

A BOUND FOR THE THEOREM OF DE FRANCHIS

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ABSTRACT. We give a bound on the number of nonconstant holomorphic maps between compact Riemann surfaces of genera > 1 .

1. INTRODUCTION

Holomorphic maps of Riemann surfaces have a lot of rigid properties. It is known that the action of a holomorphic map on the fundamental groups, or on the homology groups strongly restricts the behaviour of the holomorphic map. Using these rigid properties, we may show finiteness of the number of holomorphic maps in the case of compact Riemann surfaces.

Let \tilde{X} be a compact Riemann surface of genus $\tilde{g} (> 1)$. Related to finiteness of the number of holomorphic maps, de Franchis [F] stated the following:

Theorem of de Franchis. (1) *For a fixed compact Riemann surface X of genus > 1 , the number of nonconstant holomorphic maps $\tilde{X} \rightarrow X$ is finite.*

(2) *There are only finitely many compact Riemann surfaces X_i of genus > 1 which admit a nonconstant holomorphic map from \tilde{X} .*

For algebraic proofs of this theorem, see e.g. Kani [K], Martens [Mr2], [Mr3], and Howard and Sommese [H-S]. Imayoshi [I1] [I2] gave the analytic proofs for Riemann surfaces of finite types.

It is an interesting problem to find an upper bound of the number of nonconstant holomorphic maps of \tilde{X} to X . When these genera are the same, any nonconstant holomorphic map must be conformal. Thus the number of such holomorphic maps is just the number of holomorphic automorphisms of \tilde{X} , and it is not greater than $84(\tilde{g}-1)$ by Hurwitz's theorem [H]. Furthermore, Machbeath [Mc] showed that the bound is sharp for an infinite number of integers \tilde{g} .

Let $g (> 1)$ denote the genus of a target surface of a holomorphic map from \tilde{X} , and we consider the case that $\tilde{g} > g$. Let $h_i : \tilde{X} \rightarrow X_i$ be a nonconstant holomorphic map ($i = 1, 2$). We say that h_1 and h_2 are isomorphic if and only if there is a conformal map $f : X_1 \rightarrow X_2$ such that $f \circ h_1 = h_2$. Let $\mathcal{I}_g(\tilde{X})$ be the set of all isomorphic classes of nonconstant holomorphic maps into compact Riemann surfaces of genus $g > 1$, and let $\mathcal{I}(\tilde{X}) = \bigcup_{\tilde{g} > g > 1} \mathcal{I}_g(\tilde{X})$. By the theorem of de Franchis, we see that $\#\mathcal{I}(\tilde{X})$ is finite. Howard and Sommese [H-S] first showed that

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there is a bound on $\#\mathcal{I}(\tilde{X})$ depending only on \tilde{g} . Kani [K] showed that

$$\#\mathcal{I}_g(\tilde{X}) < 2^{2\tilde{g}^2-1}(2^{2\tilde{g}^2-1} - 1),$$

and

$$\#\mathcal{I}(\tilde{X}) < (\tilde{g} - 1)2^{2\tilde{g}^2-2}(2^{2\tilde{g}^2-1} - 1).$$

To the best knowledge of the author, it is the smallest bound depending only on \tilde{g} .

In this paper, we will estimate the number of nonconstant holomorphic maps of \tilde{X} into a fixed Riemann surface X of genus g . In this case, Martens [Mr2] gave the first proof that the number of all nonconstant holomorphic maps into a fixed Riemann surface has a bound depending only on \tilde{g} (explicitly computable and it seems that the bound was given by $(c\tilde{g})^{2\tilde{g}^2}$ for some constant c independent of \tilde{g}). Using Kani's estimate, we see immediately that the number of all nonconstant holomorphic maps into a fixed Riemann surface X of genus $g > 1$ is less than

$$2^{2\tilde{g}^2-1}(2^{2\tilde{g}^2-1} - 1) \times 84(g - 1)$$

where $84(g-1)$ is the upper bound of $\#\text{Aut}(X)$. It was the smallest bound depending only on \tilde{g} and g when the author started to consider this problem, although Kani's estimate is counting maps into other Riemann surfaces of the same genus g .

Here, we will show

Theorem. *Let \tilde{X}, X be compact Riemann surfaces of genera $\tilde{g}, g (> 1)$. Let $\text{Hol}(\tilde{X}, X)$ be the set of all nonconstant holomorphic maps of \tilde{X} into X . Then, we have*

$$\#\text{Hol}(\tilde{X}, X) \leq (4(\tilde{g} - 1)/(g - 1) + 1)^{2\tilde{g}} \times 2(\tilde{g} - 1) \times (2g - 1).$$

If one wants to have a bound without using g , one may write

$$\#\text{Hol}(\tilde{X}, X) \leq (4\tilde{g} - 3)^{2\tilde{g}} \times 6(\tilde{g} - 1)$$

since the left-hand side of the inequality of the Theorem is monotone decreasing for $g > 1$.

By the Theorem, we may say that $\#\text{Hol}(\tilde{X}, X) \leq (c\tilde{g})^{2\tilde{g}}$ for some constant c , while $\#\mathcal{I}(\tilde{X}) \leq c'\tilde{g}^2$ for some constant c' by Kani's estimate.

2. PRELIMINARIES

Let \tilde{X}, X be compact Riemann surfaces of genera $\tilde{g}, g (> 1)$. We denote by $H_1(X)$ the first homology group with \mathbb{Z} -coefficients of X . Any basis for $H_1(X)$ (say $\{\chi_1, \dots, \chi_{2g}\}$), with intersection matrix (that is a matrix whose (k, j) -entry is given by the intersection number $\chi_k \cdot \chi_j$),

$$J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

will be called a canonical homology basis, where E is the $g \times g$ identity matrix.

We denote by H the space of harmonic differentials on X . For a canonical homology basis $\{\chi_1, \dots, \chi_{2g}\}$, there is a unique dual basis $\{\alpha_1, \dots, \alpha_{2g}\}$ ($\int_{\chi_j} \alpha_k = \delta_{jk}$ ($j, k = 1, \dots, 2g$)) of H , where each α_k is real. Similarly for \tilde{X} .

Let $h : \tilde{X} \rightarrow X$ be a nonconstant holomorphic map. Then h induces a homomorphism $h_* : H_1(\tilde{X}) \rightarrow H_1(X)$. Let $M = (m_{kj}) \in M(2g, 2\tilde{g}; \mathbb{Z})$, where $h_*(\tilde{\chi}_j) = \sum_{k=1}^{2g} m_{kj}\chi_k$. (We denote by $M(m, n; K)$ the set of $m \times n$ matrices

with K -coefficients.) We will call M the matrix representation of h with respect to $\{\tilde{\chi}_1, \dots, \tilde{\chi}_{2\tilde{g}}\}$ and $\{\chi_1, \dots, \chi_{2g}\}$. We have another interpretation of M . Considering an equality $\int_{\tilde{\chi}_j} \alpha_k \circ h = \int_{h_*(\tilde{\chi}_j)} \alpha_k$, we may write the pull-back of α_k as

$$\alpha_k \circ h = \sum_{j=1}^{2\tilde{g}} m_{kj} \tilde{\alpha}_j.$$

Thus, we may consider M as the representation of h on the space of harmonic differentials.

We denote by $*$ the conjugation operator, that is, for a 1-form $w = f dx + g dy$ ($z = x + iy$ is a local coordinate)

$$*w = -g dx + f dy.$$

We represent $*$ by a $2g \times 2g$ (resp. $2\tilde{g} \times 2\tilde{g}$) real matrix \mathcal{G} (resp. $\tilde{\mathcal{G}}$) with respect to the basis $\{\alpha_1, \dots, \alpha_{2g}\}$ ($\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_{2\tilde{g}}\}$) of H (resp. \tilde{H}),

$$\mathcal{G} = (g_{jk}) \quad (j, k = 1, \dots, 2g),$$

$$\text{(resp. } \tilde{\mathcal{G}} = (\tilde{g}_{jk}) \quad (j, k = 1, \dots, 2\tilde{g}),$$

that is, $*\alpha_k = \sum_{j=1}^{2g} g_{jk} \alpha_j$ (resp. $*\tilde{\alpha}_k = \sum_{j=1}^{2\tilde{g}} \tilde{g}_{jk} \tilde{\alpha}_j$). It is easy to see that $\mathcal{G}^2 = -E$. Let a $2g \times 2g$ real matrix

$$\Gamma = (\gamma_{jk}) = \left(\int_X \alpha_j \wedge * \alpha_k \right) \quad (j, k = 1, \dots, 2g).$$

It is well-known that Γ is symmetric, positive definite, and $\Gamma = J\mathcal{G}$ (see e.g. [F-K, chapter 3]). Similarly for \tilde{X} .

For an arbitrary 1-form w on X , we have

$$*w \circ h = *(w \circ h).$$

For $\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_{2\tilde{g}}\}$ and $\{\alpha_1, \dots, \alpha_{2g}\}$, considering the matrix representation of the above equality, we obtain

$$(1) \quad M^t \tilde{\mathcal{G}} = {}^t \mathcal{G} M.$$

(${}^t \mathcal{G}$ denotes the transposition of \mathcal{G} .) Indeed,

$$*(\alpha_j \circ h) = * \left(\sum_k m_{jk} \tilde{\alpha}_k \right) = \sum_k m_{jk} * \tilde{\alpha}_k = \sum_k m_{jk} \left(\sum_{l=1}^{2\tilde{g}} \tilde{g}_{lk} \tilde{\alpha}_l \right) = \sum_l \sum_k m_{jk} \tilde{g}_{lk} \tilde{\alpha}_l,$$

and

$$*\alpha_j \circ h = \left(\sum_k g_{kj} \alpha_k \right) \circ h = \sum_k g_{kj} \left(\sum_l m_{kl} \tilde{\alpha}_l \right) = \sum_l \sum_k g_{kj} m_{kl} \tilde{\alpha}_l.$$

Since the equality $*(\alpha_j \circ h) = *\alpha_j \circ h$ holds, we have (1).

It is easy to see that the set of all $M \in M(2g, 2\tilde{g}; \mathbb{Z})$ satisfying (1) is a \mathbb{Z} -module. We denote it by $\mathcal{M}(\tilde{\mathcal{G}}, \mathcal{G})$.

Remark. For the case that M is $2g \times 2g$ sized, that is, $M^t \mathcal{G} = {}^t \mathcal{G} M$, Weyl [W] investigated the relation algebraically to give conditions that M satisfies for a given \mathcal{G} .

Lemma 1. *Let $D \in \mathcal{M}(\tilde{\mathcal{G}}, \mathcal{G})$. Then, we have*

$$(2) \quad D\tilde{\Gamma}^t D\Gamma^{-1} = D\tilde{J}^t D J^{-1},$$

In particular, when M is the matrix representation of a nonconstant holomorphic map $h : \tilde{X} \rightarrow X$ of degree d ,

$$(3) \quad M\tilde{\Gamma}^t M\Gamma^{-1} = M\tilde{J}^t M J^{-1} = dE.$$

Proof. Indeed,

$$D\tilde{\Gamma}^t D\Gamma^{-1} = D\tilde{J}\tilde{\mathcal{G}}^t D\Gamma^{-1} = D\tilde{J}^t D\mathcal{G}\Gamma^{-1} = D\tilde{J}^t D J^{-1}$$

since $\Gamma = J\mathcal{G} = {}^t\mathcal{G}^t J$. When M is the matrix representation of a nonconstant holomorphic map of degree d , equality $M\tilde{J}^t M J^{-1} = dE$ holds (see e.g. [Mr3], [T]). Thus, we have (3). \square

Remark. If \tilde{X} and X are conformally equivalent, a matrix $M \in M(2g, 2g; \mathbb{Z})$ exists which satisfies formula (3) (Torelli's theorem, see e.g. [Mr1]).

Lemma 2. *Let*

$$D' = \tilde{J}^t D J^{-1} D,$$

where $D \in \mathcal{M}(\tilde{\mathcal{G}}, \mathcal{G})$. Then we have

$$(4) \quad D' {}^t\tilde{\mathcal{G}} = {}^t\tilde{\mathcal{G}} D',$$

and

$$(5) \quad D'\tilde{\Gamma}^t D'\tilde{\Gamma}^{-1} = D'\tilde{J}^t D' \tilde{J}^{-1}.$$

In particular, when $M' = \tilde{J}^t M J^{-1} M$ where M is the matrix representation of a nonconstant holomorphic map $h : \tilde{X} \rightarrow X$ of degree d , we have

$$(6) \quad M'\tilde{\Gamma}^t M'\tilde{\Gamma}^{-1} = dM'.$$

Proof. Indeed,

$$\begin{aligned} D' {}^t\tilde{\mathcal{G}} &= \tilde{J}^t D J^{-1} D {}^t\tilde{\mathcal{G}} = \tilde{J}^t D J^{-1} {}^t\mathcal{G} D \\ &= \tilde{J}^t D \mathcal{G} J D = \tilde{J}\tilde{\mathcal{G}}^t D J D = {}^t\tilde{\mathcal{G}}\tilde{J}^t D J^{-1} D = {}^t\tilde{\mathcal{G}} D'. \end{aligned}$$

By the same consideration as in the proof of relation (1), we have (5). When $M' = \tilde{J}^t M J^{-1} M$ where M is the matrix representation of a nonconstant holomorphic map of degree d , $M\tilde{J}^t M J^{-1} = dE$ holds and we see (6) holds by easy calculation. \square

Definition. Define the constant $I(X)$ by

$$I(X) = \inf_{\mathbf{n} \in \mathbb{Z}^{2g} \setminus \{0\}} \mathbf{n}\Gamma^t \mathbf{n}.$$

It is easy to see that $I(X)$ does not depend on the choice of the canonical homology basis. We see that $I(X) = \min_{\mathbf{n} \in \mathbb{Z}^{2g} \setminus \{0\}} \mathbf{n}\Gamma^t \mathbf{n}$ since Γ is positive definite. We define a norm $|\cdot|_{\Gamma}$ by

$$|\mathbf{n}|_{\Gamma}^2 = \mathbf{n}\Gamma^t \mathbf{n} \quad (\mathbf{n} \in \mathbb{Z}^{2g}).$$

3. PROOF OF THE THEOREM

Before proving the Theorem, we give a proposition.

Proposition. *Let $h_i : \tilde{X} \rightarrow X$ be a nonconstant holomorphic map, and M_i be the matrix representation ($i = 1, 2$). Let $e = \sum_{j=1}^{2g} e_j \alpha_j$ ($e = (e_j) \in \mathbb{Z}^{2g} \setminus \{0\}$) be a harmonic differential with $e\Gamma^t e = I(X)$. Suppose that there is an integer $l > 4(\tilde{g} - 1)/(g - 1)$ such that $eM_1 \equiv eM_2 \pmod{l}$ holds. Then $e \circ h_1 = e \circ h_2$.*

Proof. Let $D = M_2 - M_1$. Then,

$$(7) \quad eD\tilde{J}^t DJ^{-1} \equiv 0 \pmod{l}$$

by the assumption. We will first show that the equality (7) implies $eD\tilde{J}^t DJ^{-1} = 0$. Let $\tilde{e} = eD$. If $\tilde{e}\tilde{J}^t DJ^{-1} \neq 0$, then we have

$$l^2 I(X) \leq |\tilde{e}\tilde{J}^t DJ^{-1}|_{\Gamma}^2 \leq \{|\tilde{e}\tilde{J}^t M_2 J^{-1}|_{\Gamma} + |\tilde{e}\tilde{J}^t M_1 J^{-1}|_{\Gamma}\}^2.$$

The second inequality is obtained by using the triangle inequality. By formula (3) in Lemma 1, we have

$$|\tilde{e}\tilde{J}^t M_i J^{-1}|_{\Gamma}^2 = d_i^{-1} |\tilde{e}M'_i|_{\tilde{\Gamma}}^2 \quad (i = 1, 2).$$

Thus,

$$(8) \quad l^2 I(X) \leq (d_2^{-1/2} |\tilde{e}M'_2|_{\tilde{\Gamma}} + d_1^{-1/2} |\tilde{e}M'_1|_{\tilde{\Gamma}})^2.$$

Here we have

$$(9) \quad |\tilde{e}M'_i|_{\tilde{\Gamma}} \leq d_i |\tilde{e}|_{\tilde{\Gamma}}.$$

To show this, we will take a normal orthogonal basis in $\mathbb{R}^{2\tilde{g}}$ with respect to the inner product defined by $\tilde{\Gamma}$. First, let $\mathbf{u}_1 = \tilde{e}/|\tilde{e}|$. Next, we apply the Schmidt orthogonalization to obtain a normal orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_{2\tilde{g}}\}$. Then, we can write

$$\tilde{e}M'_i = \sum_{j=1}^{2\tilde{g}} x_{ij} \mathbf{u}_j, \quad x_{ij} \in \mathbb{R} \ (j = 1, \dots, 2\tilde{g}).$$

By formula (6) in Lemma 2, we have $\tilde{e}M'_i \tilde{\Gamma}^t M'_i{}^t \tilde{e} = d_i \tilde{e}M'_i \tilde{\Gamma}^t \tilde{e}$, or

$$\sum_{j=1}^{2\tilde{g}} x_{ij}^2 = d_i |\tilde{e}| x_{i1},$$

or

$$(x_{i1} - d_i |\tilde{e}|/2)^2 + \sum_{j=2}^{2\tilde{g}} x_{ij}^2 = (d_i |\tilde{e}|/2)^2.$$

Hence, $(x_{i1}, \dots, x_{i2\tilde{g}})$ must be on a $2\tilde{g} - 1$ -dimensional sphere whose radius is $d_i |\tilde{e}|/2$, centre is $(d_i |\tilde{e}|/2, 0, \dots, 0)$. Thus, $|\tilde{e}M'_i|_{\tilde{\Gamma}}^2 = \sum_{j=1}^{2\tilde{g}} x_{ij}^2 \leq (d_i |\tilde{e}|_{\tilde{\Gamma}})^2$.

Next, we will give an upper bound of \tilde{e} . By the triangle inequality, we have

$$(10) \quad |\tilde{e}|_{\tilde{\Gamma}} = |eD|_{\tilde{\Gamma}} \leq |eM_1|_{\tilde{\Gamma}} + |eM_2|_{\tilde{\Gamma}}.$$

By formula (3),

$$(11) \quad |eM_i|_{\tilde{\Gamma}}^2 = d_i |e|_{\Gamma}^2 = d_i I(X) \quad (i = 1, 2)$$

holds. Combining (10) and (11), we obtain

$$(12) \quad |\tilde{\mathbf{e}}|_{\Gamma}^2 \leq 4dI(X),$$

where $d = \max.(d_1, d_2)$. By applying (9) and (12) to (8),

$$l^2I(X) \leq 4^2d^2I(X).$$

Therefore, we must have $l \leq 4d \leq 4(\tilde{g}-1)/(g-1)$ (the right-hand side is obtained by the Riemann-Hurwitz formula), but it contradicts the assumption. Consequently, we see that (7) implies $\mathbf{e}D\tilde{J}^tDJ^{-1} = \mathbf{0}$.

Using formula (2), we have

$$\mathbf{e}D\tilde{\Gamma}^tD^te = \mathbf{e}D\tilde{J}^tDJ^{-1}\Gamma^te = 0.$$

This implies that $\mathbf{e}D = \mathbf{0}$ or $e \circ h_1 = e \circ h_2$ since $\tilde{\Gamma}$ is positive definite. □

Now, we prove the Theorem.

Proof of the Theorem. First, recall that, for an arbitrary 1-form w on X , we have

$$*w \circ h = *(w \circ h),$$

where $h \in Hol(\tilde{X}, X)$. Then, letting

$$\phi = e + i^*e,$$

we have

$$e \circ h_i = e \circ h_j \Leftrightarrow \phi \circ h_i = \phi \circ h_j,$$

where $h_i, h_j \in Hol(\tilde{X}, X)$. Thus,

$$(13) \quad \#\{\phi \circ h_i | h_i \in Hol(\tilde{X}, X)\} \leq (4(\tilde{g}-1)/(g-1) + 1)^{2\tilde{g}}$$

by the Proposition.

Let $p_0 \in X$ be one of the zeroes of ϕ . Suppose that $\phi \circ h_i = \phi \circ h_j$ and $h_i^{-1}(p_0) \cap h_j^{-1}(p_0) \neq \emptyset$. Then, let \tilde{p}_0 be such a zero of $\tilde{\phi} = \phi \circ h_i$. There are at most $2(\tilde{g}-1)$ possible points that \tilde{p}_0 can be since the number of zeroes of a holomorphic differential is $2(\tilde{g}-1)$ with multiplicity. Taking sufficiently small neighbourhood $U_{\tilde{p}_0}$ (resp. U_{p_0}) of \tilde{p}_0 (resp. p_0), we may assume that there is no zero of $\tilde{\phi}$ (resp. ϕ) on $U_{\tilde{p}_0}$ (resp. U_{p_0}) except \tilde{p}_0 (resp. p_0), and $h_k(U_{\tilde{p}_0}) \subset U_{p_0}$ ($k = i, j$). We may take a local coordinate \tilde{z} (resp. z) on $U_{\tilde{p}_0}$ (resp. U_{p_0}) such that \tilde{p}_0 (resp. p_0) corresponds to 0 (resp. 0) and

$$\tilde{\phi} = \tilde{z}^m d\tilde{z} \quad (\text{resp. } \phi = z^n dz).$$

For an arbitrary $\tilde{p} \in U_{\tilde{p}_0} \setminus \{\tilde{p}_0\}$,

$$(14) \quad \int_0^{\tilde{p}} \tilde{z}^m d\tilde{z} = \int_0^{h_i(\tilde{p})} z^n dz = \int_0^{h_j(\tilde{p})} z^n dz.$$

Thus, there are at most $n+1 \leq 2g-1$ possible points on U_{p_0} that (14) holds. By the identity theorem, $h_i(\tilde{p}) = h_j(\tilde{p})$ for an arbitrary $\tilde{p} \in U_{\tilde{p}_0} \setminus \{\tilde{p}_0\}$ implies $h_i = h_j$.

Let us sum up the argument above. First, we have the estimate (13). For a zero p_0 of ϕ and a $\tilde{\phi} = \phi \circ h \in \{\phi \circ h_i | h_i \in Hol(\tilde{X}, X)\}$, there are at most $2(\tilde{g}-1)$ possible points that a point \tilde{p}_0 with $h(\tilde{p}_0) = p_0$ can be. After determining $\tilde{\phi}$ and \tilde{p}_0 , we have at most $2g-1$ possible nonconstant holomorphic maps. Now, we obtain

$$\#Hol(\tilde{X}, X) \leq (4(\tilde{g}-1)/(g-1) + 1)^{2\tilde{g}} \times 2(\tilde{g}-1) \times (2g-1). \quad \square$$

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