PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 9, Pages 2665–2670
S 0002-9939(99)04738-3
Article electronically published on April 23, 1999

ON A QUESTION IN THE THEORY OF ALMOST PERIODIC DIFFERENTIAL EQUATIONS

ZUO SHENG HU AND ANGELO B. MINGARELLI

(Communicated by Hal L. Smith)

Abstract. We show that there exists a real homogeneous differential equation of order n with classical almost periodic coefficients such that all solutions are uniformly bounded on the real line yet no non-trivial solution is almost periodic. This now appears to make the search for a Floquet theory of such equations a futile enterprise.

1. Introduction

One of the greatest impediments to a thorough understanding of the nature of solutions of linear differential equations with (Bohr) almost periodic coefficients [1] is the lack of a Floquet theory.

It is well-known [3, p. 101] that if the $p_i(t), i = 1, 2, \dots , n$, are piecewise continuous periodic functions on $\mathbb{R}$, then every bounded solution of

$$x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0 \tag{1.1}$$

is almost periodic on account of Floquet theory. The analogous result for almost periodic coefficients $p_i(t)$ is now known to be *false*. For example, Conley and Miller [2] gave an example of an equation (1.1) with $n = 1$ where a bounded solution is not almost periodic; however, this result did not extend to $n > 1$. In [4], Mingarelli, Pu and Zheng constructed an example, for each $n > 1$, of an equation (1.1) with almost periodic coefficients in which there existed a bounded solution which is not almost periodic: For each such case $n > 1$ there is always another unbounded solution. As a result of this peculiarity the second author raised the following question:

If $p_i(t)$ are all almost periodic and all the solutions of (1.1) are bounded, does it necessarily follow that all solutions are almost periodic?

This is a question whose answer has eluded us for some time. In this paper we answer this simple question in the negative thus annihilating any hope of a Floquet-type theory for linear almost periodic differential equations. We will show that, for each $n \geq 2$, there exists an equation of form (1.1) for which *every* solution is bounded on $\mathbb{R}$ but yet no solution (except the trivial solution) is almost periodic. Indeed, the situation is worse than we thought originally, as we had hoped for a dichotomy, at the very least.

Received by the editors October 7, 1997.
1991 *Mathematics Subject Classification.* Primary 34C27.
Key words and phrases. Second-order, almost periodic, boundedness.
The second author was partially supported by an NSERC research grant.

2. General result

First, we give two lemmas in order to establish our general result.

Lemma 2.1. *Suppose that $a_i(t)$ $(i = 1, 2, \dots, n)$ and $g(t)$ have continuous $(n-1)$th-order derivatives. Then the following equation holds:*

(2.1)
$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_1' + a_1 g & a_2' + a_2 g & \cdots & a_n' + a_n g \\ (a_1' + a_1 g)' & (a_2' + a_2 g)' & \cdots & (a_n' + a_n g)' \\ \cdots & \cdots & \cdots & \cdots \\ (a_1' + a_1 g)^{(n-2)} & (a_2' + a_2 g)^{(n-2)} & \cdots & (a_n' + a_n g)^{(n-2)} \end{vmatrix} = W(a_1, \dots, a_n)(t)$$

where $W(a_1, \dots, a_n)(t)$ is the Wronskian determinant of $a_1, \dots, a_n$.

Proof. We use an induction argument. For $n = 2$, we have

$$\begin{vmatrix} a_1 & a_2 \\ a_1' + a_1 g & a_2' + a_2 g \end{vmatrix} = a_1 a_2' + a_1 a_2 g - a_2 a_1' - a_1 a_2 g = \begin{vmatrix} a_1 & a_2 \\ a_1' & a_2' \end{vmatrix}.$$

We assume that (2.1) is true for some n. Now, we verify that (2.1) is true for $n+1$. In fact,

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_{n+1} \\ a_1' + a_1 g & a_2' + a_2 g & \cdots & a_{n+1}' + a_{n+1} g \\ \cdots & \cdots & \cdots & \cdots \\ (a_1' + a_1 g)^{(n-2)} & (a_2' + a_2 g)^{(n-2)} & \cdots & (a_{n+1}' + a_{n+1} g)^{(n-2)} \\ (a_1' + a_1 g)^{(n-1)} & (a_2' + a_2 g)^{(n-1)} & \cdots & (a_{n+1}' + a_{n+1} g)^{(n-1)} \end{vmatrix}$$

$$= \sum_{i=1}^{n+1} (-1)^{n+1+i} (a_i' + a_i g)^{(n-1)}$$

$$\times \begin{vmatrix} a_1 & \cdots & a_{i-1} & a_{i+1} & \cdots & a_{n+1} \\ a_1' + a_1 g & \cdots & a_{i-1}' + a_{i-1} g & a_{i+1}' + a_{i+1} g & \cdots & a_{n+1}' + a_{n+1} g \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (a_1' + a_1 g)^{(n-2)} & \cdots & (a_{i-1}' + a_{i-1} g)^{(n-2)} & (a_{i+1}' + a_{i+1} g)^{n-2} & \cdots & (a_{n+1}' + a_{n+1} g)^{(n-2)} \end{vmatrix}$$

$$= \sum_{i=1}^{n+1} (-1)^{n+1+i} (a_i' + a_i g)^{(n-1)} \cdot W(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{n+1})(t)$$

$$= \begin{vmatrix} a_1 & a_2 & \cdots & a_{n+1} \\ a_1' & a_2' & \cdots & a_{n+1}' \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{(n-1)} & a_2^{(n-1)} & \cdots & a_{n+1}^{(n-1)} \\ (a_1' + a_1 g)^{(n-1)} & (a_2' + a_2 g)^{(n-1)} & \cdots & (a_{n+1}' + a_{n+1} g)^{(n-1)} \end{vmatrix}$$

$$= W(a_1, \ldots, a_{(n+1)})(t) + \begin{vmatrix} a_1 & \cdots & a_{n+1} \\ a_1' & \cdots & a_{n+1}' \\ \cdots & \cdots & \cdots \\ a_1^{(n-1)} & \cdots & a_{n+1}^{(n-1)} \\ (a_1 g)^{(n-1)} & \cdots & (a_{n+1} g)^{(n-1)} \end{vmatrix}$$

$$= W(a_1, \cdots, a_{n+1})(t).$$

So, by induction, (2.1) holds for any n. □

The following lemma also follows by induction on account of Lemma 2.1.

Lemma 2.2. *Suppose that $a_i(t)$ $(i = 1, 2, \ldots, n)$ and $g(t)$ are the same as in Lemma 2.1. If*

$$\phi_i(t) = a_i(t) \exp(\int_0^t g(s)ds), i = 1, 2, \ldots, n,$$

then

$$W(\phi_1, \ldots, \phi_n)(t) = W(a_1, \ldots, a_n)(t) \exp(n \int_0^t g(s)ds).$$

We now state our general result.

Theorem. *If there exist functions $a_i(t)$ $(i = 1, 2, \ldots, n)$ and $g(t)$ such that*

(i) $a_i(t)$ $a_i^{(k)}(t)$ $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, n-1)$ *are all almost periodic on* $\mathbb{R}$*;*

(ii) $\inf_{t \in \mathbb{R}} |W(a_1, \ldots, a_n)(t)| \neq 0$ *for all* $t \in \mathbb{R}$*;*

(iii) $g(t), g^{(k)}(t)$ $(k = 1, 2, \ldots, n-1)$ *are all almost periodic on* $\mathbb{R}$*;*

(iv) $\int_0^t g(s)ds \leq 0$ *for all* $t \in \mathbb{R}$*;*

(v) $\inf_{t \in \mathbb{R}} \int_0^t g(s)ds = -\infty$,

then there exist almost periodic functions $p_i(t)$ $(i = 1, 2, \ldots, n)$ such that the all solutions of the equation

$$x^{(n)} + p_n(t)x^{(n-1)} + \cdots + p_1(t)x = 0 \tag{2.2}$$

are bounded on $\mathbb{R}$*, but any non-trivial solution of (2.2) is not almost periodic.*

Proof. Let

$$\phi_i(t) = a_i(t) \exp\big(\int_0^t g(s)ds\big), i = 1, 2, \ldots, n.$$

By Lemma 2.2 and condition (ii), we have

$$W(\phi_1, \ldots, \phi_n)(t) = W(a_i, \ldots, a_n)(t) \exp\big(n \int_0^t g(s)ds\big) \neq 0$$

for all $t \in \mathbb{R}$. Therefore, the $p_i(t)$ $(i = 1, 2, \ldots, n)$ can be determined by the following linear system:

$$\begin{Bmatrix} \phi_1 p_1(t) & + & \phi_1' p_2(t) & + \cdots + & \phi_1^{(n-1)} p_n(t) & = -\phi_1^{(n)} \\ \cdots & & \cdots & \cdots & \cdots & \cdots \\ \phi_n p_1(t) & + & \phi_n' p_2(t) & + \cdots + & \phi_n^{(n-1)} p_n(t) & = -\phi_n^{(n)} \end{Bmatrix} \tag{2.3}$$

and

$$p_i(t) = W_i(t)/W(a_1, \dots, a_n)(t) \exp\big(n \int_0^t g(s)ds\big),$$

where

$$W_i(t) = \begin{vmatrix} \phi_1 & \cdots & \phi_1^{(i-1)} & -\phi_1^{(n)} & \phi_1^{(i+1)} & \cdots & \phi_1^{(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_n & \cdots & \phi_n^{(i-1)} & -\phi_n^{(n)} & \phi_n^{(i+1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}, i = 1, 2, \dots, n.$$

We also have

$$\phi_i^{(k)}(t) = b_{i,k}(t) \exp\big(\int_0^t g(s)ds\big), i = 1, 2, \dots, n,\ \ k = 1, 2, \dots, n-1,$$

where $b_{i,k}(t)$ are some algebraic combination of $a_i^{(k)}(t)$ and $g^{(k)}(t)$. In fact $b_{i,k}(t) = a_i^{(k)}(t) + c_{1,k}(t)a_i^{(k-1)}(t) + \cdots + c_{k,k}(t)a_i(t)$ for $i = 1, 2, \dots, n$, where the $c_{j,k}(t)$ are given by $c_{j,k}(t) = d_{1,j,k}g^{(j-1)} + d_{2,j,k}g^{(j-2)}g + \cdots + d_{m_j,j,k}g^j$ and $d_{l,j,k}$ are constants $(j = 1, 2, \dots, k, k = 1, 2, \dots, n-1)$. So,

$$W_i(t) = B_i(t) \exp\big(n \int_0^t g(s)ds\big),\ \ i = 1, 2, \dots, n,$$

where $B_i(t)$ $(i = 1, 2, \dots, n)$ are still some algebraic combinations of $a_i^{(k)}(t)$ and $g^{(k)}(t)$ $(i = 1, 2, \dots, n;\ k = 1, 2, \dots, n-1)$. It follows that

$$p_i(t) = B_i(t)/W(a_1, \dots, a_n)(t),\ \ i = 1, 2, \dots, n,$$

are almost periodic [1]. For these $p_i(t)$, (2.2) is an almost periodic differential equation and $\{\phi_2, \dots, \phi_n\}$ is a fundamental system of solutions. From the assumptions on $a_i(t)$ and $g(t)$, it is obvious that $\phi_i(t)$ $(i = 1, 2, \dots, n)$ are all bounded on $\mathbb{R}$, and thus all solutions of (2.2) are bounded.

Next, we show that any non-trivial solution of (2.2) is not almost periodic. Let $x(t)$ be a non-trivial solution; then there are constants $C_1, \dots, C_n$ such that

$$\begin{aligned} x(t) &= C_1\phi_1(t) + \cdots + C_n\phi_n(t) \\ &= \big(C_1a_1(t) + \cdots + C_na_n(t)\big) \exp\big(\int_0^t g(s)ds\big). \end{aligned}$$

Let $b(t) = C_1a_1(t) + \cdots + C_na_n(t)$; then $b(t) \not\equiv 0$ $(t \in \mathbb{R})$ and it inherits the same properties as $a_i(t)$. We write $x^{(k)}(t)$ in the form

$$x^{(k)}(t) = b_k(t) \exp\big(\int_0^t g(s)ds\big), k = 1, 2, \dots, n-1,$$

where $b_k(t)$ $(k = 1, 2, \dots, n-1)$ are some algebraic combinations of $b(t), b^{(k)}(t), g(t)$ and $g^{(k)}(t)$. We see that the $b_k(t)$ are almost periodic and thus bounded $(k = 1, 2, \dots, n-1)$.

Let $X(t) = Col\big(x(t), x'(t), \dots, x^{(n-1)}(t)\big)$; then $X(t)$ is a solution of the system

$$X'(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -p_1(t) & -p_2(t) & -p_3(t) & \dots & -p_n(t) \end{pmatrix} X(t). \tag{2.4}$$

Now, (2.4) is an almost periodic system and

$$\inf_{t\in\mathbb{R}} |X(t)|^2 = \inf_{t\in\mathbb{R}} \sum_{k=1}^{n} |b_k(t)|^2 \exp(2\int_0^t g(s)ds) = 0,$$

$$X(t) \not\equiv 0, \quad \text{for all} \quad t \in \mathbb{R},$$

since $b_k(t)$ $(k = 1, 2, \ldots, n-1)$ are bounded and $\inf_{t\in\mathbb{R}} \int_0^t g(s)ds = -\infty$. So, $x(t)$ and $x^{(k)}(t)$ $(k = 1, 2, \ldots, n-1)$ are all not almost periodic because if $x(t)$ is almost periodic, so is $x^{(k)}(t)$ for each k (because of the form of $x(t)$) and thus $X(t)$ is almost periodic, contradicting Theorem 5.7 in [3] (see [3, p. 85]), since $X(t)$ is a non-trivial solution. This ends the proof of the theorem.

3. Construction of $a_i(t)$, $(i = 1, 2, \ldots, n)$ and $g(t)$

In this section, we will give the construction of the functions $a_i(t)$ $(i = 1, 2, \ldots, n)$ and $g(t)$ for any n.

For any n, we define the function $g(t)$ as

$$g(t) = \sum_{k=1}^{\infty} f_k(t) \tag{3.1}$$

where

$$f_k(t) = -\frac{1}{k^n} \sin \frac{t}{k^{n+1}}, t \in \mathbb{R}.$$

Then $g(t)$ satisfies the conditions (iii)–(v) (see [4]) of the theorem.

In order to construct the functions $a_i(t)$, $(i = 1, 2, \ldots, n)$ we give the following lemma, easily proved by induction.

Lemma 3.1. *Let λ_i $(i = 1, 2, \ldots, m)$ be any real numbers. Then the Wronskian determinant of the functions*

$$\{\cos\lambda_1 t, \sin\lambda_1 t, \ldots, \cos\lambda_m t, \sin\lambda_m t\}$$

satisfies

$$W(\cos\lambda_1 t, \sin\lambda_1 t, \ldots, \cos\lambda_m t, \sin\lambda_m t) = \prod_{i=1}^{m} \lambda_i \prod_{1\le i<j\le m} (\lambda_i^2 - \lambda_j^2)^2. \tag{3.2}$$

Corollary. *Let λ_i $(i = 1, 2, \ldots, m)$ be any real numbers. Then*

$$W(\cos\lambda_1 t, \sin\lambda_1 t, \ldots, \cos\lambda_m t, \sin\lambda_m t) = 0$$

if and only if there are i and j such that $i \neq j$ but $\lambda_i = \pm\lambda_j$ or there is an i such that $\lambda_i = 0$.

Now, we can construct the functions $a_i(t)$ $(i = 1, 2, \ldots, n)$ satisfying the conditions (i) and (ii) of the theorem for any n.

If n is even, we let

$$a_{2k-1}(t) = \cos\lambda_k t, a_{2k}(t) = \sin\lambda_k t, k = 1, 2, \ldots, n/2,$$

where $\lambda_k \neq 0$ $(k = 1, 2, \ldots, n/2)$ and $\lambda_i \neq \pm\lambda_j$ as $i \neq j$. By the corollary, we have that the $a_k(t)$ $(k = 1, 2, \ldots, n)$ satisfy the conditions (i) and (ii) of the theorem.

If n is odd, we still take the λ_i such that $\lambda_i \neq 0, i = 1, 2, \dots, \frac{n-1}{2}$ and $\lambda_i \neq \lambda_j$ as $i \neq j$. And we let

$$a_1(t) = 1, a_{2k}(t) = \cos \lambda_k t, a_{2k+1}(t) = \sin \lambda_k t, k = 1, 2, \dots, \frac{n-1}{2}.$$

Since $W(a_1, a_2, \dots, a_n)(t) = W(a_2, \dots, a_n)(t)$ and $\inf_{t\in\mathbb{R}} |W(a_2, \dots, a_n)(t)| \neq 0$, the $a_k(t)$ still satisfy the conditions (i) and (ii) of the theorem. In particular, when $n = 2$, we take $\lambda_1 = 1$, i.e., $a_1(t) = \cos(t), a_2(t) = \sin t$, and we get

$$\phi_1(t) = \cos t \ \exp\Big(\int_0^t g(s)ds\Big), \phi_2(t) = \sin t \ \exp\Big(\int_0^t g(s)ds\Big)$$

and $W(\phi_1, \phi_2)(t) \equiv 1$ for all $t \in \mathbb{R}$. $p_1(t) = 1 + g^2 - g', p_2(t) = -2g$. We consider the second-order linear equation

$$x'' - 2g(t)x' + (1 + g^2(t) - g'(t))x = 0. \tag{3.3}$$

Obviously, (3.3) is an almost periodic equation and has the two linear independent solutions $\phi_1(t)$ and $\phi_2(t)$. So, all solutions are bounded, but no non-trivial solution is almost periodic.

References

[1] H. Bohr, Almost periodic functions, Chelsea, New York, 1947. MR **8:**512a

[2] C.C. Conley and R.K. Miller, Asymptotic stability without uniform stability: almost periodic coefficients, J. Differential Equations, 1 (1965), 333-336. MR **34:**1619

[3] M.A. Fink, Almost periodic differential equations, Springer-Verlag, New York-Berlin, 1974.

[4] A.B. Mingarelli, F.Q. Pu and L. Zheng, A counter-example in the theory of almost periodic differential equations, Rocky Mountain J. of Math. 25 (1995), 437-440. MR **96e:**34070

School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6

E-mail address: `angelo_mingarelli@carleton.ca`