] can also be employed to define the quantized enveloping algebra of a contragredient Lie superalgebra (see [2, 3]), a natural question is whether one can also define crystal bases for the quantized enveloping algebras of these Lie superalgebras, especially for the classical ones. In this paper, we discuss a construction of the crystal bases for the quantized enveloping algebra U of the Lie superalgebra $sl(2,1)$.

Since the finite dimensional representations of U are not completely reducible, one should not expect to have a complete analog of the crystal bases defined by Kashiwara. In our construction, we require the following properties for the crystal bases:

1) A crystal basis of a U-module M must be a crystal basis for the even part of U when M is viewed as a module for the even part of U.

2) If a U-module M has a crystal basis, then any quotient module of M has a crystal basis obtained by taking the image of the crystal basis of M.

Our idea is to construct these bases for the indecomposable projective modules in the category of finite dimensional weight modules. Then based on the fact that any finite dimensional weight module is a quotient of a direct sum of these projective modules, we can obtain a crystal basis for any finite dimensional weight module by taking the quotient of the crystal basis of the corresponding projective cover.

2. The algebra U and its highest weight modules

Let q be an indeterminate over the complex number field $\mathbb{C}$. Let A be the ring of rational functions in q without pole at $q = 0$ (the localization of $\mathbb{C}[q]$ at $q = 0$). Let $G = sl(2,1)$, and let $(a_{ij})_{2\times 2}$ be defined by $a_{11} = 2$, $a_{22} = 0$ and $a_{12} = a_{21} = -1$. Let $U = U_q(G)$ be the associative $\mathbb{Z}_2$-graded algebra over $\mathbb{C}(q)$ (with 1) generated by e_i, f_i, $t_i^{\pm 1}$, $i = 1, 2$, with grading given by $\deg(e_1) = \deg(f_1) = \deg(t_1^{\pm 1}) = \deg(t_2^{\pm 1}) = 0$, $\deg(e_2) = \deg(f_2) = 1$, and the following relations:

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(1) $t_it_i^{-1} = t_i^{-1}t_i = 1$, $t_it_j = t_jt_i$, $t_ie_jt_i^{-1} = q^{a_{ij}}e_j$, $t_if_jt_i^{-1} = q^{-a_{ij}}f_j$,
(2) $e_if_j - (-1)^{ab}f_je_i = \delta_{ij}(t_i - t_i^{-1})/(q - q^{-1})$, $a = \deg(e_i)$, $b = \deg(f_j)$,
(3) $e_1^2e_2 - (q + q^{-1})e_1e_2e_1 + e_2e_1^2 = 0$, $f_1^2f_2 - (q + q^{-1})f_1f_2f_1 + f_2f_1^2 = 0$,
(4) $e_2^2 = 0$, $f_2^2 = 0$.

The algebra U is a $\mathbb{Z}_2$-graded Hopf algebra, but we do not need the Hopf algebra structure in this paper. There exists an anti-automorphism $\theta : U \to U$ defined by

$$\theta e_i = f_i, \theta f_i = e_i, \theta t_i = t_i^{-1}, \theta q = q^{-1},$$

and $\theta(xy) = \theta(y)\theta(x)$ for any $x, y \in U$.

We let $e_3 = qe_1e_2 - e_2e_1$, $f_3 = \theta(e_3) = -f_1f_2 + q^{-1}f_2f_1$, and let $H_i = (t_i - t_i^{-1})/(q - q^{-1})$, $i = 1, 2$. Then the following identities hold in U ([9, Lemma 2.1 and Lemma 2.2]):
(2.1) $e_3^2 = 0$, $f_3^2 = 0$; $f_3e_3 + e_3f_3 = t_2H_1 + t_1^{-1}H_2$;
(2.2) $e_1e_3 = qe_3e_1$, $e_2e_3 = -qe_3e_2$; $f_1f_3 = qf_3f_1$, $f_2f_3 = -qf_3f_2$;
(2.3) $f_1e_3 - e_3f_1 = t_1^{-1}e_2$, $f_3e_1 - e_1f_3 = f_2t_1$;
(2.4) $f_2e_3 + e_3f_2 = qt_2e_1$, $f_3e_2 + e_2f_3 = t_2^{-1}f_1$.

Let U^+, U^- and U^0 be the subalgebras (with 1) of U generated by the e_i, the f_i and the $t_i^{\pm 1}$ $(i = 1, 2)$ respectively. Let $U^{\geq 0} = U^+U^0$, let P be generated by $U^{\geq 0}$ together with f_1, and let U_0 be generated by e_1, f_1, $t_1^{\pm 1}$ and $t_2^{\pm 1}$. Then the elements $e_3^{d_3}e_2^{d_2}e_1^k$ (resp. $f_3^{d_3}f_2^{d_2}f_1^k$), $d_3, d_2 \in \{0, 1\}$, $k \in \mathbb{Z}_+$, form a basis of U^+ (resp. U^-), and the monomials $t_2^{m_2}t_1^{m_1}$, $m_1, m_2 \in \mathbb{Z}$, form a basis of U^0. Let $r = (d_1, d_2, k) \in \{0, 1\} \times \{0, 1\} \times \mathbb{Z}_+$, $m = (m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$, and let $e^r = e_3^{d_1}e_2^{d_2}e_1^k$, $f^r = f_3^{d_1}f_2^{d_2}f_1^k$, $t^m = t_1^{m_1}t_2^{m_2}$. Then the elements $f^{r'}t^me^r$ form a basis of U.

Let $G = N^- + H + N^+$ be the standard triangular decomposition. Then $H = \langle h_1 = E_{11} - E_{22}, h_2 = E_{22} + E_{33}\rangle$. We use ϵ_1, ϵ_2 and δ_1 to express the roots of G and choose $\alpha = \epsilon_1 - \epsilon_2$, $\beta = \epsilon_2 - \delta_1$ to be a simple root system. The positive even and odd root sets are $R_0^+ = \{\alpha\}$, $R_1^+ = \{\beta, \alpha + \beta\}$ respectively. For $\lambda \in H^*$, let $a = \lambda(h_1)$, $b = \lambda(h_2)$, and write $\lambda = (a, b)$. Note that $\alpha = (2, -1)$, $\beta = (-1, 0)$.

By [6], $\lambda \in H^*$ is typical if and only if $b \neq 0$ and $a + b + 1 \neq 0$. Let $K(\lambda)$ be the Kac G-module with highest weight λ and let $L(\lambda)$ be the simple G-module with highest weight λ. We call $\lambda = (a, b) \in H^*$ dominant integral if $a \in \mathbb{Z}_+$, $b \in \mathbb{Z}$.

Let $Q = \{m_1\alpha + m_2\beta : m_1, m_2 \in \mathbb{Z}\}$ and let $Q^+ = \{m_1\alpha + m_2\beta : m_1, m_2 \in \mathbb{Z}_+\}$. We identify Q as a subset of $\mathbb{Z} \times \mathbb{Z}$, and denote the elements of Q by (a, b) as before.

By a weight of U, we mean an element $\omega = (\omega_1, \omega_2) \in (\mathbb{C}(q)^\times)^2$. If ω' is another weight, we write $\omega' \leq \omega$ if ${\omega'}_1^{-1}\omega_1 = q^a$ and ${\omega'}_2^{-1}\omega_2 = q^b$ for some $(a, b) \in Q^+$. If V is a U-module, then its weight spaces are just the non-zero $\mathbb{C}(q)$-linear subspaces of the form $V_\omega = \{v \in V : t_iv = \omega_iv, i = 1, 2\}$. The nonzero vectors in V_ω are called weight vectors. A weight vector v is called maximal if $U^+v = 0$. A U-module is called a weight U-module if it is a direct sum of weight subspaces. We call a weight ω integral if $\omega = (q^a, q^b)$ with $(a, b) \in \mathbb{Z} \times \mathbb{Z}$.

A highest weight U-module V is a U-module having a maximal vector v such that $V = U \cdot v$. For a weight ω, one can define the Verma module $V(\omega)$ by letting $V(\omega) = U/J(\omega)$, where $J(\omega)$ is the left ideal of U generated by e_i and $t_i - \omega_i$, $i = 1, 2$. The module $V(\omega)$ has a unique simple quotient $L(\omega)$ and every simple highest weight U-module is isomorphic to some $L(\omega)$. One can also define Kac modules by first taking the simple highest weight U_0-module $L_0(\omega)$ and extending to a P-module by letting e_2 act trivially on it, then setting $K(\omega) = U \otimes_P L_0(\omega)$.

The unique simple quotient of $K(\omega)$ is isomorphic to $L(\omega)$. If $\omega = (\omega_1, \omega_2)$, then the U-module $L(\omega)$ is finite dimensional if and only if $\omega_1 = \pm q^a$ for some $a \in \mathbb{Z}_+$.

By [9], every finite dimensional highest weight G-module with integral weight admits a deformation. If $\lambda = (a, b)$ is dominant integral, we call the U-module $K(\omega)$ (resp. $L(\omega)$) with $\omega = (q^a, q^b)$ type 1 deformation of the G-module $K(\lambda)$ (resp. $L(\lambda)$). We will restrict our attention to the category of U-modules whose composition factors are type 1 deformations of the corresponding simple finite dimensional G-modules.

Let U_i be the subalgebra of U generated by e_i, f_i, $t_i^{\pm 1}$, $i = 1, 2$. For $\lambda = (a, b) \in \mathbb{Z}^2$, let $M_\lambda = \{m \in M : t_1 m = q^a m, t_2 m = q^b m\}$. We call a U-module M integrable if (i) $M = \bigoplus_{\lambda \in \mathbb{Z}^2} M_\lambda$ with $\dim M_\lambda < \infty$; and (ii) as a U_1-module, M is a direct sum of finite dimensional U_1-submodules. For $\lambda = (a, b) \in \mathbb{Z}^2$, H_1 and H_2 act on the weight subspace M_λ by the scalars $[a] = (q^a - q^{-a})/(q - q^{-1})$ and $[b] = (q^b - q^{-b})/(q - q^{-1})$ respectively.

For an integrable U_1-module M, Kashiwara [7,8] has defined the operators $\tilde{e}_1$ and $\tilde{f}_1$ by letting $\tilde{e}_1 f_1^{(r)} u = f_1^{(r-1)} u$ and $\tilde{f}_1 f_1^{(r)} u = f_1^{(r+1)} u$. where $u \in \ker e_1$ and $f_1^{(k)} = [k]!^{-1} f_1^k$. In the next section, we will define $\tilde{e}_2$ and $\tilde{f}_2$.

3. Weight U_2-modules, the operators $\tilde{e}_2$ and $\tilde{f}_2$

By a weight U_2-module, we mean a U_2-module M such that

$$M = \bigoplus_{b \in \mathbb{Z}} M_b, \quad \text{where} \quad M_b = \{v \in M : t_2 v = q^b v\} \quad \text{and} \quad \dim M_b < \infty.$$

We assume that all the U_2-modules are weight U_2-modules in this paper. The simple U_2-modules $L(b)$ for $b \in \mathbb{Z}$ are given by:

$$e_2 \to \begin{pmatrix} 0 & [b] \\ 0 & 0 \end{pmatrix}, f_2 \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, t_2 \to \begin{pmatrix} q^b & 0 \\ 0 & q^b \end{pmatrix}, \quad \text{if} \quad b \neq 0. \tag{3.1}$$

$$e_2 \to 0, f_2 \to 0, t_2 \to 1, \quad \text{if} \quad b = 0. \tag{3.2}$$

Lemma 3.1. *Let M be a U_2-module. Then $M = \bigoplus_{b \in \mathbb{Z}} M_b$ is a direct sum of U_2-submodules.*

Proof. We only need to verify that each weight subspace M_b is actually a U_2-submodule. But this fact follows from the relations $t_2 e_2 t_2^{-1} = e_2$ and $t_2 f_2 t_2^{-1} = f_2$. □

Lemma 3.2. *For $b \in \mathbb{Z}$, $b \neq 0$, M_b is a direct sum of simple U_2-modules.*

Proof. For any $v \in \ker e_2|_{M_b}$, the linear span of v and $f_2 v$, denoted by (v), is a U_2-submodule isomorphic to $L(b)$ (since $e_2 f_2 v = H_2 v = [b] v$ and $[b] \neq 0$). Hence as a U_2-module, M_b is isomorphic to the direct sum of $(1/2) \dim M_b$ copies of $L(b)$. □

In order to control the case $b = 0$, we introduce an indecomposable U_2-module $B(0)$ as follows. Let U_2^0 be the subalgebra generated by $t_2^{\pm 1}$. Let V_0 be the one dimensional U_2^0-module defined by $t_2 v = v$ for any $v \in V_0$, and let $B(0) = U_2 \otimes_{U_2^0} V_0$. Then $\dim B(0) = 4$ and for any nonzero $v_0 \in V_0$, the vectors v_0 (identify v_0 with

$1 \otimes v_0)$, $v_1 = f_2 v_0$, $v_2 = e_2 v_0$, $v_3 = f_2 e_2 v_0$ form a basis of $B(0)$. With respect to the basis $\{v_0, v_1, v_2, v_3\}$, the actions of e_2, f_2, t_2 on $B(0)$ are given by

$$e_2 \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, f_2 \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, t_2 \to id. \tag{3.3}$$

Lemma 3.3. *The U_2-module $B(0)$ is an indecomposable projective object in the category of U_2-modules.*

Proof. The U_2-module $B(0)$ is projective because for any U_2^0-module V and any U_2-module M, we have that $Hom_{U_2}(U_2 \otimes_{U_2^0} V, M) \cong Hom_{U_2^0}(V, M)$ and that the one dimensional U_2^0-module V_0 is projective in the category of weight U_2^0-modules. To see that $B(0)$ is indecomposable, one just needs to note that any nonzero submodule of $B(0)$ contains the one dimensional submodule (v_3). In fact, if $N \neq (0)$ is a submodule of $B(0)$, let $0 \neq v = \sum_{0 \leq i \leq 3} c_i v_i \in N$; then one can always get v_3 from v by using the action of U_2. For example, if $c_0 \neq 0$, then $v_3 = c_0^{-1} f_2 e_2 v$. □

Let M_0 be a U_2-module such that t_2 acts on it as the identity. Then by Lemma 3.3, M_0 is a quotient of a direct sum of copies of $B(0)$; hence one can understand the action of e_2 on M_0 through the quotient. Thus for a U_2-module $M = \bigoplus_{b \in \mathbb{Z}} M_b$, we define operators $\tilde{f}_2$, $\tilde{e}_2$ on M by letting

$$\tilde{f}_2 = f_2 \quad \text{and} \quad \tilde{e}_2 v = \begin{cases} q^{-1} t_2 e_2 v, & \text{if } v \in M_b,\ b > 0, \\ e_2 v, & \text{if } v \in M_0, \\ -q^{-1} t_2^{-1} e_2, & \text{if } v \in M_b,\ b < 0. \end{cases} \tag{3.4}$$

If we decompose M_b $(b \neq 0)$ into a direct sum of copies of $L(b)$ and choose $\{v, f_2 v\}$ as a basis for each $L(b)$, where $v \neq 0$ is such that $e_2 v = 0$, then the action of $\tilde{e}_2$ on M_b is given by:

$$\tilde{e}_2 v = 0, \quad \tilde{e}_2(f_2 v) = \begin{cases} (q^{2b} - 1)/(q^2 - 1) v, & b > 0, \\ (1 - q^{-2b})/(1 - q^2) v, & b < 0. \end{cases} \tag{3.5}$$

4. The category of finite dimensional integrable U-modules

Let $\Sigma^+ = \mathbb{Z}_+ \times \mathbb{Z}$ (i.e. Σ^+ is the subset of $\mathbb{Z}^2$ consisting of those λ such that $L(\lambda)$ is a finite dimensional $U(G)$-module), and let $\Omega^+ = \{\omega = (q^a, q^b) : (a, b) \in \Sigma^+\}$. Let $\mathfrak{F}$ be the category formed by finite dimensional weight U-modules M whose simple subquotients are those $L(\omega)$ with $\omega \in \Omega^+$. For a U-module M and an element $v \in M$, we denote by (v) the submodule of M generated by v.

For $\omega \in \Omega^+$, we set $Q(\omega) = U \otimes_{U_0} L_0(\omega)$. Then $Q(\omega)$ is a projective object in $\mathfrak{F}$. Fix a highest weight vector v_0 of $L_0(\omega)$, write the element $u \otimes v$ of $Q(\omega)$ as uv, and let

$$Q^0 = Q(\omega), \quad Q^1 = (e_2 v_0), \quad Q^2 = (e_3 v_0), \quad Q^3 = (e_3 e_2 v_0); \tag{4.1}$$

$$\omega^0 = \omega = (q^a, q^b), \quad \omega^1 = (q^{a-1}, q^b), \quad \omega^2 = (q^{a+1}, q^{b-1}), \quad \omega^3 = (q^a, q^{b-1}).$$

Then if $a = b = 0$, we have $Q^1 = Q^2$, $Q^3 \cong K(\omega^3)$, $Q(\omega)/Q^1 \cong K(\omega)$ and $Q^1/Q^3 \cong K(\omega^2)$; otherwise, we have $Q^0 \supsetneq Q^1 \supsetneq Q^2 \supsetneq Q^3 \supsetneq (0) = Q^4$ and $Q^i/Q^{i+1} \cong K(\omega^i)$, $0 \leq i \leq 3$.

The U-module $Q(\omega)$ is a direct sum of indecomposable objects in $\mathfrak{F}$. We are interested in the indecomposable direct summand $I(\omega)$ of $Q(\omega)$ which has $K(\omega)$ as its quotient. Consider the weight subspace $Q(\omega)_\omega$. Let $\{v_k\}_{0\le k\le a}$ be a basis of $L_0(\omega)$ such that (set $v_{-1} = v_{a+1} = 0$)

$$t_1 v_k = q^{a-2k} v_k, e_1 v_k = [a-k+1] v_{k-1}, f_1 v_k = [k+1] v_{k+1}. \tag{4.2}$$

Then $\{v_0, f_2e_2v_0, f_3e_3v_0, f_2e_3v_1, f_3f_2e_3e_2v_0\}$ is a basis of $Q(\omega)_\omega$.

To find a generator of $I(\omega)$, let

$$v_\omega = c_0v_0 + c_1f_2e_2v_0 + c_2f_3e_3v_0 + c_3f_2e_3v_1 + c_4f_3f_2e_3e_2v_0. \tag{4.3}$$

Then the condition for v_ω to be a maximal vector is $e_1v_\omega = e_2v_\omega = 0$, which leads to a homogeneous linear system on the c_i's with a one-dimensional solution space, we choose the following solution:

$$c_0 = -q^{-1}[b][a+b+1], c_1 = q^{-2}[b], c_2 = q^{-2}[b], c_3 = -q^{-b-2}, c_4 = 1. \tag{4.4}$$

If $b \ne 0$, $a+b+1 \ne 0$, then (4.3) and (4.4) provide a unique (up to multiple) maximal vector v_ω which generates a copy of $K(\omega)$ as a direct summand of $Q(\omega)$.

Lemma 4.1. *Let $\omega = (q^a, q^b) \in \Omega^+$ be such that $b \ne 0$ and $a+b+1 \ne 0$. Let v_ω be the maximal vector provided by (4.3) and (4.4). Then $Q(\omega) = (v_\omega) \oplus Q^1$ as U-modules, where Q^1 is defined in (4.1).*

Proof. Under the assumption of the lemma, the coefficient c_0 of v_ω defined in (4.5) is not zero; hence $v_0 \in (v_\omega) + Q^1$ and therefore $Q(\omega) = (v_\omega) + Q^1$. To see that the sum is a direct sum, we only need to note that the sum is a direct sum as U^--modules. □

If $b = 0$ or $a+b+1 = 0$, then the maximal vector v_ω does not generate a direct summand of $Q(\omega)$. However, we have the following lemma.

Lemma 4.2. *Let ω^1 and ω^2 be defined as in (4.1). Then we have*

i) For $b = a = 0$, there exists a weight vector v_ω of weight ω such that

1) $e_1v_\omega = 0$, $(e_3e_2v_\omega) = (e_3e_2v_0)$;

2) (v_ω) is a direct summand of $Q(\omega)$ and $(v_\omega)/(e_3e_2v_\omega) \cong K(\omega)$.

ii) For $b = 0$, $a > 0$, there exists a weight vector v_ω of weight ω such that:

1) $e_1v_\omega = e_3v_\omega = 0$, $e_2v_\omega \ne 0$;

2) (v_ω) is a direct summand of $Q(\omega)$, e_2v_ω is maximal, $(e_2v_\omega) \cong K(\omega^1)$ and $(v_\omega)/(e_2v_\omega) \cong K(\omega)$.

iii) For $a+b+1 = 0$, there exists a weight vector v_ω of weight ω such that:

1) $e_1v_\omega = 0$, $e_3v_\omega \ne 0$,

2) (v_ω) is a direct summand of $Q(\omega)$, e_3v_ω is maximal, $(e_3v_\omega) \cong K(\omega^2)$ and $(v_\omega)/(e_3v_\omega) \cong K(\omega)$.

Proof. i) For $a = b = 0$, let $v_\omega = v_0 - qf_2e_2v_0 - q^{-1}f_3e_3v_0$, $v^2 = e_3v_0 + f_2e_2e_3v_0$. Then v^2 is maximal, $(v^2) \cong K(\omega^2)$, $e_2v_\omega = f_3e_3e_2v_0$, and $e_3v_\omega = qf_2e_3e_2v_0$. Hence the statement follows with the U-module decomposition $Q(\omega) = (v_\omega) \oplus (v^2)$.

ii) If $b = 0$, $a > 0$, let $v_\omega = q[a+1]v_0 - f_3e_3v_0 - q^a[a]^{-1}f_2e_3v_1$; then $e_1v_\omega = e_3v_\omega = 0$ and $e_2v_\omega \ne 0$. It follows that e_2v_ω is maximal, $(e_2v_\omega) \cong K(\omega^1)$, and $(v_\omega)/(e_2v_\omega) \cong K(\omega)$. Furthermore, $Q(\omega) = (v_\omega) \oplus Q^2$. In fact, since $Q^2 = (e_3v_0)$, it can be verified that $v_0 \in (v_\omega) + Q^2$; hence $Q(\omega) = (v_\omega) + Q^2$. Since the coefficient of v_0 is not zero, the sum is a direct sum as U^--modules.

iii) If $a+b+1=0$, let $u_\omega = -[b]v_0 + f_2e_2v_0$, and let

$$v^1 = -q(q+[b])[a]e_2v_0 + (1+q^{-1}[b])e_3v_1 + q^{b+1}[a]f_3e_3e_2v_0 + f_2e_3e_2v_1.$$

Then $e_2u_\omega = 0$, $e_3u_\omega \neq 0$, and e_3u_ω is maximal, so $(e_3u_\omega) \cong K(\omega^2)$ and $(u_\omega)/(e_3u_\omega) \cong K(\omega)$ (since $e_1u_\omega = [1-b]^{-1}f_2e_3u_\omega \in (e_3u_\omega)$). Also v^1 is maximal and $Q(\omega) = (u_\omega) \oplus (v^1) \oplus (e_3e_2v_0)$. Now let $v_\omega = [a+2]u_\omega - f_1e_1u_\omega$; then $(v_\omega) = (u_\omega)$ and $e_1v_\omega = 0$. So iii) follows. □

Let the direct summand of $Q(\omega)$ provided by Lemma 4.1 and Lemma 4.2 be $I(\omega)$ (i.e. $I(\omega) = (v_\omega)$); then $I(\omega)$ is a projective object in $\mathfrak{F}$. If $b \neq 0$, $a+b+1 \neq 0$, then $I(\omega) = K(\omega)$ is indecomposable. For $b=0$ or $a+b+1=0$, $I(\omega)$ is also indecomposable. Because for $b=0$, any submodule of $I(\omega)$ contains the submodule $(f_2f_3e_2e_3v_\omega)$ (for $a=0$) or $(f_2e_2v_\omega) \cong L(\omega)$ (for $a>0$); and for $a+b+1=0$, any submodule of $I(\omega)$ contains the submodule $(ue_3v_\omega) \cong L(\omega)$, where $u = [a]f_3 + q^af_2f_1$. Hence $I(\omega)$ is the projective cover of $L(\omega)$ in $\mathfrak{F}$. On the other hand, if P is an indecomposable projective object of $\mathfrak{F}$, then it must be the projective cover of some $L(\omega)$ and hence must be one of the $I(\omega)$. Hence we have the following theorem:

Theorem 4.3. *The indecomposable projective objects of $\mathfrak{F}$ are indexed by the elements of Σ^+, and their structures are given by Lemma 4.1 and Lemma 4.2.*

5. Crystal bases

Let $\mathfrak{F}$ be the category of U-modules defined in section 4, and let $M \in \mathfrak{F}$. Following [7], we define a crystal basis for M to be a pair (L, B) satisfying the following conditions:

(5.1) L is a free A-module such that $M \cong \mathbb{C}(q) \otimes_A L$.
(5.2) B is a basis for L/qL.
(5.3) L is stable by $\tilde{e}_1$ and $\tilde{f}_i$, $i=1,2$.
(5.4) $\tilde{e}_1B$, $\tilde{f}_iB \subset B \cup \{0\}$, $i=1,2$.
(5.5) $L = \bigoplus L_\lambda$, $B = \coprod B_\lambda$, where $L_\lambda = L \cap M_\lambda$, $B_\lambda = B \cap (L_\lambda/qL_\lambda)$.
(5.6) For b, $b' \in B$, $b' = \tilde{f}_1b$ if and only if $\tilde{e}_1b' = b$.

We first consider the crystal basis for a highest weight U-module in $\mathfrak{F}$. Let $\omega = (q^a, q^b) \in \Omega^+$, and let $\{v_k\}_{0\le k\le a}$ be the basis of $L_0(\omega)$ as described by (4.2).

Let $\mathcal{L}_K(\omega)$ be the lattice in $K(\omega)$ consisting all A-linear combinations of the elements of

$$\mathcal{B}_K(\omega) = \{v_k, q^{-k}f_2v_k, -q^{k-a}f_3v_k, -q^{-a}f_2f_3v_k\}_{0\le k\le a}. \tag{5.7}$$

By abusing notation, we also let $\mathcal{B}_K(\omega) \subset \mathcal{L}_K(\omega)/q\mathcal{L}_K(\omega)$ be the set of the equivalence classes of the elements in $\mathcal{B}_K(\omega)$. Then we have

Theorem 5.1. *The pair $(\mathcal{L}_K(\omega), \mathcal{B}_K(\omega))$ is a crystal basis for $K(\omega)$.*

In order to prove this theorem, we need two lemmas.

Lemma 5.2. *As a U_1-module, $K(\omega)$ decomposes as follows:*

i) For $a=0$, $K(\omega) \cong \mathbb{C} \oplus \mathbb{C} \oplus L_0(q, q^b)$ with generators v_0, $f_3f_2v_0$ and f_2v_0 respectively.

ii) For $a>0$, $K(\omega) \cong L_0(\omega) \oplus L_0(\omega_1) \oplus L_0(\omega_2) \oplus L_0(\omega_3)$, where $\omega_1 = (q^{a+1}, q^b)$, $\omega_2 = (q^a, q^{b+1})$, $\omega_3 = (q^{a-1}, q^{b+1})$, with generators v_0, f_2v_0, $f_3f_2v_0$, $q^a[a]^{-1}f_2v_1 + f_3v_0$ respectively.

Proof. We just need to note that vectors listed in the lemma are in the kernel of e_1 and the dimensions add up right. □

Let

$$B_1 = \{v_k, \tilde{f}_1^{k_1} f_2 v_0, -\tilde{f}_1^{k_2} q^{-a} f_2 f_3 v_0, -\tilde{f}_1^{k_3}([a]^{-1} f_2 v_1 + q^{-a} f_3 v_0) : \\ 0 \le k \le a, 0 \le k_1 \le a+1, 0 \le k_2 \le a, 0 \le k_3 \le a-1\}. \tag{5.8}$$

Lemma 5.3. *The linear span of B_1 over A is $\mathcal{L}_K(\omega)$ and $\mathcal{B}_K \equiv B_1(\bmod q\mathcal{L}_K(\omega))$.*

Proof. In the proof, the congruences will always be modulo $q\mathcal{L}_K(\omega)$. We first note that when $a = 0$, $B_1 = \{v_0, f_2 v_0, -f_2 f_3 v_0, -f_3 v_0\} = \mathcal{B}_K(\omega)$. For $a > 0$, we claim that

$$\tilde{f}_1^k f_2 v_0 \equiv \begin{cases} q^{-k} f_2 v_k, & k < a+1, \\ -f_3 v_a, & k = a+1; \end{cases} \quad \tilde{f}_1^k f_2 f_3 v_0 \equiv f_2 f_3 v_k, \quad 0 \le k \le a; \tag{5.9}$$

$$\tilde{f}_1^k([a]^{-1} f_2 v_1 + q^{-a} f_3 v_0) \equiv q^{k-a} f_3 v_k, \quad 0 \le k \le a-1.$$

These relations can be proved by using induction on k. For example, for the first formula, note that it holds for $k = 0$, and assume that it holds for $k - 1 \ge 0$; then

$$\begin{aligned} \tilde{f}_1^k f_2 v_0 &= [k]^{-1} f_1 \tilde{f}_1^{k-1} f_2 v_0 \equiv [k]^{-1} q^{-k+1} f_1 f_2 v_{k-1} \\ &= [k]^{-1} q^{-k+1}(q^{-1} f_2 f_1 - f_3) v_{k-1} = q^{-k} f_2 v_k - [k]^{-1} q^{-k+1} f_3 v_{k-1} \\ &\equiv \begin{cases} q^{-k} f_2 v_k, & k < a+1, \\ -f_3 v_a, & k = a+1. \end{cases} \qquad \text{(by (5.7))} \end{aligned}$$

Now the lemma follows from (5.9). □

Proof of Theorem 5.1. By Lemma 5.2 and Lemma 5.3, $(\mathcal{L}_K(\omega), B_1)$ is a crystal basis for $K(\omega)$ as a U_1-module (a lower basis, see [8]); hence (5.1)–(5.6) hold for $i = 1$. Since it is clear from the definition of $\mathcal{B}_K(\omega)$ that $f_2 \mathcal{B}_K(\omega) \subset \mathcal{B}_K(\omega) \cup \{0\}$, the theorem follows. □

Remark. In our definition of the crystal bases, we do not require $\tilde{e}_2 B \subset B \cup \{0\}$. This reflects the fact that in order to require a crystal basis be a crystal basis for U_1 via restriction, we need to give up something, due to the presence of the relations $e_2^2 = f_2^2 = 0$, which lead to $e_1 e_2 e_1 e_2 = e_2 e_1 e_2 e_1$ and $f_1 f_2 f_1 f_2 = f_2 f_1 f_2 f_1$ in U, and in these last two relations, the indeterminate q is not involved.

Corollary 5.4. *Let $\pi : K(\omega) \to L(\omega)$ be the quotient map, let $\mathcal{L}_L(\omega) = \pi(\mathcal{L}_K(\omega))$ and let $\mathcal{B}_L(\omega) = \pi'(\mathcal{B}_K(\omega)) \backslash \{0\}$, where $\pi' : \mathcal{L}_K(\omega)/q\mathcal{L}_K(\omega) \to \mathcal{L}_L(\omega)/q\mathcal{L}_L(\omega)$ is the induced map. Then $(\mathcal{L}_L(\omega), \mathcal{B}_L(\omega))$ is a crystal basis for $L(\omega)$.*

Proof. One just needs to note that the kernel of π is given by: i) $\{0\}$ if $a+b+1 \ne 0$ and $b \ne 0$; or ii) $(f_2 f_3 v_0)$ if $a = b = 0$, or $(f_2 v_0)$ if $a > 0$ and $b = 0$; or iii) $([a]^{-1} f_2 v_1 + q^{-a} f_3 v_0)$ if $a + b + 1 = 0$. □

In order to construct crystal bases for any U-module $M \in \mathfrak{F}$; we need to construct crystal bases for the indecomposable projective modules $I(\omega)$. By Theorem 4.3, $I(\omega)$ is generated by a single vector of weight ω; we fix such a generator provided by Lemma 4.1 and Lemma 4.2, and denote it by v_ω. If $a = b = 0$, we have $\tilde{e}_2 v_\omega = e_2 v_\omega$, $\tilde{e}_1 \tilde{e}_2 v_\omega = e_3 v_\omega$, $\tilde{e}_2 \tilde{e}_1 \tilde{e}_2 v_\omega = e_2 e_3 v_\omega$, with $e_2 e_3 v_\omega$ a vector of maximal weight in $I(\omega)$. If $b = 0$, $a > 0$, then $\tilde{e}_2 v_\omega = e_2 v_\omega$ is a vector of maximal weight in $I(\omega)$. If $a + b + 1 = 0$, direct computation shows that $\tilde{e}_1 \tilde{e}_2 v_\omega = -q^{a-1}(q^{-a} - [a]q)^{-1} e_3 v_\omega$ is a vector of maximal weight in $I(\omega)$. For all these three cases, we use v to denote the vector of maximal weight we just described. Also if $I(\omega) = K(\omega)$, we let $v = 0$. For our convenience, we also denote $v_0 = v_\omega$ and $v_k = \tilde{f}_1^k v_0$. Set

$$\mathcal{B}'_I(\omega) = \{v_k, q^{-k} f_2 v_k, -q^{k-a} f_3 v_k, -q^{-a} f_2 f_3 v_k, \tilde{f}_1^{k_1} v, q^{-k_1} f_2 \tilde{f}_1^{k_1} v, \\ - q^{k_1-d} f_3 \tilde{f}_1^{k_1} v, -q^{-d} f_2 f_3 \tilde{f}_1^{k_1} v : 0 \le k \le a, 0 \le k_1 \le d\},$$

where $d = 0$ if $a = b = 0$, or $a - 1$ if $b = 0$ and $a > 0$, or $a + 1$ if $a + b + 1 = 0$. Let $\mathcal{L}_I(\omega)$ be the linear span of $\mathcal{B}'_I(\omega)$ over A, and let $\mathcal{B}_I(\omega)$ be the image of $\mathcal{B}'_I(\omega)$ in $\mathcal{L}_I(\omega)/q\mathcal{L}_I(\omega)$.

Theorem 5.5. *The pair $(\mathcal{L}_I(\omega), \mathcal{B}_I(\omega))$ is a crystal basis of $I(\omega)$, and if M is a quotient of $I(\omega)$, $\pi : I(\omega) \to M$ is the projection, then $(\pi(\mathcal{L}_I(\omega)), \pi'(\mathcal{B}_I(\omega))\backslash\{0\})$ is a crystal basis of M, where $\pi' : \mathcal{L}_I(\omega)/q\mathcal{L}_I(\omega) \to \pi(\mathcal{L}_I(\omega))/q\pi(\mathcal{L}_I(\omega))$ is the map induced by π.*

Proof. Since $e_1 v_\omega = 0$ and $e_1 v = 0$, by Lemma 5.2, Lemma 5.3 and the definition of $\mathcal{B}_I(\omega)$, $(\mathcal{L}_I(\omega), \mathcal{B}_I(\omega))$ is a crystal basis for $I(\omega)$ as a U_1-module. Since $f_2\mathcal{B}_I(\omega) \subset \mathcal{B}_I(\omega) \cup \{0\}$ by the definition of $\mathcal{B}_I(\omega)$, we see that $(\mathcal{L}_I(\omega), \mathcal{B}_I(\omega))$ is a crystal basis for $I(\omega)$. Hence the first statement follows.

Now let M be a quotient of $I(\omega)$ and let $I(\omega)/N = M$. We may assume that $N \neq (0)$. By Corollary 5.4, we may also assume that $I(\omega) \neq K(\omega)$. We consider the three cases listed in Lemma 4.2 separately.

Case i): $a = b = 0$. In this case, note that by the proof of Lemma 4.2, $f_2 f_3 e_2 e_3 v_\omega$ is maximal, $f_2 v_\omega$ is primitive and

$$\begin{aligned} e_2 f_2 v_\omega &= (H_2 - f_2 e_2) v_\omega = -f_2 e_2 (v_0 - q f_2 e_2 v_0 - q^{-1} f_3 e_3 v_0) \\ &= f_2 f_3 e_2 e_3 v_0 = f_2 f_3 e_2 e_3 v_\omega = f_2 f_3 v. \end{aligned}$$

We see that N must be one of the submodules $(f_2 f_3 v)$, (v), $(f_2 v_\omega)$ and $(f_2 v_\omega, v)$. Hence from the structure of $I(\omega)$ and the definition of $\mathcal{B}_I(\omega)$, we can conclude that the theorem holds in this case.

Case ii): $a > 0$, $b = 0$. In this case, $f_2 v$ is maximal, $f_2 v_\omega$ is primitive and $e_2 f_2 v_\omega = f_2 e_2 v_\omega = f_2 v$, and the theorem holds in this case.

Case iii): $a + b + 1 = 0$. Let

$$u = q^a [a]^{-1} f_2 v_1 + f_3 v_0, \quad u_1 = (q^{a+1}[a+1]^{-1} f_2 f_1 + f_3) v.$$

Then N is one of the submodules (u), (u_1), (v) and (u, v), and the theorem also holds in this case. □

Corollary 5.6. *Every projective object in $\mathfrak{F}$ has a crystal basis.*

Theorem 5.7. *Every module $M \in \mathfrak{F}$ has a crystal basis. In fact, if P is the projective cover of M in $\mathfrak{F}$, (L_P, B_P) is a crystal basis of P, $\pi : P \to M$ is the quotient map, then $(\pi(L_P), \pi'(B_P)\backslash\{0\})$ is a crystal basis for M, where π' is the induced map from L_P/qL_P to $\pi(L_P)/q\pi(L_P)$.*

Proof. Let $P = \bigoplus_{1 \le i \le k} I(\omega_i)$. We use induction on k. The case $k = 1$ is covered by Theorem 5.5. Assume the result for $1 \le k < n$ and consider the case $k = n$.

Let $I_1 = I(\omega_1)$, $I_2 = \bigoplus_{2 \le i \le n} I(\omega_i)$. Let $V = \pi(I_2)$ and let $\pi_1 : V \oplus I_1 \to M$ be defined by $\pi_1 = id \oplus \pi|_{I_1}$. Then by the induction assumption, V has a crystal basis formed by the images of the crystal basis of I_2; let this crystal basis be (L, B).

Let the kernel of π_1 be N. Then we may assume that $N \neq \{0\}$. Otherwise $M \cong V \oplus I_1$, and there is nothing to prove. Note that $N \cap V = \{0\}$ and $\pi_1(I_1)$ is not a subset of $\pi_1(V)$ (since P is the projective cover of M, n must be minimal). So there exists a proper submodule N' of I_1 such that $N \cong N'$. Hence in order to find a crystal basis of M we only need to take L and a crystal basis of I_1/N'. Now the proof of the second statement of Theorem 5.5 applies, and we can complete the proof of the theorem. ☐

Corollary 5.8. *Let $M \in \mathfrak{F}$, let (L, B) be a crystal basis of M provided by Theorem 5.7, and let $\pi : M \to M'$ be a quotient map. Then $(\pi(L), \pi'(B)\backslash\{0\})$ is a crystal basis of M', where $\pi' : L/qL \to \pi(L)/q\pi(L)$ is the induced map.*

Proof. This is clear from Theorem 5.7. ☐

It is clear from the construction of the crystal basis of $K(\omega)$ (see also (5.9)) that crystal bases for $L(\omega)$ are not unique. If $a > 0$ and $b = 0$, then $K(\omega)/(f_2v_0) \cong L(\omega)$; we let

$$\mathcal{B}'_L(\omega) = \{v_k, q\tilde{f}_1^{k_1}f_3v_0 : 0 \leq k \leq a, 0 \leq k_1 \leq a-1\}, \tag{5.10}$$

and if $a + b + 1 = 0$, then $K(\omega)/(q^a[a]^{-1}f_2v_1 + f_3v_0) \cong L(\omega)$, we let

$$\mathcal{B}'_L(\omega) = \{v_k, \tilde{f}_1^{k_1}f_2v_0 : 0 \leq k \leq a, 0 \leq k_1 \leq a+1\}. \tag{5.11}$$

Then in each of these cases, $(\mathcal{L}_L(\omega), \mathcal{B}_L(\omega))$ is also a crystal basis for $L(\omega)$.

6. A crystal basis for U^-

As in the Lie algebra case, we now try to melt a crystal basis.

Lemma 6.1. *For every $u \in U^-$ with $deg(u) = b$, there exist unique elements u^+ and $u^- \in U^-$ such that*

$$e_iu - (-1)^{ab}ue_i = (t_iu^+ - t_i^{-1}u^-)/(q - q^{-1}), \quad i = 1, 2, \quad b = deg(e_i). \tag{6.1}$$

Proof. This follows from the following identity:

$$e_1f_1^k - f_1^ke_1 = (t_1q^k[k+1]f_1^{k-1} - t_1^{-1}q^{-k}[k+1]f_1^{k-1})/(q - q^{-1}),$$

the relations $e_1f_2 = f_2e_1$, $e_2f_1 = f_1e_2$, and (2) in section 2. ☐

By Lemma 6.1, we can define endomorphisms e_i^+ and e_i^-, $i = 1, 2$, of U^- by letting $e_i^+u = u^+$ and $e_i^-u = u^-$ for all $u \in U^-$ with u^+ and u^- as in (6.1). Then U^- becomes a U'-module, where U' is generated by the endomorphisms e_i^- and f_i (f_i acts on U^- by left multiplication), $i = 1, 2$, with the relations

$$e_i^-f_j = q^{-a_{ij}}f_je_i^- + \delta_{ij},$$

$$(e_1^-)^2e_2^- - (q + q^{-1})e_1^-e_2^-e_1^- + e_2^-(e_1^-)^2 = 0,$$
$$f_1^2f_2 - (q + q^{-1})f_1f_2f_1 + f_2f_1^2 = 0.$$

Lemma 6.2. *We have $K = \{1, f_2, f_2f_3, f_3\} \subset \ker e_1^-$, and $\{f_1^ku : u \in K, k \in \mathbb{Z}_+\}$ is a basis of U^-.*

Proof. The first statement follows from a direct computation and the second statement follows from the PBW theorem. ☐

Thus, we can define endomorphisms $\tilde{e}_1$ and $\tilde{f}_1$ (note that these operators are defined not just for the finite dimensional representations of U_1) of U^- by

$$\tilde{e}_1((f_1)^{(r)}u) = (f_1)^{(r-1)}u, \quad \tilde{f}_1((f_1)^{(r)}u) = (f_1)^{(r+1)}u, \quad u \in \ker e_1^-.$$

We also define the operators $\tilde{e}_2$ and $\tilde{f}_2$ as in (3.4).

Now imitating the definition of the crystal basis for $K(\omega)$, we consider the following set of elements of U^-:

$$B(\infty) = \{\tilde{f}_1^k \cdot 1, \tilde{f}_1^k f_2 \cdot 1, q\tilde{f}_1^k f_3 \cdot 1, -\tilde{f}_1^k f_2 f_3 \cdot 1 : k \in \mathbb{Z}_+\}.$$

Let $L^-(\infty)$ be the linear span of $B(\infty)$ over A, let $B^-(\infty)$ be the images of the elements of $B(\infty)$ in $L^-(\infty)/qL^-(\infty)$. Then we have

Theorem 6.1. *With the above notation, (i) The pair $(L^-(\infty), B^-(\infty))$ is a crystal basis for U^-.*

(ii) Let $\omega = (q^a, q^b) \in \Omega^+$ be such that (a, b) is atypical, let v_0 be a highest weight vector of $L(\omega)$, let L be the linear span of $B' = \{b \cdot v_0 : b \in B(\infty)\}$ over A, and let B be the nonzero images of the elements of B' in L/qL. Then $(L, B) = (\mathcal{L}_L(\omega), \mathcal{B}_L(\omega))$, where $\mathcal{B}_L(\omega)$ is chosen as in (5.10) or (5.11).

Proof. From the definition of $B(\infty)$, in order to verify (i), we only need to verify (5.3) and (5.4) for $\tilde{f}_2$. By using induction on k, one can prove the following identity in U^-:

$$f_2 f_1^{(k)} = [k]^{-1}(\sum_{i=0}^{k-1} q^{k-2i}) f_1^{(k-1)} f_3 + q^k f_1^{(k)} f_2.$$

Thus we have $f_2\tilde{f}_1^k \cdot 1 \equiv q\tilde{f}_1^{k-1} f_3 \cdot 1$, $f_2\tilde{f}_1^k f_2 \cdot 1 = -\tilde{f}_1^{k-1} f_2 f_3 \cdot 1$, $f_2 q\tilde{f}_1^k f_3 \cdot 1 = 0$, and $f_2(-\tilde{f}_1^k f_2 f_3 \cdot 1) = 0$. So (i) follows. To verify (ii), we consider the case that $a + b + 1 = 0$. The other cases can be considered similarly (and more easily). We need to verify that we get the images of the elements in (5.11) from B'. By comparing these two sets, we see that we only need to note that the images of $q\tilde{f}_1^k f_3 v_0$ and $-\tilde{f}_1^k f_2 f_3 v_0$ are 0 in L/qL. But by the statement preceding (5.11), we have

$$q\tilde{f}_1^k f_3 v_0 = q^{a+1}[a]^{-1}\tilde{f}_1^k f_2 v_1 \equiv 0 \pmod{qL},$$

and

$$-\tilde{f}_1^k f_2 f_3 v_0 = -q^a[a]^{-1}\tilde{f}_1^k f_2 f_2 v_1 = 0.$$

Thus (ii) follows. □

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