PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 9, Pages 2759–2766
S 0002-9939(99)04784-X
Article electronically published on April 23, 1999

ONE-TO-ONE BOREL SELECTION THEOREMS

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(Communicated by Frederick W. Gehring)

Abstract. For $X = [0,1]$ we obtain new theorems stating that a Borel set in X^2 with large sets of large vertical and large horizontal sections admits a one-to-one Borel selection with large domain and large range. Largeness is meant mainly in measure or category sense. Our proofs combine a result of Graf and Mauldin with a modified result of Sarbadhikari.

1. Introduction

Measurable selections have been extensively studied in recent years. For an expository survey, see [W]. Interesting problems concerning one-to-one Borel selections were investigated in [M1], [M2], [GM], [MS], [S] and [DSR]. Some results contained in these papers can be formulated by the use of σ-ideals.

Let $X = [0,1]$. Assume that $A \subseteq X^2$. Recall that f forms a (one-to-one) *selection* of a set A if f is a (one-to-one) function such that $f \subseteq A$. We denote by $\operatorname{dom} A$ and $\operatorname{ran} A$ the projections of A on the first and the second axis, respectively. For $x, y \in X$ we write

$$A_x = \{t \in X : \ \langle x,t\rangle \in A\}, \quad A^y = \{t \in X : \ \langle t,y\rangle \in A\}.$$

These are vertical and horizontal *sections* of A. If f is a selection of $A \subseteq X^2$, one usually requires that $\operatorname{dom} f = \operatorname{dom} A$. For a one-to-one selection f of $A \subseteq X^2$, we will require that the differences $\operatorname{dom} A \setminus \operatorname{dom} f$ and $\operatorname{ran} A \setminus \operatorname{ran} f$ are small in the respective sense. This standpoint was used in [M1], [GM], [S]. The reason is that in several cases a Borel set has all vertical and horizontal sections large and it does not admit a one-to-one Borel selection defined everywhere. (See [M1, Example, p.828], [GM, Remark, p.422] and [DSR, Th.3].) The largeness of sections is described in the language of σ-ideals, mainly those of meager sets or of measure zero sets. Note, that the case when σ-ideals establishing the largeness of sections can vary also seems interesting. This was used in [GM] and [M1] for probability transition kernels. We will consider *Borel σ-ideals* of subsets of X, i.e., σ-ideals $\mathcal{I}$ such that each set $A \in \mathcal{I}$ is contained in a Borel set $B \in \mathcal{I}$.

Let $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ be Borel σ-ideals of subsets of X. A Borel set $A \subseteq X^2$ will be called $\langle \mathcal{I}_1, \mathcal{I}_2\rangle$-*wide* if there are Borel sets $E \in \mathcal{I}_1$, $F \in \mathcal{I}_2$ and a Borel isomorphism f from $X \setminus E$ onto F such that $f \subseteq A$. We say that A is $\langle \mathcal{I}_1, \mathcal{I}_2\rangle$-*tall* if there are Borel sets $E \in \mathcal{I}_1$, $F \in \mathcal{I}_2$ and a Borel isomorphism f from E onto $X \setminus F$ such that $f \subseteq A$. We say that A is $\langle \mathcal{I}_1, \mathcal{I}_2\rangle$-*large* if there are Borel sets $E \in \mathcal{I}_1, F \in \mathcal{I}_2$

Received by the editors November 3, 1997.
1991 *Mathematics Subject Classification.* Primary 04A15, 28A05, 54H05.
Key words and phrases. One-to-one selection, Borel set, meager set, Lebesgue null set.

and a Borel isomorphism f from $X \setminus E$ onto $X \setminus F$ such that $f \subseteq A$. It can be easily checked that any of the following notions: $\langle \mathcal{I}_1, \mathcal{I}_2 \rangle$-wideness, -tallness and -largeness, is monotonic with respect to each of variables. For instance, if $\mathcal{I}_1 \subseteq \mathcal{I}_3$ and A is $\langle \mathcal{I}_1, \mathcal{I}_2 \rangle$-wide, then it is $\langle \mathcal{I}_3, \mathcal{I}_2 \rangle$-wide. Obviously, those notions can be generalized to the case where $A \subseteq Z \times Y$ and Z, Y are some abstract (e.g. Polish, analytic) topological spaces.

The following lemma shows how one can infer the largeness of a set from its wideness and a kind of tallness. That idea was used several times in the literature. (See [M1], [GM], [S].)

Lemma 1. *Assume that a Borel set $A \subseteq X^2$ satisfies the conditions:*

(1) *A is $\langle \mathcal{I}_1, \mathcal{I}_2 \rangle$-wide,*
(2) *for each Borel set $F \in \mathcal{I}_2$ there are Borel sets $C \subseteq X$, $D \in \mathcal{I}_2$, $D \supseteq F$, and a Borel isomorphism g from C onto $X \setminus D$ such that $g \subseteq A$.*

Then A is $\langle \mathcal{I}_1, \mathcal{I}_2 \rangle$-large.

Proof. By (1) we find Borel sets $E \in \mathcal{I}_1$, $F \in \mathcal{I}_2$ and a Borel isomorphism f from $X \setminus E$ onto F such that $f \subseteq A$. Then pick sets C, D and a function g satisfying condition (2). Put

$$h(x) = \begin{cases} f(x) & \text{for } x \in X \setminus E \setminus C, \\ g(x) & \text{for } x \in C. \end{cases}$$

Then h is a one-to-one Borel function and $h \subseteq A$. The sets $\operatorname{dom} h = (X \setminus E) \cup C$ and $\operatorname{ran} h = (F \setminus f[C]) \cup (X \setminus D)$ are Borel and their complements are in $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. □

Corollary 1. *If a Borel set $A \subseteq X^2$ is $\langle \mathcal{I}_1, \mathcal{I}_2 \rangle$-wide and $A \setminus (X \times F)$ is $\langle \mathcal{I}_3, \mathcal{I}_2 \rangle$-tall for each Borel set $F \in \mathcal{I}_2$, then A is $\langle \mathcal{I}_1, \mathcal{I}_2 \rangle$-large.*

Proof. It suffices to check that condition (2) of Lemma 1 holds true. Let $F \in \mathcal{I}_2$ be a Borel set. Since $A \setminus (X \times F)$ is $\langle \mathcal{I}_3, \mathcal{I}_2 \rangle$-tall, there are Borel sets $C \in \mathcal{I}_3$, $D \in \mathcal{I}_2$ and a Borel isomorphism g from C onto $X \setminus D$ such that $g \subseteq A \setminus (X \times F)$. Obviously, $D \supseteq F$. □

In the sequel, if we consider a theorem about a Borel set $A \subseteq X^2$, we can formulate its (equivalent) *dual version* where the order of the coordinates in X^2 is reversed. Then the notions of a horizontal section and a vertical section, a domain and a range, $\langle \mathcal{I}_1, \mathcal{I}_2 \rangle$-wideness and $\langle \mathcal{I}_2, \mathcal{I}_1 \rangle$-tallness, $\langle \mathcal{I}_1, \mathcal{I}_2 \rangle$-largeness and $\langle \mathcal{I}_2, \mathcal{I}_1 \rangle$-largeness are interchanged in both theorems.

Let $\mathcal{M}, \mathcal{N}, \mathcal{C}$ denote, respectively, the σ-ideals of all meager, Lebesgue null and countable subsets of X. The following facts are known:

Fact 1 ([S]). Let $A \subseteq X^2$ be a Borel set such that $\{x \in X : A_x \in \mathcal{M}\} \in \mathcal{M}$ and $\{y \in X : A^y \in \mathcal{M}\} \in \mathcal{M}$. Then A is $\langle \mathcal{M}, \mathcal{M} \rangle$-large.

Fact 2 (cf. [GM, Th.4.4]). Let $A \subseteq X^2$ be a Borel set such that $\{x \in X : A_x \in \mathcal{C}\} \in \mathcal{N}$ and $\{y \in X : A^y \in \mathcal{C}\} \in \mathcal{N}$. Then A is $\langle \mathcal{N}, \mathcal{N} \rangle$-large.

The exact measure analogue of Fact 1 was proved earlier in [M1], and Fact 2 is its stronger version. Now, it seems natural to ask whether the category analogue of Fact 2 holds true.

Problem 1. Let $A \subseteq X^2$ be a Borel set such that $\{x \in X : A_x \in \mathcal{C}\} \in \mathcal{M}$ and $\{y \in X : A^y \in \mathcal{C}\} \in \mathcal{M}$. Can one conclude that A is $\langle \mathcal{M}, \mathcal{M} \rangle$-large? (We do not know.)

Facts 1 and 2 were proved by the use of the scheme presented in Corollary 1. In particular, the proof of Fact 1 was based on the following result and its dual version (which we also formulate to show an example of a dual theorem).

Fact 3 ([S, Th.2])**.** Let $A \subseteq X^2$ be a Borel set such that $\{x \in X : A_x \in \mathcal{M}\} \in \mathcal{M}$. Then A is $\langle \mathcal{M}, \mathcal{M} \rangle$-wide.

Dual version: Let $A \subseteq X^2$ be a Borel set such that $\{y \in X : A^y \in \mathcal{M}\} \in \mathcal{M}$. Then A is $\langle \mathcal{M}, \mathcal{M} \rangle$-tall.

Let us give (in a general fashion taken from [GM]) a tallness criterion needed to derive Fact 2 from Corollary 1. That statement will be useful in Section 3. Namely, we reformulate Theorem 4.1 from [GM] with additional information that the respective domain is K_σ (a countable union of compact sets). That, however, follows at once from [GM, Th.3.1] and from the proof of Theorem 4.1 presented in [GM]. In our version we use finite measures instead of probability measures.

Fact 4 (cf. [GM, Thms 4.1 and 3.1])**.** Let Z and Y be analytic spaces. Let μ_Z and μ_Y be finite Borel measures on Z and Y, respectively, and let $\mathcal{N}_Z$, $\mathcal{N}_Y$ be the families of μ_Z-, μ_Y-null sets. Assume that $A \subseteq Z \times Y$ is a Borel set such that $\{y \in Y : A^y \in \mathcal{C}\} \in \mathcal{N}_Z$. Then there are a Borel set $E \in \mathcal{N}_Z$ contained in a K_σ set, a Borel set $F \in \mathcal{N}_Y$ and a Borel isomorphism f from E onto $Y \setminus F$. Thus, if Z is a Polish space and if μ_Z does not vanish on nonvoid open sets in Z, then $E \in \mathcal{M}_Z \cap \mathcal{N}_Z$ (where $\mathcal{M}_Z$ stands for the family of all meager sets in Z) and so, A is $\langle \mathcal{M}_Z \cap \mathcal{N}_Z, \mathcal{N}_Y \rangle$-tall.

Finally, note that in aiming to solve Problem 1 by the use of Corollary 1 it is enough to answer the following question affirmatively.

Problem 2. Let $A \subseteq X^2$ be a Borel set such that $\{x \in X : A_x \in \mathcal{C}\} \in \mathcal{M}$. Can one conclude that A is $\langle \mathcal{M}, \mathcal{M} \rangle$-wide?

2. Strengthened theorem of Sarbadhikari

In this section we will improve Fact 3. Our auxiliary Proposition 1 strengthens slightly Theorem 1 from [S]. The proof is similar but we give it with details for the reader's convenience.

Proposition 1. *Let $B \subseteq X^2$ be a Borel set such that $\{x \in X : B_x$ is comeager$\}$ is comeager in X. Then there is a comeager Borel set $E \subseteq X$, a nowhere dense, Lebesgue null Borel set $F \subseteq X$, and a Borel isomorphism f from E onto F such that $f \subseteq B$.*

Proof. Fix a countable base $\{U_n : n \geq 1\}$ of nonempty intervals open in X. Note that B is comeager in X^2 by the assumption and by the converse of the Kuratowski-Ulam theorem [O, Th.15.4]. Hence, there exist dense open sets $V_1 \subseteq V_2 \subseteq \dots$ in X^2 with $\bigcap_n V_n \subseteq B$.

By induction on n, we shall define a sequence $\{B_{ni}\}_{i=1}^{\infty}$ of subsets of X and a sequence $\{(a_{ni}, b_{ni})\}_{i=1}^{\infty}$ of nonvoid open intervals such that for all n the following conditions hold:

(1) for $H_n = \bigcup_{i=1}^{\infty} B_{ni} \times (a_{ni}, b_{ni})$ we have $H_{n+1} \subseteq H_n \subseteq V_n$ and $\operatorname{cl}((H_{n+1})_x) \subseteq (H_n)_x$ for all $x \in X$;

(2) the sets B_{ni} $(i = 1, 2, \dots)$ are pairwise disjoint nonmeager G_δ with $B_{ni} \subseteq (k_{ni}/2^n, (k_{ni}+1)/2^n)$ for a positive integer k_{ni}, and $\bigcup_{i=1}^{\infty} B_{ni}$ is comeager in X;

(3) the intervals $[a_{ni}, b_{ni}]$ $(i = 1, 2, \dots)$ are pairwise disjoint, $\sum_{i=1}^{\infty}(b_{ni} - a_{ni}) < 1/2^n$, and there exists a nonempty open set $W_n \subseteq U_n \setminus \bigcup_{i=1}^{\infty} [a_{ni}, b_{ni}]$.

Before the construction we shall show that $H = \bigcap_{n=1}^{\infty} H_n$ is the graph of the required function f.

Since $\operatorname{dom} H \subseteq \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni}$, we have $H_x = \emptyset$ for $x \notin \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni}$. Let $x \in \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni}$. We shall prove that there is a unique $y \in X$ such that $y \in H_x$ (then $\bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni} = \operatorname{dom} f$ for our function f). Observe that by (2) there exists a unique sequence $\{i_n(x)\}_{n=1}^{\infty} = \{i_n\}_{n=1}^{\infty}$ such that $x \in \bigcap_{n=1}^{\infty} B_{n,i_n}$. Hence, for every n, by (1), we have

$$\operatorname{cl}\left((H_{n+1})_x\right) \subseteq (H_n)_x \subseteq \operatorname{cl}\left((H_n)_x\right) = [a_{n,i_n}, b_{n,i_n}].$$

Since (3) implies that $b_{n,i_n} - a_{n,i_n} < 1/2^n$, there is a unique $y \in X$ such that $\{y\} = \bigcap_{n=1}^{\infty} [a_{n,i_n}, b_{n,i_n}]$. Then it is not hard to check that $\langle x, y \rangle \in H$. In fact, $H_x = \{y\}$, and we put $y = f(x)$. Next, note that $\operatorname{dom} f = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni}$ is a comeager Borel set, by (2).

To show that f is one-to-one, consider $x_1, x_2 \in \operatorname{dom} f$, $x_1 \neq x_2$. Let $\{i_n(x_j)\}_{n=1}^{\infty} = \{i_n^{(j)}\}_{n=1}^{\infty}$ be the sequence chosen for x_j $(j = 1, 2)$ as above. From $x_1 \neq x_2$ it follows that $\{i_n^{(1)}\}_{n=1}^{\infty} \neq \{i_n^{(2)}\}_{n=1}^{\infty}$ and consequently,

$$\bigcap_{n=1}^{\infty} \left[a_{n,i_n^{(1)}}, b_{n,i_n^{(1)}}\right] \neq \bigcap_{n=1}^{\infty} \left[a_{n,i_n^{(2)}}, b_{n,i_n^{(2)}}\right],$$

by (3). Hence, $f(x_1) \neq f(x_2)$.

Now, let U be open in X. The set $A = (X \times U) \cap H$ is Borel and $\operatorname{dom} A$ is Borel as the image of the Borel set A by the one-to-one continous function $\operatorname{pr}_1 : H \to X$. (See [Ke, Th.15.1].) Since $\operatorname{dom} A = f^{-1}[U]$, it follows that f is Borel measurable. Similarly, we show that the converse function f^{-1} is Borel. Thus, in particular, $F = f\left[\bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni}\right]$ is a Borel set.

The condition (3) implies that $\bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} [a_{ni}, b_{ni}]$ is a nowhere dense set. Moreover,

$$\lambda\left(\bigcup_{i=1}^{\infty} [a_{ni}, b_{ni}]\right) = \sum_{i=1}^{\infty} (b_{ni} - a_{ni}) < \frac{1}{2^n}$$

for each positive integer n. So $F \subseteq \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} [a_{ni}, b_{ni}]$ is a nowhere dense, Lebesgue null set.

The construction. Assume that n is a positive integer and that we have defined sequences $\{B_{ni}\}_{i=1}^{\infty}$ and $\{(a_{ni}, b_{ni})\}_{i=1}^{\infty}$ fulfilling conditions (1), (2), (3).

Fix i. Note that $V_{n+1} \cap (B_{ni} \times (a_{ni}, b_{ni}))$ is comeager in $B_{ni} \times (a_{ni}, b_{ni})$. Hence, by the Kuratowski-Ulam theorem [O, Th.15.1], the set

$$\{y \in (a_{ni}, b_{ni}) : \ V_{n+1}^{y} \cap B_{ni} \text{ is comeager in } B_{ni}\}$$

is comeager in (a_{ni}, b_{ni}). Pick $y_n^{(i)}, z_n^{(i)}$ from this set with $y_n^{(i)} < z_n^{(i)}$. If $U_{n+1} \cap (a_{ni}, b_{ni}) \neq \emptyset$, let l_{ni} denote the length of the interval $U_{n+1} \cap (a_{ni}, b_{ni})$, and, in the opposite case, let $l_{ni} = 1$. Let $M_n^{(i)}$ be a positive integer such that

$$\frac{1}{M_n^{(i)}} < \min\left\{z_n^{(i)} - y_n^{(i)}, l_{ni}, \frac{1}{2}(b_{ni} - a_{ni})\right\}$$

and put

$$A_{nj}^{i} = \left\{x \in B_{ni} : \ \left[y_n^{(i)} - \frac{1}{4j}, y_n^{(i)} + \frac{1}{4j}\right] \cup \left[z_n^{(i)} - \frac{1}{4j}, z_n^{(i)} + \frac{1}{4j}\right] \subseteq (V_{n+1})_x \cap (a_{ni}, b_{ni})\right\} \qquad \text{for } j \geq M_n^{(i)}.$$

Then every set A_{nj}^{i} is of type G_δ. Indeed, $X \setminus A_{nj}^{i}$ is the union of an F_σ set $X \setminus B_{ni}$ and a closed set in X (being the projection of the respective compact set in X^2).

Next, observe that

$$\bigcup_{j=M_n^{(i)}}^{\infty} A_{nj}^{i} = V_{n+1}^{y_n^{(i)}} \cap V_{n+1}^{z_n^{(i)}} \cap B_{ni}$$

which follows from the definition of A_{nj}^{i}. We also have that $V_{n+1}^{y_n^{(i)}} \cap V_{n+1}^{z_n^{(i)}} \cap B_{ni}$ is comeager in B_{ni}.

Find pairwise disjoint Borel sets $G_{nj}^{i} \subseteq A_{nj}^{i}$, $j \geq M_n^{(i)}$, with $\bigcup_{j=M_n^{(i)}}^{\infty} G_{nj}^{i} = \bigcup_{j=M_n^{(i)}}^{\infty} A_{nj}^{i}$. Put

$$C_{nj}^{i} = G_{nj}^{i} \cap \left(\frac{k_{ni}}{2^n}, \frac{2k_{ni}+1}{2^{n+1}}\right), \quad D_{nj}^{i} = G_{nj}^{i} \cap \left(\frac{2k_{ni}+1}{2^{n+1}}, \frac{k_{ni}+1}{2^n}\right).$$

We modify sets $C_{nj}^{i}, D_{nj}^{i}, j \geq M_n^{(i)}$, as follows. First, we ignore those which are meager. Next, we throw out, from the remaining sets C_{nj}^{i}, D_{nj}^{i}, meager parts to get G_δ sets. In that way, we obtain nonmeager G_δ sets. For simplicity, we suppose that all our sets $C_{nj}^{i}, D_{nj}^{i}, j \geq M_n^{(i)}$, have these properties.

Finally, we arrange sequences $\{C_{nj}^{i}\}_{i\geq 1, j \geq M_n^{(i)}}$ and $\{D_{nj}^{i}\}_{i\geq 1, j\geq M_n^{(i)}}$ into one sequence $\{B_{n+1,i}\}_{i\geq 1}$, and by the same method we arrange sequences $\left\{\left(y_n^{(i)} - \frac{1}{4j}, y_n^{(i)} - \frac{1}{4j+1}\right)\right\}_{i\geq 1, j\geq M_n^{(i)}}$ and $\left\{\left(z_n^{(i)} - \frac{1}{4j}, z_n^{(i)} - \frac{1}{4j+1}\right)\right\}_{i\geq 1, j \geq M_n^{(i)}}$ into one sequence $\{(a_{n+1,i}, b_{n+1,i})\}_{i\geq 1}$.

Now, it is not difficult to check that conditions (1), (2), (3) are satisfied for $n+1$.

The construction for $n = 1$ is similar; we use $V_1 \cap (X \times (0,1))$ instead of $V_{n+1} \cap (B_{ni} \times (a_{ni}, b_{ni}))$ applied in the above proof. □

Corollary 2. *The statement of Proposition 1 remains true if $B \subseteq Y \times Z$ where Y is an uncountable dense-in-itself Polish space and Z is a closed nondegenerate interval.*

Proof. Suppose first that $Z = X$ and Y equals the set X_0 of all irrationals from X. Since X_0 is a comeager G_δ set in X, we can apply Proposition 1 to our set $B \subseteq Y \times Z$. We obtain the respective function f which is good.

Now, consider a general assumption about Y and Z. We remove from Y the boundaries of open sets from a fixed countable base in Y. Thus, we get a zero-dimensional Polish space Y_* comeager in Y. Next, we consider a G_δ set $Y_0 \subseteq Y_*$ which is dense and boundary in Y_*. Obviously, $Y \setminus Y_0$ is meager. By the theorem of Mazurkiewicz [Ku, §36, II, Th.3], there is a homeomorphism h from Y_0 onto X_0. Let l be a linear function from Z onto X. We apply Proposition 1 to the set

$$g\,[B \cap (Y_0 \times Z)] \quad \text{where} \quad g\,(s,t) = \langle h\,(s)\,, l\,(t)\rangle \quad \text{for} \quad \langle s,t\rangle \in Y_0 \times Z.$$

Then we obtain the respective function f from E onto F. It is easy to check that the function $\left(l^{-1} \circ f \circ h\right) |h^{-1}[E]$ from $h^{-1}[E]$ onto $l^{-1}[F]$ is good. □

Proposition 2. *Let $B \subseteq X^2$ be a Borel set such that $\{x \in X :\ B_x \in \mathcal{M}\} \in \mathcal{M}$. Then B is $\langle \mathcal{M}, \mathcal{M} \cap \mathcal{N}\rangle$-wide.*

Proof. Let $\{U_n\}_{n\geq 1}$ be a countable open base for X. For each $n \geq 1$ we denote

$$D_n = \{x \in X :\ B_x \cap U_n \text{ is comeager in } U_n\}.$$

Then put

$$A_1 = D_1, \quad A_n = D_n \setminus \bigcup_{i=1}^{n-1} D_i \quad (n > 1).$$

Since every D_n is Borel (cf. [Ke, Exercise 22.22]), so is A_n. Notice that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} D_n = \{x \in X :\ B_x \notin \mathcal{M}\}$, thus $\bigcup_{n=1}^{\infty} A_n$ is comeager in X. Next we ignore those sets A_n which are meager, and we throw out from the remaining sets A_n their meager parts to get nonmeager dense-in-itself G_δ sets. So, we may assume that each A_n is a nonmeager dense-in-itself G_δ set.

We use induction on n to define the required function f. For $k \leq n$ suppose a Borel isomorphism f_k from $E_k \subseteq A_k$ onto $F_k \subseteq U_k$ has been defined so that E_k is a Borel comeager set in A_k and F_k is null and nowhere dense. Since $\bigcup_{i=1}^{n} F_i$ is nowhere dense, we can pick a closed nondegenerate interval $V_{n+1} \subseteq U_{n+1} \setminus \bigcup_{i=1}^{n} F_i$. Put $B_{n+1} = (A_{n+1} \times V_{n+1}) \cap B$. Notice that B_{n+1} is a Borel subset of the nonmeager set $A_{n+1} \times V_{n+1}$ and $\{x \in A_{n+1} :\ (B_{n+1})_x \text{ is comeager in } V_{n+1}\} = A_{n+1}$. By applying Corollary 2, we find a comeager Borel set E_{n+1} in A_{n+1} and a Borel isomorphism f_{n+1} from E_{n+1} onto $F_{n+1} \subseteq V_{n+1}$ such that F_{n+1} is a meager null set and $f_{n+1} \subseteq B$.

Since $E_n,\ n \geq 1$, are pairwise disjoint, we can define a function f from $\bigcup_{i=1}^{\infty} E_i$ onto $\bigcup_{i=1}^{\infty} F_i$ by $f(x) = f_n(x)$ for $x \in E_n$. Thus, f witnesses that B is $\langle \mathcal{M}, \mathcal{M} \cap \mathcal{N}\rangle$-wide. □

3. Applications

Theorem 1. *Assume that $A \subseteq X^2$ is a Borel set such that*

(a) $\{x \in X : A_x \in \mathcal{M}\} \in \mathcal{M}$,
(b) $\{y \in X : A^y \in \mathcal{C}\} \in \mathcal{N}$.

Then A is $\langle \mathcal{M}, \mathcal{N} \rangle$-large.

Proof. Method 1. By (a) and Proposition 2, the set A is $\langle \mathcal{M}, \mathcal{M} \cap \mathcal{N} \rangle$-wide; thus, also $\langle \mathcal{M}, \mathcal{N} \rangle$-wide. Let $F \in \mathcal{N}$ be a Borel set. By (b) we have

$$(*) \qquad \{y \in X : (A \setminus (X \times F))^y \in \mathcal{C}\} = F \cup \{y \in X \setminus F : A^y \in \mathcal{C}\} \in \mathcal{N}.$$

Hence, $A \setminus (X \times F)$ is $\langle \mathcal{M} \cap \mathcal{N}, \mathcal{N} \rangle$-tall by Fact 4. Now, the assertion follows from Corollary 1.

Method 2. We will prove the dual theorem. So, instead of (a) and (b) we assume that:

(a*) $\{y \in X : A^y \in \mathcal{M}\} \in \mathcal{M}$,
(b*) $\{x \in X : A_x \in \mathcal{C}\} \in \mathcal{N}$.

From (b*) and the dual version of Fact 4 we infer that A is $\langle \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle$-wide. Let $F \in \mathcal{M}$ be Borel. Then similarly as in (*) we check that

$$\{y \in X : (A \setminus (X \times F))^y \in \mathcal{M}\} \in \mathcal{M}.$$

Hence, $A \setminus (X \times F)$ is $\langle \mathcal{M}, \mathcal{M} \rangle$-tall by the dual version of Fact 3. Now, by Corollary 1, the set A is $\langle \mathcal{N}, \mathcal{M} \rangle$-large. So, the dual version of Theorem 1 has been proved. Hence, Theorem 1 is also true. □

Proposition 3. *Let $A \subseteq X^2$ be a Borel set such that $\{x \in X : A_x \in \mathcal{M}\} \in \mathcal{M} \cap \mathcal{N}$. Then A is $\langle \mathcal{M} \cap \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle$-wide.*

Proof. Since $\{x \in X : A_x \in \mathcal{M}\} \in \mathcal{M}$, the set A is $\langle \mathcal{M}, \mathcal{M} \cap \mathcal{N} \rangle$-wide, by virtue of Proposition 2. Thus, there are Borel sets $E \in \mathcal{M}$ and $F \in \mathcal{M} \cap \mathcal{N}$ and a Borel isomorphism f from $X \setminus E$ onto F such that $f \subseteq A$. We may assume that E is of type F_σ. If $E \in \mathcal{N}$, the set A is $\langle \mathcal{M} \cap \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle$-wide. Thus, assume that $E \notin \mathcal{N}$ and apply the dual Fact 4 to the set $A \cap (E \times (X \setminus F))$ in the space $E \times X$ where in E we consider Lebesgue measure restricted to subsets of E. The respective assumptions are satisfied since

$$\begin{aligned} \{x \in E : (A \cap (E \times (X \setminus F)))_x \in \mathcal{C}\} &= E \cap \{x \in X : A_x \setminus F \in \mathcal{C}\} \\ &\subseteq \{x \in X : A_x \setminus F \in \mathcal{M}\} = \{x \in X : A_x \in \mathcal{M}\} \in \mathcal{N}. \end{aligned}$$

Thus, there are Borel sets $U \in \mathcal{N}$, $U \subseteq E$, and $V \in \mathcal{M} \cap \mathcal{N}$, and a Borel isomorphism g from $E \setminus U$ onto V such that $g \subseteq A \cap (E \times (X \setminus F))$. Put

$$h(x) = \begin{cases} f(x) & \text{for } x \in X \setminus E, \\ g(x) & \text{for } x \in E \setminus U. \end{cases}$$

Then h is Borel one-to-one included in A with $\operatorname{dom} h = X \setminus U$, where $X \setminus \operatorname{dom} h = U \in \mathcal{M} \cap \mathcal{N}$, and $\operatorname{ran} h = F \cup V \in \mathcal{M} \cap \mathcal{N}$. Hence, A is $\langle \mathcal{M} \cap \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle$-wide. □

Theorem 2. *If a Borel set $A \subseteq X^2$ satisfies the conditions*

$$\{x \in X : A_x \in \mathcal{M}\} \in \mathcal{M} \cap \mathcal{N}, \quad \{y \in X : A^y \in \mathcal{M}\} \in \mathcal{M} \cap \mathcal{N},$$

then A is $\langle \mathcal{M} \cap \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle$-large.

Proof. It suffices to apply Proposition 3, its dual version and Corollary 1. □

Remark (added in proof). Roman Pol has shown us the following short proof of Proposition 1: Find a G_δ dense set $E \subseteq X$ and a perfect null set $F \subseteq X$ such that $E \times F \subseteq B$; then a Borel isomorphism f from E onto F is as desired. In a recent note by Roman Pol and the authors, submitted for publication, Problem 1 has been solved in the negative.

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