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THE MOD 2 COHOMOLOGY OF THE LINEAR GROUPS OVER THE RING OF INTEGERS

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Abstract. This paper completely determines the Hopf algebra structure of the mod 2 cohomology of the linear groups $GL(\mathbb{Z})$, $SL(\mathbb{Z})$ and $St(\mathbb{Z})$ as a module over the Steenrod algebra, and provides an explicit description of the generators.

1. Introduction

Recently, J. Rognes and C. Weibel deduced from V. Voevodsky's proof [V] of the Milnor conjecture the complete calculation of the 2-torsion of the algebraic K-theory of the ring of integers $\mathbb{Z}$ (see Table 1 of [W] and Theorem 0.6 of [RW]). Of course, this has immediate consequences on the mod 2 cohomology of the infinite general linear group $GL(\mathbb{Z})$ and more generally on the understanding of the space $BGL(\mathbb{Z})^+$.

In [Bok], M. Bökstedt tried to construct a 2-adic model for the space $BGL(\mathbb{Z})^+$: he considered any prime number $p \equiv 3$ or $5 \bmod 8$ and introduced a space $J(p)$ which is defined by the pull-back diagram

$$\begin{tikzcd}
J(p) \arrow[r, "h'"] \arrow[d, "f'_p"] & BO \arrow[d, "c"] \\
F\Psi^p \arrow[r, "b"] & BU\,,
\end{tikzcd}$$

where $F\Psi^p$ is the fiber of $(\Psi^p - 1) : BU \to BU$ (recall that $F\Psi^p \simeq BGL(\mathbb{F}_p)^+$ by Theorem 7 of [Q2]), b is the Brauer lifting and c is the complexification. The fibers of the horizontal maps are homotopy equivalent to the unitary group U. He was actually more precisely interested in the covering space $JK(\mathbb{Z}, p)$ of $J(p)$ corresponding to the cyclic subgroup of order 2 of $\pi_1 J(p) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. Bökstedt's definition of the space $JK(\mathbb{Z}, p)$ (see [Bok], Definition 1.7 and the proof of Lemma 2.1) is based on the Adams conjecture and on the calculation of the 2-primary part of the homotopy groups of $(F\Psi^p)\hat{}_2$ which is the same, in dimensions $\equiv 3 \bmod 4$, for

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all primes $p \equiv 3$ or $5 \bmod 8$ (this explains the choice of p; see Section 3 of [Au] for more details). Notice that the space $JK(\mathbb{Z},p)$, in the case $p=3$, appears also in Section 4 of [DF] and in [M]. After completion at the prime 2, Bökstedt constructed a map

$$\varphi : (BGL(\mathbb{Z})^+)^{\wedge}_2 \longrightarrow JK(\mathbb{Z},p)^{\wedge}_2$$

which induces a split surjection on all homotopy groups (see [Bok], Diagram 1.9). Recall that the localization exact sequence in K-theory implies that

$$(BGL(\mathbb{Z}[\tfrac{1}{2}])^+)^{\wedge}_2 \simeq (BGL(\mathbb{Z})^+)^{\wedge}_2 \times (S^1)^{\wedge}_2 \, .$$

Therefore, φ provides a map

$$\widetilde{\varphi} : (BGL(\mathbb{Z}[\tfrac{1}{2}])^+)^{\wedge}_2 \longrightarrow J(p)^{\wedge}_2$$

which also induces a split surjection on all homotopy groups. Bökstedt's idea was indeed excellent because now the 2-torsion of $K_*(\mathbb{Z})$ is known and turns out to be isomorphic to the 2-torsion of $\pi_* JK(\mathbb{Z},p)$ (according to Table 1 of [W] and Theorem 0.6 of [RW]); therefore, φ and $\widetilde{\varphi}$ are actually homotopy equivalences. Observe in particular that the homotopy type of $(JK(\mathbb{Z},p))^{\wedge}_2$ does not depend on p (for $p \equiv 3$ or $5 \bmod 8$). Consequently, we obtain for all primes $p \equiv 3$ or $5 \bmod 8$ the pull-back diagram (see also Corollary 8 of [W])

$$\begin{tikzcd}
(BGL(\mathbb{Z})^+)^{\wedge}_2 \times (S^1)^{\wedge}_2 \arrow[r, "h'"] \arrow[d, "f'_p"] & BO^{\wedge}_2 \arrow[d, "c"] \\
(F\Psi^p)^{\wedge}_2 \arrow[r, "b"] & BU^{\wedge}_2
\end{tikzcd}$$

and the commutative diagram (where both rows are fibrations)

(*)
$$\begin{tikzcd}
SU^{\wedge}_2 \arrow[r, "\eta"] \arrow[d, "\zeta"] & (BGL(\mathbb{Z})^+)^{\wedge}_2 \arrow[r, "h"] \arrow[d, "f_p"] & BO^{\wedge}_2 \arrow[d, "c"] \\
U^{\wedge}_2 \arrow[r, "\theta"] & (F\Psi^p)^{\wedge}_2 \arrow[r, "b"] & BU^{\wedge}_2 \, ,
\end{tikzcd}$$

in which f_p and h denote the composition of the inclusion

$$(BGL(\mathbb{Z})^+)^{\wedge}_2 \hookrightarrow (BGL(\mathbb{Z})^+)^{\wedge}_2 \times (S^1)^{\wedge}_2$$

with f'_p and h' respectively, and ζ the 2-completion of the inclusion $SU \hookrightarrow U \simeq SU \times S^1$. According to Section 2 of [Bok], the map h is induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ and for all odd primes p, the diagram

$$\begin{tikzcd}
(BGL(\mathbb{Z})^+)^{\wedge}_2 \arrow[r, "\simeq", "\varphi"'] \arrow[d, "\mathrm{red}_p"] & JK(\mathbb{Z},p)^{\wedge}_2 \arrow[d, "f_p"] \\
(BGL(\mathbb{F}_p)^+)^{\wedge}_2 \arrow[r, "\simeq"] & (F\Psi^p)^{\wedge}_2 \, ,
\end{tikzcd}$$

where red_p is the map induced by the reduction mod $p : GL(\mathbb{Z}) \to GL(\mathbb{F}_p)$, is homotopy commutative. Thus, we may assume that the map f_p in the diagram $(*)$ is induced by the reduction mod p.

S. Mitchell computed the mod 2 homology of the space $JK(\mathbb{Z},3)$ in Theorem 4.3 of [M]; because of the above homotopy equivalence $(BGL(\mathbb{Z})^+)^{\wedge}_2 \simeq JK(\mathbb{Z},3)^{\wedge}_2$, this provides the calculation of $H_*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$ and by dualization the determination of the Hopf algebra structure of $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$ as a module over the Steenrod algebra $\mathcal{A}$ (see [M], Remark 4.5). However, Mitchell's argument does not give explicit generators of $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$. The first goal of the present paper is to use the above commutative diagram $(*)$ in order to get a direct proof of Michell's result.

Theorem. *There is an isomorphism of Hopf algebras and of modules over the Steenrod algebra*

$$\alpha : H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \cong H^*(BO;\mathbb{Z}/2) \otimes H^*(SU;\mathbb{Z}/2)\,.$$

Recall that $H^*(BO;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1,w_2,\dots]$ and $H^*(SU;\mathbb{Z}/2) \cong \Lambda(v_3,v_5,\dots)$, where $\deg(w_j)=j$ and $\deg(v_{2k-1})=2k-1$.

In fact, the main objective of this paper is to describe explicitly the generators of $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$. The generators of the polynomial part are the Stiefel-Whitney classes, also denoted by w_j, coming from $H^*(BO;\mathbb{Z}/2)$ via the homomorphism induced by h. On the other hand, we identify precisely (see Definitions 5 and 10 and Remark 14) the exterior generators u_{2k-1} of degree $2k-1$ in $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$, corresponding to $1\otimes v_{2k-1}$ under the above isomorphism α, in terms of the image of the homomorphism

$$f_p^* : H^*(F\Psi^p;\mathbb{Z}/2) \cong H^*(BGL(\mathbb{F}_p)^+;\mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$$

induced by the reduction mod p for $p \equiv 5 \bmod 8$ (they actually do not depend on the choice of p). We show that the classes u_{2k-1} are primitive cohomology classes and compute the action of the Steenrod squares on them. Therefore, we get an isomorphism of Hopf algebras and of modules over the Steenrod algebra

$$H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1,w_2,\dots] \otimes \Lambda(u_3,u_5,\dots)$$

and we deduce that the isomorphism α is unique (see Theorem 11). We also obtain an explicit formula relating the classes u_{2k-1} to the image of the homomorphism f_p^* for all primes $p \equiv 3 \bmod 8$ (see Theorem 13).

This provides a complete description of the mod 2 cohomology of the infinite general linear group $GL(\mathbb{Z})$. In the remainder of the paper we compute the mod 2 cohomology of the infinite special linear group $SL(\mathbb{Z})$ and of the infinite Steinberg group $St(\mathbb{Z})$ (see Corollary 15, Theorem 17 and Remark 18).

2. The mod 2 cohomology of the linear groups $GL(\mathbb{Z})$ and $SL(\mathbb{Z})$

Theorem 1. *There is an isomorphism of Hopf algebras and of modules over the Steenrod algebra*

$$\alpha : H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \cong H^*(BO;\mathbb{Z}/2) \otimes H^*(SU;\mathbb{Z}/2)\,.$$

Proof. As mentioned in the introduction, this follows indirectly from [M], Theorem 4.3 and Remark 4.5. Here is a direct argument. Let Q denote the subgroup of diagonal matrices in $GL(\mathbb{Z})$ and let $\lambda : BQ \to BGL(\mathbb{Z})^+$ be the map induced by the inclusion $Q \hookrightarrow GL(\mathbb{Z})$. It is known by Theorem 22.7 of [Bor] that the composition $h\,\lambda : BQ \to BO$ induces an injective homomorphism $\lambda^*\,h^* : H^*(BO;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1,w_2,\dots] \to H^*(BQ;\mathbb{Z}/2) \cong \varprojlim_m \mathbb{Z}/2[z_1,z_2,\dots,z_m]$ (with $\deg(z_i)=1$) and that $\lambda^*\,h^*(w_j)=\sigma_j$, where σ_j is the element of $H^j(BQ;\mathbb{Z}/2)$ whose restriction to $\mathbb{Z}/2[z_1, z_2, \dots, z_m]$ is the j-th elementary symmetric function in the m variables $z_1, \dots, z_m$, for all $m \geq j$. This implies that the infinite loop map h induces an injective homomorphism $h^* : H^*(BO; \mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$. Therefore, Theorem 15.2 of [Bor] shows that the Serre spectral sequence of the fibration

$$SU\hat{}_2 \xrightarrow{\eta} (BGL(\mathbb{Z})^+)\hat{}_2 \xrightarrow{h} BO\hat{}_2$$

collapses (see also Corollary 4.3 of [DF]) and we get additively the desired isomorphism. Since $(BGL(\mathbb{Z})^+)\hat{}_2$ is an H-space, the maps λ and η produce an H-map

$$\psi : BQ \times SU\hat{}_2 \longrightarrow (BGL(\mathbb{Z})^+)\hat{}_2$$

which induces an injective $\mathcal{A}$-module Hopf algebra homomorphism

$$\psi^* : H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \longrightarrow H^*(BQ; \mathbb{Z}/2) \otimes H^*(SU; \mathbb{Z}/2)\,.$$

Moreover, the fact that $\lambda^* : H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \to H^*(BQ; \mathbb{Z}/2)$ also satisfies $\lambda^*(w_j) = \sigma_j$ (see Lemma 1.1 of [Ar1]) implies that the image of ψ^* is isomorphic to $R \otimes H^*(SU; \mathbb{Z}/2)$, where R is the subalgebra of $H^*(BQ; \mathbb{Z}/2)$ generated by the elementary symmetric functions σ_j. On the other hand, the image of the injective $\mathcal{A}$-module Hopf algebra homomorphism

$$\lambda^* h^* \otimes 1 : H^*(BO; \mathbb{Z}/2) \otimes H^*(SU; \mathbb{Z}/2) \longrightarrow H^*(BQ; \mathbb{Z}/2) \otimes H^*(SU; \mathbb{Z}/2)$$

is also $R \otimes H^*(SU; \mathbb{Z}/2)$. This provides the statement of the theorem. □

In order to get a more precise picture of $H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$, let us identify its generators and understand the action of the Steenrod algebra on them. For $j \geq 1$ let us write $w_j \in H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$ for the image of the j-th universal Stiefel-Whitney class in $H^*(BO; \mathbb{Z}/2)$ under the homomorphism $h^* : H^*(BO; \mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$. The action of the Steenrod algebra on the Stiefel-Whitney classes is known by Wu's formula (see for instance [MT], Part I, p. 141). It remains to identify the exterior generators of $H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$. This will be done by using the homomorphism $f_p^* : H^*(F\Psi^p; \mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$ induced by the map f_p.

Let us first recall some properties of $H^*(F\Psi^p; \mathbb{Z}/2) \cong H^*(BGL(\mathbb{F}_p)^+; \mathbb{Z}/2)$. According to Quillen's calculation and notation (see [Q2]), if p is a prime$\equiv 5 \bmod 8$, then

$$H^*(F\Psi^p; \mathbb{Z}/2) \cong \mathbb{Z}/2[c_1, c_2, \dots] \otimes \Lambda(e_1, e_2, \dots)\,,$$

where $\deg c_j = 2j$ and $\deg e_k = 2k-1$; if p is a prime$\equiv 3 \bmod 8$, then $H^*(F\Psi^p; \mathbb{Z}/2)$ is also generated by the classes c_j and e_k ($j \geq 1$, $k \geq 1$), but one has the relations

$$e_k^2 = c_{2k-1} + \sum_{j=1}^{k-1} c_j c_{2k-1-j}$$

for $k \geq 1$, and $H^*(F\Psi^p; \mathbb{Z}/2)$ is polynomial:

$$H^*(F\Psi^p; \mathbb{Z}/2) \cong \mathbb{Z}/2[e_1, e_2, \dots, c_2, c_4, \dots]$$

(see also Section IV.8 of [FP]). In both cases, c_j is the image under $b^* : H^*(BU; \mathbb{Z}/2) \to H^*(F\Psi^p; \mathbb{Z}/2)$ of the reduction mod 2 of the j-th universal Chern class in $H^{2j}(BU; \mathbb{Z})$ and a spectral sequence argument shows that

$$\theta^* : H^*(F\Psi^p; \mathbb{Z}/2) \to H^*(U; \mathbb{Z}/2) \cong \Lambda(v_1, v_2, \dots)$$

satisfies $\theta^*(e_k) = v_{2k-1}$ for $k \geq 1$. For a prime $p \equiv 3$ or 5 mod 8, consider the homomorphism $f_p^* : H^*(F\Psi^p;\mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$ induced by f_p. For all $j \geq 1$, it is well known (see also Lemma 1.4 of [Ar1]) that

$$f_p^*(c_j) = w_j^2$$

and we established in [Ar2] for $k \geq 2$ the nonvanishing of the exterior class $f_p^*(e_k)$ if $p \equiv 5$ mod 8, respectively of the exterior class

$$\gamma_k = f_p^*(e_k) + w_{2k-1} + \sum_{j=1}^{k-1} w_j w_{2k-j-1}$$

of degree $2k-1$ if $p \equiv 3$ mod 8.

Let us mention the effect of the Steenrod squares on these cohomology classes.

Lemma 2. *(a) In $H^*(SU;\mathbb{Z}/2)$, $Sq^{2i}v_{2k-1} = \binom{k-1}{i} v_{2k+2i-1}$ for $k \geq 2$, $1 \leq i < k$, and $Sq^{2i-1}v_{2k-1} = 0$ for $k \geq 2$, $1 \leq i \leq k$.*

(b) In $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$, for any odd prime p, for $k \geq 1$ and $1 \leq i < k$,

$$Sq^{2i} f_p^*(e_k) = \tbinom{k-1}{i} f_p^*(e_{k+i}) + \sum_{j=1}^{i} \tbinom{k-j-1}{i-j} \left(w_j^2 f_p^*(e_{k+i-j}) + w_{k+i-j}^2 f_p^*(e_j) \right).$$

(c) In $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$, for $k \geq 1$ and $1 \leq i \leq k$,

$$Sq^{2i-1} f_p^*(e_k) = \begin{cases} 0, & \text{if } p \equiv 1 \text{ mod } 4 \text{ or if } p \equiv 3 \text{ mod } 4 \text{ and } k-i \text{ is odd}, \\ \displaystyle\sum_{j=0}^{i-1} \tbinom{k-j-1}{i-j-1} w_j^2 w_{k+i-j-1}^2, & \text{if } p \equiv 3 \text{ mod } 4 \text{ and } k-i \text{ is even.} \end{cases}$$

(d) In $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$, for any prime $p \equiv 3$ mod 8 and for $k \geq 1$,

$$Sq^{2i}\gamma_k = \tbinom{k-1}{i} \gamma_{k+i} + \sum_{j=1}^{i} \tbinom{k-j-1}{i-j} \left(w_j^2 \gamma_{k+i-j} + w_{k+i-j}^2 \gamma_j \right)$$

for $1 \leq i < k$ and $Sq^{2i-1}\gamma_k = 0$ for $1 \leq i \leq k$.

Proof. Lemma 4 of [Ar2] gives the following information on the action of the Steenrod squares on the classes $e_k \in H^*(F\Psi^p;\mathbb{Z}/2)$ for $k \geq 1$: for any odd prime p and for $1 \leq i < k$,

$$Sq^{2i} e_k = \tbinom{k-1}{i} e_{k+i} + \sum_{j=1}^{i} \tbinom{k-j-1}{i-j} \left(c_j\, e_{k+i-j} + c_{k+i-j}\, e_j \right),$$

and for $1 \leq i \leq k$,

$$Sq^{2i-1} e_k = \begin{cases} 0, & \text{if } p \equiv 1 \bmod 4 \text{ or if } p \equiv 3 \bmod 4 \text{ and } k-i \text{ is odd}, \\ \displaystyle\sum_{j=0}^{i-1} \tbinom{k-j-1}{i-j-1} c_j c_{k+i-j-1}, & \text{if } p \equiv 3 \bmod 4 \text{ and } k-i \text{ is even.} \end{cases}$$

The formula (a) is well known but can be deduced from the previous equalities because the composition $\zeta^*\,\theta^* : H^*(F\Psi^p;\mathbb{Z}/2) \to H^*(SU;\mathbb{Z}/2)$ satisfies $\zeta^*\,\theta^*(e_k) = v_{2k-1}$ for $k \geq 2$ and $\zeta^*\,\theta^*(c_j) = 0$ for $j \geq 1$. The statements (b) and (c) follow directly since $Sq^{2i} f_p^*(e_k) = f_p^*(Sq^{2i}e_k)$ and $f_p^*(c_j) = w_j^2$ for $j \geq 1$. In order to get (d), let us consider again the homomorphism $\lambda^* : H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \to$

$H^*(BQ;\mathbb{Z}/2)$ which is injective on $\mathbb{Z}/2[w_1, w_2, \dots]$ and trivial on exterior classes because $H^*(BQ;\mathbb{Z}/2)$ is polynomial. If $p \equiv 3$ mod 8, one has by the definition of γ_k

$$Sq^{2i}\gamma_k = Sq^{2i} f_p^*(e_k) + Sq^{2i} w_{2k-1} + \sum_{j=1}^{k-1} Sq^{2i}(w_j w_{2k-j-1})$$

for $1 \le i < k$. According to (b),

$$\begin{aligned} Sq^{2i}\gamma_k = \tbinom{k-1}{i} f_p^*(e_{k+i}) + \sum_{j=1}^{i} \tbinom{k-j-1}{i-j} \left(w_j^2 f_p^*(e_{k+i-j}) + w_{k+i-j}^2 f_p^*(e_j)\right) \\ + (\text{element of } \mathbb{Z}/2[w_1, w_2, \dots]) \end{aligned}$$

and consequently,

$$\begin{aligned} Sq^{2i}\gamma_k = \tbinom{k-1}{i} \gamma_{k+i} + \sum_{j=1}^{i} \tbinom{k-j-1}{i-j} \left(w_j^2 \gamma_{k+i-j} + w_{k+i-j}^2 \gamma_j\right) \\ + (\text{element of } \mathbb{Z}/2[w_1, w_2, \dots]) . \end{aligned}$$

Since the classes γ_k are exterior, they belong to the kernel of λ^* and $\lambda^*(Sq^{2i}\gamma_k) = 0$. However, the injectivity of λ^* on Stiefel-Whitney classes implies that the element of $\mathbb{Z}/2[w_1, w_2, \dots]$ in the last formula vanishes. The assertion (c) shows that $Sq^{2i-1}\gamma_k$ is an element of $\mathbb{Z}/2[w_1, w_2, \dots]$ and one deduces similarly that $Sq^{2i-1}\gamma_k = 0$. $\square$

Our argument will be based on the understanding of the homomorphism

$$\begin{aligned} \mu^* : H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) &\to H^*(BGL(\mathbb{Z})^+ \times BGL(\mathbb{Z})^+;\mathbb{Z}/2) \\ &\cong H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \otimes H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \end{aligned}$$

induced by the H-space structure μ of $BGL(\mathbb{Z})^+$.

Lemma 3. *(a) For any $j \ge 1$,*

$$\mu^*(w_j) = \sum_{s=0}^{j} w_s \otimes w_{j-s} .$$

(b) For any prime $p \equiv 5$ mod 8 and any integer $k \ge 2$,

$$\mu^*(f_p^*(e_k)) = f_p^*(e_k) \otimes 1 + 1 \otimes f_p^*(e_k) + \sum_{\ell=1}^{k-2} \left(w_\ell^2 \otimes f_p^*(e_{k-\ell}) + f_p^*(e_{k-\ell}) \otimes w_\ell^2\right) .$$

(c) For any prime $p \equiv 3$ mod 8 and any integer $k \ge 2$,

$$\mu^*(\gamma_k) = \gamma_k \otimes 1 + 1 \otimes \gamma_k + \sum_{\ell=1}^{k-2} \left(w_\ell^2 \otimes \gamma_{k-\ell} + \gamma_{k-\ell} \otimes w_\ell^2\right) .$$

Proof. Assertion (a) is known (see for instance [MT], Part I, p. 140). If ν denotes the H-space structure of $F\Psi^p$, Proposition 2 of [Q2] implies that

$$\begin{aligned} \mu^*(f_p^*(e_k)) = f_p^*(\nu^*(e_k)) &= f_p^*\Big(\sum_{\ell=0}^{k} (c_\ell \otimes e_{k-\ell} + e_{k-\ell} \otimes c_\ell)\Big) \\ &= \sum_{\ell=0}^{k} \left(w_\ell^2 \otimes f_p^*(e_{k-\ell}) + f_p^*(e_{k-\ell}) \otimes w_\ell^2\right) \end{aligned}$$

for any odd prime p. If $p \equiv 5 \bmod 8$, $f_p^*(e_1)$ vanishes since e_1 is exterior and one gets immediately (b). If $p \equiv 3 \bmod 8$, the definition of γ_k,

$$\gamma_k = f_p^*(e_k) + w_{2k-1} + \sum_{j=1}^{k-1} w_j w_{2k-j-1}\,,$$

shows that

$$\mu^*(\gamma_k) = \sum_{\ell=0}^{k} \big(w_\ell^2 \otimes f_p^*(e_{k-\ell}) + f_p^*(e_{k-\ell}) \otimes w_\ell^2\big) + (\text{element of } \mathbb{Z}/2[w_1, w_2, \dots])\,.$$

Since $p \equiv 3 \bmod 8$, it turns out that $f_p^*(e_1) = w_1$ and consequently that

$$\begin{aligned}\mu^*(\gamma_k) = \gamma_k \otimes 1 + 1 \otimes \gamma_k + \sum_{\ell=1}^{k-2} \big(w_\ell^2 \otimes \gamma_{k-\ell} + \gamma_{k-\ell} \otimes w_\ell^2\big) \\ + (\text{element of } \mathbb{Z}/2[w_1, w_2, \dots])\,.\end{aligned}$$

However, the element of $\mathbb{Z}/2[w_1, w_2, \dots]$ in that formula must be trivial since $\mu^*(\gamma_k)$ is exterior. This implies the last assertion. □

Now, let p be a prime $\equiv 5 \bmod 8$ and k an integer ≥ 2. Consider an integer $m \geq k$, C the cyclic group of order $p-1$ and

$$H^*(BC^m; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, \dots, x_m] \otimes \Lambda(y_1, y_2, \dots, y_m)$$

with $\deg(x_i) = 2$ and $\deg(y_i) = 1$ for $1 \leq i \leq m$, endowed with the differential d defined by $d(x_i) = y_i$ and $d(y_i) = 0$. Then, look at the homomorphism $\rho : H^*(F\Psi^p; \mathbb{Z}/2) \to H^*(BC^m; \mathbb{Z}/2)$, introduced in [Q2], p. 563–565, which is injective in dimensions $\leq 2m$ (and in particular in dimensions $\leq 2k$) since its kernel is the ideal generated by the elements c_j and e_j for $j > m$, and which fulfills $\rho(c_j) = s_j$ and $\rho(e_j) = d(s_j)$ for $1 \leq j \leq m$, where s_j denotes the j-th elementary symmetric function in $x_1, x_2, \dots, x_m$. For $k \geq 1$, define the exterior class

$$\xi_k = \sum_{j=1}^{m} x_j^{k-1} y_j \in H^{2k-1}(BC^m; \mathbb{Z}/2)\,.$$

Since $s_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$, one has

$$d(s_k) = \sum_{i_1 < i_2 < \cdots < i_k} \sum_{\ell} x_{i_1} \cdots \widehat{x_{i_\ell}} \cdots x_{i_k} y_{i_\ell}.$$

Then, consider the difference

$$\begin{aligned} d(s_k) - s_{k-1}\xi_1 &= \sum_{i_1 < i_2 < \cdots < i_k} \sum_{\ell} x_{i_1} \cdots \widehat{x_{i_\ell}} \cdots x_{i_k} y_{i_\ell} \\ &\qquad - \sum_{i_1 < i_2 < \cdots < i_{k-1}} x_{i_1} x_{i_2} \cdots x_{i_{k-1}} \sum_i y_i \\ &= \sum_{i_1 < i_2 < \cdots < i_{k-1}} \sum_{\ell} x_{i_1} x_{i_2} \cdots x_{i_\ell} \cdots x_{i_{k-1}} y_{i_\ell} \\ &= \sum_{i_1 < i_2 < \cdots < i_{k-1}} \sum_{\ell} x_{i_1} x_{i_2} \cdots \widehat{x_{i_\ell}} \cdots x_{i_{k-1}} x_{i_\ell} y_{i_\ell}\,. \end{aligned}$$

From this formula, one may compute the difference

$$d(s_k) - s_{k-1}\xi_1 - s_{k-2}\xi_2 = \sum_{i_1<i_2<\cdots<i_{k-2}} \sum_{\ell} x_{i_1}x_{i_2}\cdots\widehat{x_{i_\ell}}\cdots x_{i_{k-2}}x_{i_\ell}^2 y_{i_\ell}$$

and obtain by induction

$$d(s_k) = \xi_k + \sum_{j=1}^{k-1} s_j\xi_{k-j}$$

for $k \geq 2$. Since $\rho(e_k) = d(s_k)$ and $\rho(c_j) = s_j$, we get

$$\rho(e_k) = \xi_k + \sum_{j=1}^{k-1} \rho(c_j)\xi_{k-j}\,.$$

This implies inductively that the exterior class ξ_k belongs to the image of ρ and the injectivity of ρ in dimensions $\leq 2k$ produces the following lemma.

Lemma 4. *For $p \equiv 5$ mod 8 and for any $k \geq 2$, the class $e_k \in H^{2k-1}(F\Psi^p;\mathbb{Z}/2)$ satisfies*

$$e_k = \rho^{-1}(\xi_k) + \sum_{j=1}^{k-1} c_j\rho^{-1}(\xi_{k-j})\,.$$

Definition 5. Let p be a prime $\equiv 5$ mod 8. For all integers $k \geq 2$, let us define the exterior class $u_{2k-1}(p) = f_p^*(\rho^{-1}(\xi_k)) \in H^{2k-1}(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$, where f_p^* denotes the homomorphism $H^*(F\Psi^p;\mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$ induced by f_p. Observe that this definition does not depend on the choice of $m \geq k$. Notice also that $f_p^*(\rho^{-1}(\xi_1)) = f_p^*(e_1) = 0$.

Proposition 6. *For any prime $p \equiv 5$ mod 8 and for $k \geq 2$, one has:*

(a) $u_{2k-1}(p) = f_p^*(e_k) + \sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}(p)$,

(b) the homomorphism $\eta^* : H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \to H^*(SU;\mathbb{Z}/2)$ fulfills $\eta^*(u_{2k-1}(p)) = v_{2k-1}$.

Proof. Lemma 4 implies that

$$f_p^*(e_k) = u_{2k-1}(p) + \sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}(p)$$

since $f_p^*(\rho^{-1}(\xi_1)) = 0$. Consequently, (b) follows directly from the commutativity of the following diagram induced by the diagram $(*)$ of the introduction

$$\begin{tikzcd}
H^*(SU;\mathbb{Z}/2) & H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \arrow[l, "\eta^*"'] & H^*(BO;\mathbb{Z}/2) \arrow[l, "h^*"'] \\
H^*(U;\mathbb{Z}/2) \arrow[u, "\zeta^*"'] & H^*(F\Psi^p;\mathbb{Z}/2) \arrow[l, "\theta^*"'] \arrow[u, "f_p^*"'] & H^*(BU;\mathbb{Z}/2), \arrow[l, "b^*"'] \arrow[u, "c^*"']
\end{tikzcd}$$

because $\eta^*\, f_p^*(e_k) = \zeta^*\,\theta^*(e_k) = v_{2k-1}$ and $\eta^*(w_j) = 0$ for all $k \geq 2$, $j \geq 1$. □

Proposition 7. *For any prime $p \equiv 5$ mod 8 and for any integer $k \geq 2$, the element $u_{2k-1}(p)$ is a primitive cohomology class in $H^{2k-1}(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$.*

Proof. We must show that the homomorphism

$$\mu^* : H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+ \times BGL(\mathbb{Z})^+;\mathbb{Z}/2)$$

satisfies $\mu^*(u_{2k-1}(p)) = u_{2k-1}(p) \otimes 1 + 1 \otimes u_{2k-1}(p)$. We proceed by induction on k. We just established in Proposition 6 that

$$u_{2k-1}(p) = f_p^*(e_k) + \sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}(p)\,.$$

For instance, $u_3(p) = f_p^*(e_2)$ and it follows from Lemma 3 (b) that $\mu^*(u_3(p)) = u_3(p) \otimes 1 + 1 \otimes u_3(p)$. We then may deduce from Lemma 3 (a) and (b) and the induction hypothesis that

$$\begin{aligned}\mu^*(u_{2k-1}(p)) = f_p^*(e_k) \otimes 1 + 1 \otimes f_p^*(e_k) &+ \sum_{\ell=1}^{k-2}\big(w_\ell^2 \otimes f_p^*(e_{k-\ell}) + f_p^*(e_{k-\ell}) \otimes w_\ell^2\big)\\ &+ \sum_{j=1}^{k-2}\Big(\sum_{s=0}^{j} w_s^2 \otimes w_{j-s}^2\Big)\big(u_{2k-2j-1}(p) \otimes 1 + 1 \otimes u_{2k-2j-1}(p)\big)\end{aligned}$$

and therefore that

$$\begin{aligned}\mu^*(u_{2k-1}(p)) = {} & u_{2k-1}(p) \otimes 1 + 1 \otimes u_{2k-1}(p)\\ &+ \sum_{\ell=1}^{k-2}\big(w_\ell^2 \otimes u_{2k-2\ell-1}(p) + u_{2k-2\ell-1}(p) \otimes w_\ell^2\big)\\ &+ \sum_{\ell=1}^{k-3}\sum_{t=1}^{k-\ell-2}\big(w_\ell^2 \otimes w_t^2 u_{2k-2\ell-2t-1}(p) + w_t^2 u_{2k-2\ell-2t-1}(p) \otimes w_\ell^2\big)\\ &+ \sum_{j=1}^{k-2}\big(w_j^2 \otimes u_{2k-2j-1}(p) + u_{2k-2j-1}(p) \otimes w_j^2\big)\\ &+ \sum_{j=1}^{k-2}\sum_{s=1}^{j-1}\big(w_s^2 u_{2k-2j-1}(p) \otimes w_{j-s}^2 + w_s^2 \otimes w_{j-s}^2 u_{2k-2j-1}(p)\big)\,.\end{aligned}$$

The last sum can be written as follows:

$$\begin{aligned}&\sum_{j=1}^{k-2}\sum_{s=1}^{j-1}\big(w_s^2 u_{2k-2j-1}(p) \otimes w_{j-s}^2 + w_s^2 \otimes w_{j-s}^2 u_{2k-2j-1}(p)\big)\\ &= \sum_{j=1}^{k-2}\sum_{s=1}^{j-1}\big(w_s^2 \otimes w_{j-s}^2 u_{2k-2j-1}(p) + w_{j-s}^2 u_{2k-2j-1}(p) \otimes w_s^2\big)\\ &= \sum_{s=1}^{k-3}\sum_{t=1}^{k-s-1}\big(w_s^2 \otimes w_t^2 u_{2k-2s-2t-1}(p) + w_t^2 u_{2k-2s-2t-1}(p) \otimes w_s^2\big).\end{aligned}$$

Consequently, $\mu^*(u_{2k-1}(p)) = u_{2k-1}(p) \otimes 1 + 1 \otimes u_{2k-1}(p)$ and $u_{2k-1}(p)$ is primitive. □

Remark 8. Since we know that the Hopf algebra structure of $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$ by Theorem 1 (or [M], Theorem 4.3 and Remark 4.5), it is obvious that there is exactly one nontrivial primitive exterior class in each odd degree ≥ 3 of $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$. However, let us show it again by a computational argument.

Lemma 9. *Consider the homomorphism* $\eta^* : H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \to H^*(SU;\mathbb{Z}/2)$. *For* $k \geq 2$, *let* u'_{2k-1} *and* u''_{2k-1} *be primitive exterior classes of degree* $2k-1$ *in* $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$ *such that* $\eta^*(u'_{2k-1}) = \eta^*(u''_{2k-1}) = v_{2k-1}$. *Then* $u'_{2k-1} = u''_{2k-1}$.

Proof. Observe first that $u'_3 = u''_3$ since there is only one exterior class of degree 3 in $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$. Then, let us define $\widetilde{u}_{2k-1} = u'_{2k-1} - u''_{2k-1}$ for all $k \geq 2$ and prove by induction on k that $\widetilde{u}_{2k-1} = 0$. Since $\widetilde{u}_{2k-1}$ is exterior and belongs to the kernel of η^*, the induction hypothesis shows that one can write

$$\widetilde{u}_{2k-1} = \sum_{s=3}^{2k-2} u'(s)w(s)\,,$$

where $u'(s)$ is an element of degree s in $\Lambda(u'_3, u'_5, \dots, u'_{2k-3})$ and $w(s)$ is an element of degree $2k-s-1$ in $\mathbb{Z}/2[w_1, w_2, \dots]$. However, the primitivity of the classes u'_{2j-1} and Lemma 3 (a) provide an explicit computation of $\mu^*(\widetilde{u}_{2k-1})$ which contradicts the primitivity of $\widetilde{u}_{2k-1}$ unless one has $\widetilde{u}_{2k-1} = 0$. ☐

Thus, we are finally able to define the exterior generators of $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$ (see also Remark 14 below).

Definition 10. Because of Proposition 7 and Remark 8, we may conclude that the classes $u_{2k-1}(p)$ do not depend on p. Therefore, for $k \geq 2$, we can define $u_{2k-1} = u_{2k-1}(p) \in H^{2k-1}(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$ for any prime $p \equiv 5 \bmod 8$. Since the image of u_{2k-1} under η^* is v_{2k-1}, the classes u_{2k-1} are nontrivial algebraically independent exterior classes in $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$. See also Remark 14 for another definition of the classes u_{2k-1}.

The following consequence follows immediately from Proposition 6 (b) and Remark 8.

Theorem 11. *The isomorphism* $\alpha : H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \xrightarrow{\cong} H^*(BO;\mathbb{Z}/2) \otimes H^*(SU;\mathbb{Z}/2)$ *given by Theorem 1 is unique and satisfies* $\alpha(u_{2k-1}) = 1 \otimes v_{2k-1}$ *for* $k \geq 2$. *Therefore, there is an isomorphism of* $\mathcal{A}$*-module Hopf algebras*

$$H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, \dots] \otimes \Lambda(u_3, u_5, \dots)\,.$$

It follows from Theorem 11 and Lemma 2 (a) that the action of the Steenrod algebra on the classes u_{2k-1} is described by the following Lemma 12. However, we mention here another proof, based on the definition of ξ_k, which provides an explicit computational argument for the existence of the isomorphism of $\mathcal{A}$-module Hopf algebras α.

Lemma 12. *For all* $k \geq 2$, $Sq^{2i}u_{2k-1} = \binom{k-1}{i}u_{2k+2i-1}$ *for* $1 \leq i < k$ *and* $Sq^{2i-1}u_{2k-1} = 0$ *for* $1 \leq i \leq k$.

Proof. It is sufficient to prove the assertion for the classes $u_{2k-1}(p)$ where p is any prime $\equiv 5$ mod 8. This follows from the injectivity of the map ρ which was explained just after the proof of Lemma 3 (if m is large enough) and from the computations $Sq^{2i-1}\xi_k = 0$ and

$$Sq^{2i}\xi_k = \sum_{j=1}^{m} Sq^{2i}x_j^{k-1}y_j = \sum_{j=1}^{m} \tbinom{k-1}{i}x_j^{k+i-1}y_j = \tbinom{k-1}{i}\xi_{k+i}\,.$$

☐

It is even possible to describe the classes u_{2k-1} in terms of the image of f_p^* for all primes $p \equiv 3$ or $5 \bmod 8$.

Theorem 13. *For $k \geq 2$, the classes $u_{2k-1} \in H^{2k-1}(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$ satisfy*

$$u_{2k-1} = \begin{cases} f_p^*(e_k) + \displaystyle\sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}\,, & \text{if } p \equiv 5 \bmod 8, \\ f_p^*(e_k) + w_{2k-1} + \displaystyle\sum_{j=1}^{k-1} w_j w_{2k-j-1} + \sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}\,, & \text{if } p \equiv 3 \bmod 8, \end{cases}$$

where f_p^* denotes the homomorphism $H^*(F\Psi^p;\mathbb{Z}/2) \cong H^*(BGL(\mathbb{F}_p)^+;\mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$ induced by the reduction mod $p : GL(\mathbb{Z}) \to GL(\mathbb{F}_p)$.

Proof. If $p \equiv 5 \bmod 8$, the statement is given by Proposition 6 (a). Observe in particular that u_{2k-1} can be written as follows: $u_{2k-1} = F_k(f_p^*(e_2), f_p^*(e_3), \ldots, f_p^*(e_k))$, where F_k is a polynomial with coefficients in $\mathbb{Z}/2[w_1, w_2, \ldots]$. If $p \equiv 3 \bmod 8$, consider again

$$\gamma_k = f_p^*(e_k) + w_{2k-1} + \sum_{j=1}^{k-1} w_j w_{2k-j-1}$$

and define $\widehat{u}_{2k-1} = F_k(\gamma_2, \gamma_3, \ldots, \gamma_k)$. It is obvious that $\widehat{u}_{2k-1}$ is an exterior class and easy to check as in the proof of Proposition 6 that $\eta^*(\widehat{u}_{2k-1}) = v_{2k-1}$. Moreover, observe that the homomorphism

$$\mu^* : H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+ \times BGL(\mathbb{Z})^+;\mathbb{Z}/2)$$

acts on γ_k (for $p \equiv 3 \bmod 8$) and on $f_p^*(e_k)$ (for $p \equiv 5 \bmod 8$) exactly in the same way, according to Lemma 3 (b) and (c). Thus, the argument of the proof of Proposition 7 implies that $\widehat{u}_{2k-1}$ is also primitive if $p \equiv 3 \bmod 8$. It finally follows from Remark 8 that

$$u_{2k-1} = \widehat{u}_{2k-1} = F_k(\gamma_2, \gamma_3, \ldots, \gamma_k) = \gamma_k + \sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}\,.$$

□

Remark 14. The formula provided by Theorem 13 can be used as an alternative recursive definition of the classes u_{2k-1} in $H^*(BGL(\mathbb{Z})^+;\mathbb{Z}/2)$.

It is known that $BGL(\mathbb{Z})^+ \simeq BSL(\mathbb{Z})^+ \times B\mathbb{Z}/2$ and one deduces immediately the calculation of the mod 2 cohomology of the space $BSL(\mathbb{Z})^+$ (recall that $H^*(BSL(\mathbb{F}_p)^+;\mathbb{Z}/2)$ is obtained from $H^*(BGL(\mathbb{F}_p)^+;\mathbb{Z}/2)$ by dividing out e_1 and c_1):

Corollary 15. *There is an isomorphism of $\mathcal{A}$-module Hopf algebras*

$$H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, \ldots] \otimes \Lambda(u_3, u_5, \ldots)\,,$$

where w_k and u_{2k-1} are also written for the image of w_k and u_{2k-1} under the homomorphism induced by the inclusion $SL(\mathbb{Z}) \hookrightarrow GL(\mathbb{Z})$. The formulas for u_{2k-1} given by Theorem 13 do still hold but observe that the first Stiefel-Whitney class of $SL(\mathbb{Z})$ is trivial.

Remark 16. The results of this section determine also the mod 2 cohomology of the groups $GL(\mathbb{Z})$ and $SL(\mathbb{Z})$ because $H^*(BG^+;\mathbb{Z}/2) \cong H^*(G;\mathbb{Z}/2)$ for $G = GL(\mathbb{Z})$ or $SL(\mathbb{Z})$.

3. The mod 2 cohomology of the Steinberg group $St(\mathbb{Z})$

The goal of this last section is to compute $H^*(St(\mathbb{Z});\mathbb{Z}/2)$ by looking at the universal central extension

$$\mathbb{Z}/2 \cong K_2(\mathbb{Z}) \rightarrowtail St(\mathbb{Z}) \overset{\pi}{\twoheadrightarrow} SL(\mathbb{Z})$$

and at the associated Serre spectral sequence

$$E_2^{*,*} \cong H^*(SL(\mathbb{Z});\mathbb{Z}/2) \otimes H^*(\mathbb{Z}/2;\mathbb{Z}/2) \Longrightarrow H^*(St(\mathbb{Z});\mathbb{Z}/2)\,.$$

Let us use the notation $Q_0 = Sq^1$ and $Q_r = Sq^{2^r}\, Q_{r-1} + Q_{r-1}\, Sq^{2^r}$ and observe that $Q_r(w_2) = Sq^{2^r}\, Sq^{2^{r-1}} \cdots Sq^1 w_2$ because $Sq^{2^r} w_2 = 0$ for $r \geq 2$ and $Sq^1\, Sq^2 w_2 = 0$.

Theorem 17. *(a) There is an isomorphism of $\mathcal{A}$-module Hopf algebras*

$$H^*(St(\mathbb{Z});\mathbb{Z}/2) \cong \mathbb{Z}/2[\bar{w}_2, \bar{w}_3, \dots]/(\bar{w}_2, Q_r(\bar{w}_2), r \geq 0) \otimes \Lambda(\bar{u}_3, \bar{u}_5, \dots)\,,$$

where $\bar{w}_k$ and $\bar{u}_{2k-1}$ denote the image of w_k and u_{2k-1} under $\pi^* : H^*(SL(\mathbb{Z});\mathbb{Z}/2) \to H^*(St(\mathbb{Z});\mathbb{Z}/2)$.

(b) For $k \geq 2$,

$$\bar{u}_{2k-1} = \begin{cases} f_p^*(e_k) + \sum_{j=4}^{k-2} \bar{w}_j^2 \bar{u}_{2k-2j-1}\,, & \text{if } p \equiv 5 \textit{ mod } 8, \\ f_p^*(e_k) + \bar{w}_{2k-1} + \sum_{j=4}^{k-1} \bar{w}_j \bar{w}_{2k-j-1} + \sum_{j=4}^{k-2} \bar{w}_j^2 \bar{u}_{2k-2j-1}\,, & \text{if } p \equiv 3 \textit{ mod } 8, \end{cases}$$

where f_p^* is written here for the homomorphism $H^*(SL(\mathbb{F}_p);\mathbb{Z}/2) \to H^*(St(\mathbb{Z});\mathbb{Z}/2)$ induced by the reduction mod $p : St(\mathbb{Z}) \to St(\mathbb{F}_p) \cong SL(\mathbb{F}_p)$.

Proof. Because $H^*(\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[z]$ with $\deg z = 1$, one can compute the differentials in the above spectral sequence:

$$d_2(z) = w_2, \quad d_3(z^2) = Sq^1 d_2(z) = Sq^1 w_2 = w_3, \quad d_5(z^4) = Sq^2 d_3(z^2) = Sq^2 w_3$$

and inductively, $d_{2^r+1}(z^{2^r}) = d_{2^r+1}(Q_{r-1}(z)) = Q_{r-1}(d_2(z)) = Q_{r-1}(w_2) = w_{2^r+1}$ + (decomposable element of $\mathbb{Z}/2[w_2, w_3, \dots]$) by Wu's formula ([MT], Part I, p. 141). Therefore, the sequence $(w_2, Q_0(w_2), Q_1(w_2), \dots)$ is regular and we obtain $E_\infty^{s,t} = 0$ if $t > 0$ and $E_\infty^{*,0} \cong H^*(SL(\mathbb{Z});\mathbb{Z}/2)/(w_2, Q_r(w_2), r \geq 0)$. This gives the mod 2 cohomology of $St(\mathbb{Z})$ as described by statement (a) and assertion (b) follows directly from Theorem 13 and Corollary 15 since $\bar{w}_2 = \bar{w}_3 = 0$. □

Remark 18. The above argument exhibits a surjective homomorphism from $H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$ to $H^*(BSt(\mathbb{Z})^+;\mathbb{Z}/2)$. However, it is actually possible to find a nice map from $BSt(\mathbb{Z})^+$ to the space $BSpin$ inducing an injective homomorphism on mod 2 cohomology. More precisely, consider the map $\varepsilon : BSL(\mathbb{Z})^+ \to BSL(\mathbb{R})^+$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ and the map $\kappa : BSL(\mathbb{R})^+ \to BSL(\mathbb{R})^{\mathrm{top}} \simeq BSO$ induced by the obvious map $SL(\mathbb{R}) \to SL(\mathbb{R})^{\mathrm{top}}$, where the first group $SL(\mathbb{R})$ is endowed with the discrete topology and $SL(\mathbb{R})^{\mathrm{top}}$ with the usual topology. Then, look at the commutative diagram

$$\begin{tikzcd}
BSt(\mathbb{Z})^+ \arrow[r, "\pi"] \arrow[d] & BSL(\mathbb{Z})^+ \arrow[r, "\beta"] \arrow[d, "\varepsilon"] & K(K_2(\mathbb{Z}),2) \arrow[d, "\bar\varepsilon"] \\
BSt(\mathbb{R})^+ \arrow[r] \arrow[d] & BSL(\mathbb{R})^+ \arrow[r, "\beta'"] \arrow[d, "\kappa"] & K(K_2(\mathbb{R}),2) \arrow[d, "\bar\kappa"] \\
BSpin \arrow[r, "\tau"] & BSO \arrow[r, "\beta''"] & K(\pi_2 BSO,2)\,,
\end{tikzcd}$$

where the rows are fibrations in which the maps β, β', β'' are the second Postnikov sections of the corresponding spaces ($BSpin$ is the fiber of β''), the maps $\bar\varepsilon$ and $\bar\kappa$ are the second Postnikov sections of ε and κ, and the vertical maps on the left are the restrictions of ε and κ to the fibers. The composition $\bar\kappa\,\bar\varepsilon$ is a homotopy equivalence because $\bar\kappa_*\,\bar\varepsilon_* : K_2(\mathbb{Z}) \to \pi_2 BSO$ is an isomorphism (see Corollary 4.6 of [Br] or p. 25-26 of [Be]). Let us denote the composition $\kappa\,\varepsilon$ by χ and its restriction to $BSt(\mathbb{Z})^+$ by $\widetilde{\chi} : BSt(\mathbb{Z})^+ \to BSpin$ (note that the 2-completion of χ is the universal cover of the map h defined in the introduction and that the fiber of the 2-completion of $\widetilde{\chi}$ is $\hat{SU_2}$ because of the diagram $(*)$). We get the commutative diagram

$$\begin{tikzcd}
B\mathbb{Z}/2 \arrow[r] \arrow[d, "\simeq"] & BSt(\mathbb{Z})^+ \arrow[r, "\pi"] \arrow[d, "\widetilde{\chi}"] & BSL(\mathbb{Z})^+ \arrow[d, "\chi"] \\
B\mathbb{Z}/2 \arrow[r] & BSpin \arrow[r, "\tau"] & BSO\,.
\end{tikzcd}$$

The ring structure of the mod 2 cohomology of $BSpin$ is known by Proposition 6.5 of [Q1]:

$$H^*(BSpin;\mathbb{Z}/2) \cong \mathbb{Z}/2[\widetilde{w}_2,\widetilde{w}_3,\dots]/(\widetilde{w}_2, Q_r(\widetilde{w}_2), r \geq 0)\,,$$

where the $\widetilde{w}_k$'s are written here for the image of the universal Stiefel-Whitney classes under the homomorphism $\tau^* : H^*(BSO;\mathbb{Z}/2) \to H^*(BSpin;\mathbb{Z}/2)$. Since $\chi^* : H^*(BSO;\mathbb{Z}/2) \to H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$ is injective, the map $\widetilde{\chi}$ induces an injective $\mathcal{A}$-module Hopf algebra homomorphism

$$\widetilde{\chi}^* : H^*(BSpin;\mathbb{Z}/2) \to H^*(BSt(\mathbb{Z})^+;\mathbb{Z}/2)\,.$$

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