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RELATIVE MODULAR THEORY FOR A WEIGHT

HIDEAKI IZUMI

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ABSTRACT. We consider the balanced weight χ of a semi-finite weight φ and a (not necessarily faithful) normal positive functional ψ on a von Neumann algebra $\mathcal{M}$, and discuss how the modular operator Δ_χ and the modular conjugation J_χ are described under the identification of the standard Hilbert space $\mathcal{H}_\chi$ with $\mathcal{H}_\varphi \oplus p\mathcal{H}_\varphi \oplus p'\mathcal{H}_\varphi \oplus pp'\mathcal{H}_\varphi$, where p is the support projection of ψ and $p' = J_\varphi p J_\varphi \in \mathcal{M}'$.

In the theory of von Neumann algebras, relative modular operators introduced by A. Connes [2] and T. Digernes [3] play important rôles as a non-commutative analogue of Radon-Nikodym derivative. In [6], H. Kosaki extended Connes' results. He considered the balanced functional of a faithful normal positive functional and a (not necessarily faithful) normal positive functional on a (σ-finite) von Neumann algebra $\mathcal{M}$, and by considering the modular operator Δ_χ, he obtained relative modular operators $\Delta_{\psi\varphi}$ and $\Delta_{\varphi\psi}$. Moreover, he showed that the modular conjugation J_χ can be described in terms of J_φ and its restrictions. His results thus obtained is used in [7] to construct $\mathcal{M}_*$-valued KMS functions, which he in turn used to prove that the L^2-space $L^2(\mathcal{M}, \varphi)$ obtained by the complex interpolation of $\mathcal{M}$ and $\mathcal{M}_*$ is exactly the standard Hilbert space $\mathcal{H}_\varphi$.

In this paper, we will extend Kosaki's results to the weight case, namely, we consider the balanced weight χ of a faithful normal semi-finite weight φ and a (not necessarily faithful) normal positive functional ψ on a von Neumann algebra $\mathcal{M}$ and get similar results as in [6]. In particular, we determine the modular conjugation J_χ for the balanced weight χ. These results will be used in [5] to construct KMS functions with respect to φ and ψ.

First, we fix the notations in modular theory ([10], [11]).

Let $\mathcal{M}$ be a von Neumann algebra and φ be a faithful normal semi-finite weight on $\mathcal{M}$. We set

$$\mathfrak{n}_\varphi = \{x \in \mathcal{M} \mid \varphi(x^*x) < \infty\}.$$

Let $\{\pi_\varphi, \mathcal{H}_\varphi, \Lambda_\varphi\}$ be the semi-cyclic representation induced by φ, $\mathcal{B}_\varphi = \Lambda_\varphi(\mathfrak{n}_\varphi)$ be left bounded vectors, $\pi_l^\varphi(\xi)$, $\xi \in \mathcal{B}_\varphi$, be the left multiplication by ξ and $\mathfrak{A}_\varphi = \Lambda_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ be the left Hilbert algebra associated to the semi-cyclic representation $\{\pi_\varphi, \mathcal{H}_\varphi, \Lambda_\varphi\}$. Next, we define $S_\varphi : \mathfrak{A}_\varphi \to \mathfrak{A}_\varphi$ by

$$S_\varphi \Lambda_\varphi(x) = \Lambda_\varphi(x^*), \ x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*.$$

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Then S_φ is a conjugate-linear preclosed operator. By polar decomposition, we have

$$\overline{S_\varphi} = J_\varphi \Delta_\varphi^{1/2},$$

where $\overline{S_\varphi}$ denotes the closure of S_φ. The operator J_φ is a conjugate-linear unitary on $\mathcal{H}_\varphi$ and is called the modular conjugation, while the operator Δ_φ is a non-singular positive self-adjoint operator on $\mathcal{H}_\varphi$ and is called the modular operator.

Next, we set

$$\mathfrak{B}'_\varphi = \left\{ \eta \in \mathcal{H}_\varphi \;\middle|\; \sup\{\|\pi_l^\varphi(\xi)\eta\| \mid \xi \in \mathfrak{A}_\varphi, \|\xi\| \leq 1\} < +\infty \right\}.$$

An element of $\mathfrak{B}'_\varphi$ is called a right bounded vector. We write the right multiplication by $\eta \in \mathcal{B}'_\varphi$ as $\pi_r^\varphi(\eta)$.

We define the right Hilbert algebra $\mathfrak{A}'_\varphi$ by

$$\mathfrak{A}'_\varphi = \mathfrak{B}'_\varphi \cap \mathcal{D}(S_\varphi^*).$$

If we write

$$\xi^\sharp = \overline{S_\varphi}\xi, \;\; \xi \in \mathcal{D}(\overline{S_\varphi});$$
$$\eta^\flat = S_\varphi^* \eta, \;\; \eta \in \mathcal{D}(S_\varphi^*),$$

then we have

$$\pi_l^\varphi(\xi^\sharp) = \pi_l^\varphi(\xi)^*, \;\; \xi \in \mathfrak{A}_\varphi;$$
$$\pi_r^\varphi(\eta^\flat) = \pi_r^\varphi(\eta)^*, \;\; \eta \in \mathfrak{A}'_\varphi.$$

Moreover, we define the positive cone $\mathcal{P}_\varphi^\natural$ by

$$\mathcal{P}_\varphi^\natural = \overline{\{\pi_l(\xi) J_\varphi \xi \mid \xi \in \mathfrak{A}_\varphi\}}.$$

Since the representation π_φ is faithful, from now on, we will identify $\pi_\varphi(\mathcal{M})$ by $\mathcal{M}$ and write x instead of $\pi_\varphi(x)$, $x \in \mathcal{M}$.

Now we fix

$$\psi \in (\mathcal{M}_*)_+.$$

Then by [1], [4] (see also [11]), there exists a unique $\xi \in \mathcal{P}_\varphi^\natural$ such that

$$\psi = \omega_\xi = (\cdot \xi | \xi).$$

We remark that the support projection $p \in \mathcal{M}$ of ψ is equal to the projection onto $[\mathcal{M}'\xi]$. Here, the notation $[\mathcal{K}]$ means the closed linear span of $\mathcal{K}$.

Let

$$\begin{aligned} \mathcal{N} &= \mathcal{M} \otimes M_2(\mathcal{C}) \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathcal{M} \right\} \end{aligned}$$

and let $\overline{p}$ be the projection in $\mathcal{N}$ such that

$$\overline{p} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

We denote the reduced algebra of $\mathcal{N}$ by the projection $\overline{p}$ by $\mathcal{N}_{\overline{p}}$, that is,

$$\mathcal{N}_{\overline{p}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a \in \mathcal{M}, \; b \in \mathcal{M}p, \; c \in p\mathcal{M}, \; d \in p\mathcal{M}p \right\}.$$

Here, we note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{N}_{\overline{p}}$ is positive, then $a \geq 0$, $d \geq 0$ and $c = b^*$.

Next, we consider the balanced weight χ on $\mathcal{N}_{\overline{p}}$ determined by

$$\chi\left(\begin{pmatrix} a & b \\ b^* & d \end{pmatrix}\right) = \varphi(a) + \psi(d),\ \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \in (\mathcal{N}_{\overline{p}})_+.$$

Then we have

$$\begin{aligned} \chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \chi\left(\begin{pmatrix} a^*a + c^*c & a^*b + c^*d \\ b^*a + d^*c & b^*b + d^*d \end{pmatrix}\right) \\ &= \varphi(a^*a + c^*c) + \psi(b^*b + d^*d) \\ &= \varphi(a^*a) + \varphi(c^*c) + \psi(b^*b) + \psi(d^*d). \end{aligned} \tag{1}$$

Hence, by [8, 3.1], [11, Lemma VIII.3.1], we have the following.

Lemma 1. (1) *The weight χ on $\mathcal{N}_{\overline{p}}$ is faithful, normal and semi-finite.*
(2) $\mathfrak{n}_\chi = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{N}_{\overline{p}} \;\middle|\; a \in \mathfrak{n}_\varphi,\ c \in p\mathfrak{n}_\varphi \right\}.$

We set $p' = J_\varphi p J_\varphi \in \mathcal{M}'$, so that p' is the projection onto $[\mathcal{M}\xi]$.

By the formula (1), for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{n}_\chi$, we have

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \|\Lambda_\varphi(a)\|^2 + \|\Lambda_\varphi(c)\|^2 + \|b\xi\|^2 + \|d\xi\|^2.$$

Here, we note that

$$\begin{array}{lll} \Lambda_\varphi(a) & \in & \mathfrak{n}_\varphi \subset \mathcal{H}_\varphi, \\ \Lambda_\varphi(c) & \in & p\Lambda_\varphi(\mathfrak{n}_\varphi) \subset p\mathcal{H}_\varphi, \\ b\xi & \in & \mathcal{M}p\xi = \mathcal{M}\xi \subset p'\mathcal{H}_\varphi, \\ d\xi & \in & p\mathcal{M}p\xi = p\mathcal{M}\xi \subset pp'\mathcal{H}_\varphi. \end{array}$$

Thus, the formula (1) means that the pre-Hilbert space $\mathfrak{n}_\chi$ equipped with the inner product induced by χ is isometrially mapped into $\mathcal{H}_\varphi \oplus p\mathcal{H}_\varphi \oplus p'\mathcal{H}_\varphi \oplus pp'\mathcal{H}_\varphi$ via

$$\eta_\chi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \Lambda_\varphi(a) \\ \Lambda_\varphi(c) \\ b\xi \\ d\xi \end{pmatrix}.$$

Since $p\Lambda_\varphi(\mathfrak{n}_\varphi) \subset p\mathcal{H}_\varphi$, $\mathcal{M}\xi \subset p'\mathcal{H}_\varphi$ and $p\mathcal{M}\xi \subset pp'\mathcal{H}_\varphi$, the image of η_χ is dense in $\mathcal{H}_\varphi \oplus p\mathcal{H}_\varphi \oplus p'\mathcal{H}_\varphi \oplus pp'\mathcal{H}_\varphi$. Hence the Hilbert space $\mathcal{H}_\chi$, the completion of $\mathfrak{n}_\chi$, can be identified with $\mathcal{H}_\varphi \oplus p\mathcal{H}_\varphi \oplus p'\mathcal{H}_\varphi \oplus pp'\mathcal{H}_\varphi$.

Now, we will examine how the representation π_χ of $\mathcal{N}_{\overline{p}}$ is described under this identification.

For

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{N}_{\overline{p}} \ \text{ and } \ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathfrak{n}_\chi,$$

we have

$$\begin{aligned}
\pi_\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\eta_\chi\left(\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) &= \eta_\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) \\
&= \eta_\chi\left(\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}\right) \\
&= \begin{pmatrix} \Lambda_\varphi(ae+bg) \\ \Lambda_\varphi(ce+dg) \\ (af+bh)\xi \\ (cf+dh)\xi \end{pmatrix} \\
&= \begin{pmatrix} a & b|_{p\mathcal{H}_\varphi} & 0 & 0 \\ c & d|_{p\mathcal{H}_\varphi} & 0 & 0 \\ 0 & 0 & a|_{p'\mathcal{H}_\varphi} & b|_{pp'\mathcal{H}_\varphi} \\ 0 & 0 & c|_{p'\mathcal{H}_\varphi} & d|_{pp'\mathcal{H}_\varphi} \end{pmatrix}\begin{pmatrix} \Lambda_\varphi(e) \\ \Lambda_\varphi(g) \\ f\xi \\ h\xi \end{pmatrix}.
\end{aligned}$$

By the identification of $\mathcal{H}_\chi$ with $\mathcal{H}_\varphi \oplus p\mathcal{H}_\varphi \oplus p'\mathcal{H}_\varphi \oplus pp'\mathcal{H}_\varphi$ via η_χ, we conclude that the representation π_χ is described as

$$\pi_\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b|_{p\mathcal{H}_\varphi} & 0 & 0 \\ c & d|_{p\mathcal{H}_\varphi} & 0 & 0 \\ 0 & 0 & a|_{p'\mathcal{H}_\varphi} & b|_{pp'\mathcal{H}_\varphi} \\ 0 & 0 & c|_{p'\mathcal{H}_\varphi} & d|_{pp'\mathcal{H}_\varphi} \end{pmatrix} \in \mathcal{L}(\mathcal{H}_\varphi \oplus p\mathcal{H}_\varphi \oplus p'\mathcal{H}_\varphi \oplus pp'\mathcal{H}_\varphi).$$

Since the representation π_χ is faithful, we identify $\pi_\chi(\mathcal{N}_{\overline{p}})$ with $\mathcal{N}_{\overline{p}}$.

We define

$$S_\chi\left(\begin{pmatrix} \Lambda_\varphi(a) \\ \Lambda_\varphi(c) \\ b\xi \\ d\xi \end{pmatrix}\right) = \begin{pmatrix} \Lambda_\varphi(a^*) \\ b^*\xi \\ \Lambda_\varphi(c^*)\xi \\ d^*\xi \end{pmatrix}, \quad \begin{pmatrix} \Lambda_\varphi(a) \\ \Lambda_\varphi(c) \\ b\xi \\ d\xi \end{pmatrix} \in \mathfrak{n}_\chi \cap \mathfrak{n}_\chi^*.$$

It is easy to see that S_χ is well-defined. Then, by (1) we have

Lemma 2.

$$\begin{pmatrix} \Lambda_\varphi(a) \\ \Lambda_\varphi(c) \\ b\xi \\ d\xi \end{pmatrix} \in \mathfrak{A}_\chi = \Lambda_\chi(\mathfrak{n}_\chi \cap \mathfrak{n}_\chi^*)$$

if and only if

$$a \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*,\ b \in \mathfrak{n}_\varphi^* p,\ c \in p\mathfrak{n}_\varphi.$$

Note that the densely-defined conjugate-linear operator S_χ is closable. By looking at each component, we have

$$\begin{array}{llll}
(1) & S_\varphi : & \Lambda_\varphi(a) \in \mathfrak{A}_\chi & \mapsto \Lambda_\varphi(a^*) \in \mathfrak{A}_\chi, \\
(2) & S_{\psi\varphi} : & \Lambda_\varphi(c) \in p\Lambda_\varphi(\mathfrak{n}_\varphi) & \mapsto c^*\xi \in \mathfrak{n}_\varphi^*\xi, \\
(3) & S_{\varphi\psi} : & b\xi \in \mathfrak{n}_\varphi^*\xi & \mapsto \Lambda_\varphi(b^*) \in p\Lambda_\varphi(\mathfrak{n}_\varphi), \\
(4) & S_\psi : & d\xi \in p\mathcal{M}\xi & \mapsto d^*\xi \in p\mathcal{M}\xi.
\end{array}$$

Then it is easily seen that

$$S_\chi = \begin{pmatrix} S_\varphi & 0 & 0 & 0 \\ 0 & 0 & S_{\varphi\psi} & 0 \\ 0 & S_{\psi\varphi} & 0 & 0 \\ 0 & 0 & 0 & S_\psi \end{pmatrix}.$$

Since S_χ is densely defined and closable, we conclude that

(a) $S_{\psi\varphi}$ is a densely-defined closable operator from $p\mathcal{H}_\varphi$ to $p'\mathcal{H}_\varphi$;
(b) $S_{\varphi\psi}$ is a densely-defined closable operator from $p'\mathcal{H}_\varphi$ to $p\mathcal{H}_\varphi$;
(c) S_ψ is a densely-defined closable operator on $pp'\mathcal{H}_\varphi$.

Furthermore, we have

$$\begin{aligned} S_\chi^*\overline{S_\chi} &= \begin{pmatrix} S_\varphi & 0 & 0 & 0 \\ 0 & 0 & S_{\varphi\psi} & 0 \\ 0 & S_{\psi\varphi} & 0 & 0 \\ 0 & 0 & 0 & S_\psi \end{pmatrix}^* \overline{\begin{pmatrix} S_\varphi & 0 & 0 & 0 \\ 0 & 0 & S_{\varphi\psi} & 0 \\ 0 & S_{\psi\varphi} & 0 & 0 \\ 0 & 0 & 0 & S_\psi \end{pmatrix}} \\ &= \begin{pmatrix} S_\varphi^* & 0 & 0 & 0 \\ 0 & 0 & S_{\psi\varphi}^* & 0 \\ 0 & S_{\varphi\psi}^* & 0 & 0 \\ 0 & 0 & 0 & S_\psi^* \end{pmatrix} \begin{pmatrix} \overline{S_\varphi} & 0 & 0 & 0 \\ 0 & 0 & \overline{S_{\varphi\psi}} & 0 \\ 0 & \overline{S_{\psi\varphi}} & 0 & 0 \\ 0 & 0 & 0 & \overline{S_\psi} \end{pmatrix} \\ &= \begin{pmatrix} S_\varphi^*\overline{S_\varphi} & 0 & 0 & 0 \\ 0 & S_{\psi\varphi}^*\overline{S_{\psi\varphi}} & 0 & 0 \\ 0 & 0 & S_{\varphi\psi}^*\overline{S_{\varphi\psi}} & 0 \\ 0 & 0 & 0 & S_\psi^*\overline{S_\psi} \end{pmatrix}. \end{aligned}$$

We define

$$\begin{aligned} \Delta_{\psi\varphi} &= S_{\psi\varphi}^*\overline{S_{\psi\varphi}} \quad \text{on } p\mathcal{H}_\varphi, \\ \Delta_{\varphi\psi} &= S_{\varphi\psi}^*\overline{S_{\varphi\psi}} \quad \text{on } p'\mathcal{H}_\varphi, \\ \Delta_\psi &= S_\psi^*\overline{S_\psi} \quad \text{on } pp'\mathcal{H}_\varphi. \end{aligned}$$

These are all non-singular positive self-adjoint operators. Among them, $\Delta_{\psi\varphi}$ and $\Delta_{\varphi\psi}$ are called relative modular operators. The above calculation shows that

$$\Delta_\chi = \begin{pmatrix} \Delta_\varphi & 0 & 0 & 0 \\ 0 & \Delta_{\psi\varphi} & 0 & 0 \\ 0 & 0 & \Delta_{\varphi\psi} & 0 \\ 0 & 0 & 0 & \Delta_\psi \end{pmatrix}.$$

Next, we will determine the modular conjugation J_χ. To do this, it is important to consider the commutant $(\mathcal{N}_{\overline{p}})'$ of $\mathcal{N}_{\overline{p}}$ acting on $\mathcal{H}_\varphi \oplus p\mathcal{H}_\varphi \oplus p'\mathcal{H}_\varphi \oplus pp'\mathcal{H}_\varphi$.

Lemma 3. *The commutant* $\mathcal{N}_{\overline{p}}'$ *is given by*

$$\mathcal{N}_{\overline{p}}' = \left\{ \begin{pmatrix} a' & 0 & b'|_{p'\mathcal{H}_\varphi} & 0 \\ 0 & a'|_{p\mathcal{H}_\varphi} & 0 & b'|_{pp'\mathcal{H}_\varphi} \\ p'c' & 0 & p'd'|_{p'\mathcal{H}_\varphi} & 0 \\ 0 & p'c'|_{p\mathcal{H}_\varphi} & 0 & p'd'|_{p'\mathcal{H}_\varphi} \end{pmatrix} \middle| \; a', b', c', d' \in \mathcal{M}' \right\}.$$

The proof is similar to that of [6, Proposition 1.2.2], so we omit the details.

Next, we set

$$J = \begin{pmatrix} J_\varphi & 0 & 0 & 0 \\ 0 & 0 & J_\varphi|_{p'\mathcal{H}_\varphi} & 0 \\ 0 & J_\varphi|_{p\mathcal{H}_\varphi} & 0 & 0 \\ 0 & 0 & 0 & J_\varphi|_{pp'\mathcal{H}_\varphi} \end{pmatrix}.$$

Our main purpose is to prove that $J_\chi = J$.

For simplicity, we write $\zeta^\sharp = \overline{S_\chi}\zeta,\ \zeta \in \mathcal{D}(\overline{S_\chi})$ and $\eta^\flat = S_\chi^*\eta,\ \eta \in \mathcal{D}(S_\chi^*)$.

Lemma 4. (1) *Suppose that* $\zeta, \eta \in \mathfrak{B}_\chi$ *satisfy*

$$\pi_l^\chi(\zeta)^* = \pi_l^\chi(\eta).$$

Then $\zeta, \eta \in \mathfrak{A}_\chi$ *and* $\zeta^\sharp = \eta$.

(2) *Suppose that* $\zeta, \eta \in \mathfrak{B}'_\chi$ *satisfy*

$$\pi_r^\chi(\zeta)^* = \pi_r^\chi(\eta)^*.$$

Then $\zeta, \eta \in \mathfrak{A}'_\chi$ *and* $\zeta^\flat = \eta$.

Proof. The proofs of (1) and (2) are similar, so we will prove only (1). Let $\zeta, \eta \in \mathfrak{B}_\chi$ satisfy $\pi_l^\chi(\zeta)^* = \pi_l^\chi(\eta)$, and let $\zeta', \eta' \in \mathfrak{A}'$. Then we have

$$\begin{aligned} (\zeta|(\zeta'\eta')^\flat) &= (\zeta|{\eta'}^\flat{\zeta'}^\flat) = (\zeta|\pi_r^\chi(\zeta')^*{\eta'}^\flat) = (\pi_r^\chi(\zeta')\zeta|{\eta'}^\flat) = (\pi_l^\chi(\zeta)\zeta'|{\eta'}^\flat) \\ &= (\zeta'|\pi_l^\chi(\zeta)^*{\eta'}^\flat) = (\zeta'|\pi_l^\chi(\eta){\eta'}^\flat) = (\zeta'|\pi_r^\chi(\eta')^*\eta) \\ &= (\pi_r^\chi(\eta')\zeta'|\eta) = (\zeta'\eta'|\eta). \end{aligned}$$

Since $(\mathfrak{A}'_\chi)^2$ is a core for the $^\flat$-operation ([11, Lemma VI.1.13]), and the $^\flat$-operation and the $^\sharp$-operation are mutually adjoint, we have $\zeta \in \mathcal{D}(\overline{S_\chi})$ and $\zeta^\sharp = \eta$. Hence both ζ and η belong to $\mathfrak{A}_\chi$. □

Lemma 5. *Suppose that* $\eta \in \mathfrak{B}'_\chi$ *satisfies*

$$\pi_r^\chi(\eta) = \begin{pmatrix} a' & 0 & b'|_{p'\mathcal{H}_\varphi} & 0 \\ 0 & a'|_{p\mathcal{H}_\varphi} & 0 & b'|_{pp'\mathcal{H}_\varphi} \\ p'c' & 0 & p'd'|_{p'\mathcal{H}_\varphi} & 0 \\ 0 & p'c'|_{p\mathcal{H}_\varphi} & 0 & p'd'|_{pp'\mathcal{H}_\varphi} \end{pmatrix} \in (\mathcal{N}_{\overline{p}})'$$

for some $a', b', c', d' \in \mathcal{M}'$ (*see Lemma 3*). *Then* η *is of the form*

$$\eta = \begin{pmatrix} \eta_1 \\ b'\xi \\ \eta_3 \\ p'd'\xi \end{pmatrix},$$

where $\eta_1, \eta_3 \in \mathfrak{B}'_\varphi,\ \pi_r(\eta_1) = a',\ \pi_r(\eta_3) = p'c'$.

Proof. Take

$$\zeta = \begin{pmatrix} \Lambda_\varphi(e) \\ \Lambda_\varphi(g) \\ f\xi \\ h\xi \end{pmatrix} \in \mathfrak{A}_\chi.$$

Then by Lemma 2, $e \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ and $f \in \mathfrak{n}_\varphi^* p$. Moreover, we have $\pi_r^\chi(\eta)\zeta = \pi_l^\chi(\zeta)\eta$ and

$$\begin{aligned}\pi_r^\chi(\eta)\zeta &= \begin{pmatrix} a' & 0 & b'|_{p'\mathcal{H}_\varphi} & 0 \\ 0 & a'|_{p\mathcal{H}_\varphi} & 0 & b'|_{pp'\mathcal{H}_\varphi} \\ p'c' & 0 & p'd'|_{p'\mathcal{H}_\varphi} & 0 \\ 0 & p'c'|_{p\mathcal{H}_\varphi} & 0 & p'd'|_{pp'\mathcal{H}_\varphi} \end{pmatrix} \begin{pmatrix} \Lambda_\varphi(e) \\ \Lambda_\varphi(g) \\ f\xi \\ h\xi \end{pmatrix} \\ &= \begin{pmatrix} a'\Lambda_\varphi(e) + b'f\xi \\ a'\Lambda_\varphi(g) + b'h\xi \\ p'c'\Lambda_\varphi(e) + p'd'f\xi \\ p'c'\Lambda_\varphi(g) + p'd'h\xi \end{pmatrix},\end{aligned}$$

and putting $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} \in \mathcal{H}_\varphi \oplus p\mathcal{H}_\varphi \oplus p'\mathcal{H}_\varphi \oplus pp'\mathcal{H}_\varphi$, we have

$$\pi_l^\chi(\zeta)\eta = \begin{pmatrix} e & f|_{p\mathcal{H}_\varphi} & 0 & 0 \\ g & h|_{p\mathcal{H}_\varphi} & 0 & 0 \\ 0 & 0 & e|_{p'\mathcal{H}_\varphi} & f|_{pp'\mathcal{H}_\varphi} \\ 0 & 0 & g|_{p'\mathcal{H}_\varphi} & h|_{pp'\mathcal{H}_\varphi} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} = \begin{pmatrix} e\eta_1 + f\eta_2 \\ g\eta_1 + h\eta_2 \\ e\eta_3 + f\eta_4 \\ g\eta_3 + h\eta_4 \end{pmatrix}.$$

Hence we have

$$\begin{pmatrix} a'\Lambda_\varphi(e) + b'f\xi \\ a'\Lambda_\varphi(g) + b'h\xi \\ p'c'\Lambda_\varphi(e) + p'd'f\xi \\ p'c'\Lambda_\varphi(g) + p'd'h\xi \end{pmatrix} = \begin{pmatrix} e\eta_1 + f\eta_2 \\ g\eta_1 + h\eta_2 \\ e\eta_3 + f\eta_4 \\ g\eta_3 + h\eta_4 \end{pmatrix}. \tag{2}$$

If we put $f = h = 0$ in (2), we get

$$a'\Lambda_\varphi(e) = e\eta_1,\ p'c'\Lambda_\varphi(e) = e\eta_3$$

for all $e \in \pi_l^\chi(\mathfrak{A}_\varphi)$. By the definition of $\mathfrak{B}'_\chi$, we have $\eta_1, \eta_3 \in \mathfrak{B}'_\chi$ with

$$\pi_r^\chi(\eta_1) = a',\ \pi_r^\chi(\eta_3) = p'c'.$$

If we put $e = g = 0$ in (2), we get

$$fb'\xi = f\eta_2,\ fp'd'\xi = f\eta_4 \tag{3}$$

for all $f \in \mathfrak{n}_\varphi^* p$. Since p commutes with b' and $p'd'$, and $\eta_2 \in p\mathcal{H}_\varphi,\ \eta_4 \in pp'\mathcal{H}_\varphi$, (3) holds also for all $f \in \mathfrak{n}_\varphi^*$. Since $\mathfrak{n}_\varphi^* (\supset \pi_l^\chi(\mathfrak{A}_\varphi))$ is strongly dense in $\mathcal{M}$, we have

$$\eta_2 = b'\xi,\ \eta_4 = p'd'\xi.$$

Hence we completed the proof. □

Lemma 6. $J\mathfrak{A}_\chi = \mathfrak{A}'_\chi$.

Proof. We first show that $J\mathfrak{A}_\chi \subset \mathfrak{A}'_\chi$.

Let

$$\eta_1 = \begin{pmatrix} \Lambda_\varphi(a) \\ \Lambda_\varphi(c) \\ b\xi \\ d\xi \end{pmatrix} \in \mathfrak{A}_\chi,\ \eta_2 = \begin{pmatrix} \Lambda_\varphi(e) \\ \Lambda_\varphi(g) \\ f\xi \\ h\xi \end{pmatrix} \in \mathfrak{A}_\chi.$$

We claim that $J\eta_1 \in \mathcal{B}'_\chi$.

We have

$$\begin{aligned}\pi_l^\chi(\eta_2)J\eta_1 &= \pi_l^\chi\left(\begin{pmatrix}\Lambda_\varphi(e)\\ \Lambda_\varphi(g)\\ f\xi\\ h\xi\end{pmatrix}\right)\begin{pmatrix}J_\varphi\Lambda_\varphi(a)\\ J_\varphi b\xi\\ J_\varphi\Lambda_\varphi(c)\\ J_\varphi d\xi\end{pmatrix}\\ &= \begin{pmatrix}e & f|_{p\mathcal{H}_\varphi} & 0 & 0\\ g & h|_{p\mathcal{H}_\varphi} & 0 & 0\\ 0 & 0 & e|_{p'\mathcal{H}_\varphi} & f|_{pp'\mathcal{H}_\varphi}\\ 0 & 0 & g|_{p'\mathcal{H}_\varphi} & h|_{pp'\mathcal{H}_\varphi}\end{pmatrix}\begin{pmatrix}J_\varphi\Lambda_\varphi(a)\\ J_\varphi b\xi\\ J_\varphi\Lambda_\varphi(c)\\ J_\varphi d\xi\end{pmatrix}\\ &= \begin{pmatrix}eJ_\varphi\Lambda_\varphi(a)+fJ_\varphi b\xi\\ gJ_\varphi\Lambda_\varphi(a)+hJ_\varphi b\xi\\ eJ_\varphi\Lambda_\varphi(c)+fJ_\varphi d\xi\\ gJ_\varphi\Lambda_\varphi(c)+hJ_\varphi d\xi\end{pmatrix}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&J\pi_l^\chi(\eta_1)J\eta_2\\ &= \begin{pmatrix}J_\varphi & 0 & 0 & 0\\ 0 & 0 & J_\varphi|_{p'\mathcal{H}_\varphi} & 0\\ 0 & J_\varphi|_{p\mathcal{H}_\varphi} & 0 & 0\\ 0 & 0 & 0 & J_\varphi|_{pp'\mathcal{H}_\varphi}\end{pmatrix}\pi_l^\chi\left(\begin{pmatrix}\Lambda_\varphi(a)\\ \Lambda_\varphi(c)\\ b\xi\\ d\xi\end{pmatrix}\right)\\ &\times\begin{pmatrix}J_\varphi & 0 & 0 & 0\\ 0 & 0 & J_\varphi|_{p'\mathcal{H}_\varphi} & 0\\ 0 & J_\varphi|_{p\mathcal{H}_\varphi} & 0 & 0\\ 0 & 0 & 0 & J_\varphi|_{pp'\mathcal{H}_\varphi}\end{pmatrix}\begin{pmatrix}\Lambda_\varphi(e)\\ \Lambda_\varphi(g)\\ f\xi\\ h\xi\end{pmatrix}\\ &= \begin{pmatrix}J_\varphi aJ_\varphi & 0 & J_\varphi bJ_\varphi|_{p'\mathcal{H}_\varphi} & 0\\ 0 & J_\varphi aJ_\varphi|_{p\mathcal{H}_\varphi} & 0 & J_\varphi bJ_\varphi|_{pp'\mathcal{H}_\varphi}\\ J_\varphi cJ_\varphi & 0 & J_\varphi dJ_\varphi|_{p'\mathcal{H}_\varphi} & 0\\ 0 & J_\varphi cJ_\varphi|_{p\mathcal{H}_\varphi} & 0 & J_\varphi dvJ_\varphi|_{pp'\mathcal{H}_\varphi}\end{pmatrix}\begin{pmatrix}\Lambda_\varphi(e)\\ \Lambda_\varphi(g)\\ f\xi\\ h\xi\end{pmatrix}\\ &= \begin{pmatrix}J_\varphi aJ_\varphi\Lambda_\varphi(e)+J_\varphi bJ_\varphi f\xi\\ J_\varphi aJ_\varphi\Lambda_\varphi(g)+J_\varphi bJ_\varphi h\xi\\ J_\varphi cJ_\varphi\Lambda_\varphi(e)+J_\varphi dJ_\varphi f\xi\\ J_\varphi cJ_\varphi\Lambda_\varphi(g)+J_\varphi dJ_\varphi h\xi\end{pmatrix} = \begin{pmatrix}eJ_\varphi\Lambda_\varphi(a)+fJ_\varphi b\xi\\ gJ_\varphi\Lambda_\varphi(a)+hJ_\varphi b\xi\\ eJ_\varphi\Lambda_\varphi(c)+fJ_\varphi d\xi\\ gJ_\varphi\Lambda_\varphi(c)+hJ_\varphi d\xi\end{pmatrix}.\end{aligned}$$

Hence we have $\pi_l^\chi(\eta_2)J\eta_1 = J\pi_l^\chi(\eta_1)J\eta_2$. Consequently, we have

$$J\eta_1 \in \mathfrak{B}'_\chi \text{ and } \pi_r^\chi(J\eta_1) = J\pi_l^\chi(\eta_1)J.$$

Replacing η_1 by $\eta_1^\sharp$, we have

$$J\eta_1^\sharp \in \mathfrak{B}'_\chi \text{ and } \pi_r^\chi(J\eta_1^\sharp) = J\pi_l^\chi(\eta_1^\sharp)J = (J\pi_l^\chi(\eta_1)J)^*.$$

By Lemma 4(2), $J\eta_1 \in \mathfrak{A}'_\chi$ with $(J\eta_1)^\flat = J\eta_1^\sharp$. Hence $J\mathfrak{A}_\chi \subset \mathfrak{A}'_\chi$

Next, we will show that $\mathfrak{A}_\chi \supset J\mathfrak{A}'_\chi$.

Take $\eta \in \mathfrak{A}'_\chi$. By Lemma 5, η is of the form

$$\eta = \begin{pmatrix}\eta_1\\ b'\xi\\ \eta_3\\ d'\xi\end{pmatrix},$$

where $\eta_1, \eta_3 \in \mathfrak{B}'_\varphi$ and $b', d' \in \mathcal{M}'$ with

$$\pi_r^\chi(\eta) = \begin{pmatrix} \pi_r^\varphi(\eta_1) & 0 & b'|_{p'\mathcal{H}_\varphi} & 0 \\ 0 & \pi_r^\varphi(\eta_1)|_{p\mathcal{H}_\varphi} & 0 & b'|_{pp'\mathcal{H}_\varphi} \\ p'\pi_r^\varphi(\eta_3) & 0 & p'd'|_{p'\mathcal{H}_\varphi} & 0 \\ 0 & p'\pi_r^\varphi(\eta_3)|_{p\mathcal{H}_\varphi} & 0 & p'd'|_{pp'\mathcal{H}_\varphi} \end{pmatrix}.$$

We claim that $J\eta \in \mathfrak{B}_\chi$. Since $J_\varphi \xi = \xi \in \mathcal{P}_\varphi^\natural$, we have

$$J\eta = \begin{pmatrix} J_\varphi \eta_1 \\ J_\varphi \eta_3 \\ J_\varphi b'\xi \\ J_\varphi d'\xi \end{pmatrix} = \begin{pmatrix} J_\varphi \eta_1 \\ J_\varphi \eta_3 \\ b\xi \\ d\xi \end{pmatrix},$$

where $b = J_\varphi b' J_\varphi \in \mathcal{M}$ and $d = J_\varphi d' J_\varphi \in \mathcal{M}$. Since $J_\varphi \eta_1, J_\varphi \eta_3 \in J_\varphi \mathfrak{B}'_\varphi = \mathfrak{B}_\varphi$, we have $J\eta \in \mathfrak{B}_\chi$ with

$$\pi_l^\chi(J\eta) = \begin{pmatrix} \pi_l^\varphi(J_\varphi\eta_1) & b|_{p\mathcal{H}_\varphi} & 0 & 0 \\ \pi_l^\varphi(J_\varphi\eta_3) & d|_{p\mathcal{H}_\varphi} & 0 & 0 \\ 0 & 0 & \pi_l^\varphi(J_\varphi\eta_1)|_{p'\mathcal{H}_\varphi} & b|_{pp'\mathcal{H}_\varphi} \\ 0 & 0 & \pi_l^\varphi(J_\varphi\eta_3)|_{p'\mathcal{H}_\varphi} & d|_{pp'\mathcal{H}_\varphi} \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} J\pi_r^\chi(\eta)J &= \begin{pmatrix} J_\varphi\pi_r^\varphi(\eta_1)J_\varphi & J_\varphi b' J_\varphi|_{p\mathcal{H}_\varphi} & 0 & 0 \\ J_\varphi\pi_r^\varphi(\eta_3)J_\varphi & J_\varphi d' J_\varphi|_{p\mathcal{H}_\varphi} & 0 & 0 \\ 0 & 0 & J_\varphi\pi_r^\varphi(\eta_1)J_\varphi|_{p'\mathcal{H}_\varphi} & J_\varphi b' J_\varphi|_{pp'\mathcal{H}_\varphi} \\ 0 & 0 & J_\varphi\pi_r^\varphi(\eta_3)J_\varphi|_{p'\mathcal{H}_\varphi} & J_\varphi d' J_\varphi|_{pp'\mathcal{H}_\varphi} \end{pmatrix} \\ &= \pi_l^\chi(J\eta). \end{aligned}$$

Replacing η by $\eta^\flat$, we have, $J\eta^\flat \in \mathfrak{B}_\chi$ with

$$\pi_l^\chi(J\eta^\flat) = J\pi_r^\chi(\eta^\flat)J = (J\pi_r^\chi(\eta)J)^*.$$

By Lemma 4(1), $J\eta \in \mathcal{D}(\overline{S_\chi})$ and $(J\eta)^\sharp = J\eta^\flat$. Hence $J\eta \in \mathfrak{A}_\chi$ and we see that $J\mathfrak{A}'_\chi \subset \mathfrak{A}_\chi$. □

We have reached the main theorem, whose proof is a modification of [1, Theorem 1].

Theorem 7. $J = J_\chi$.

Proof. First, for

$$s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{n}_\chi \cap \mathfrak{n}_\chi^*$$

with

$$\eta_\chi(s) = \begin{pmatrix} \Lambda_\varphi(a) \\ \Lambda_\varphi(c) \\ b\xi \\ d\xi \end{pmatrix} \in \mathfrak{A}_\chi,$$

we have

$$(\eta_\chi(s^*)|J\eta_\chi(s)) = \left(\begin{pmatrix} \Lambda_\varphi(a) \\ \Lambda_\varphi(c) \\ b\xi \\ d\xi \end{pmatrix}^\sharp \middle| J \begin{pmatrix} \Lambda_\varphi(a) \\ \Lambda_\varphi(c) \\ b\xi \\ d\xi \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} \Lambda_\varphi(a^*) \\ \Lambda_\varphi(b^*) \\ c^*\xi \\ d^*\xi \end{pmatrix} \middle| \begin{pmatrix} J_\varphi\Lambda_\varphi(a) \\ J_\varphi b\xi \\ J_\varphi\Lambda_\varphi(c) \\ J_\varphi d\xi \end{pmatrix} \right)$$

$$= (\Lambda_\varphi(a^*)|J_\varphi\Lambda_\varphi(a)) + (\Lambda_\varphi(b^*)|J_\varphi b\xi) + (c^*\xi|J_\varphi\Lambda_\varphi(c)) + (d^*\xi|J_\varphi d\xi)$$

$$= (\Delta_\varphi^{1/2}\Lambda_\varphi(a)|\Lambda_\varphi(a)) + \overline{(b^*J_\varphi\Lambda_\varphi(b^*)|\xi)} + (\xi|cJ_\varphi\Lambda_\varphi(c)) + (\xi|dJ_\varphi d\xi).$$

Since $\xi,\ b^*J_\varphi\Lambda_\varphi(b^*),\ cJ_\varphi\Lambda_\varphi(c)$ and $dJ_\varphi d\xi$ all belong to $\mathcal{P}_\varphi^\natural$, we conclude that

$$(\eta_\chi(s^*)|J\eta_\chi(s)) \geq 0.$$

Now, we define

$$T\eta_\chi(s) = J\eta_\chi(s^*),\ s \in \mathfrak{n}_\chi \cap \mathfrak{n}_\chi^*.$$

By the above calculations, we find that T is positive symmetric, densely defined linear operator. It is readily seen that

$$T = JS_\chi.$$

Since J preserves the norm, we have

$$\overline{T} = J\overline{S_\chi}$$

and $\mathcal{D}(\overline{T}) = \mathcal{D}(\overline{S_\chi}) = \mathcal{D}(\Delta_\chi^{1/2})$. We will show that $\overline{T}$ is self-adjoint (then $\overline{T}$ is positive self-adjoint). Since T is symmetric, we have $\overline{T} \subset T^*$. Since

$$T^* = S_\chi^* J,$$

$\mathfrak{A}'_\chi$ is a core for S_χ^* and since $J\mathfrak{A}_\chi = \mathfrak{A}'_\chi$ (by Lemma 6), we see that $\mathfrak{A}_\chi$ is a core for T^*. On the other hand, $\mathfrak{A}_\chi$ is a core also for $\overline{T}$. Hence $\overline{T} = T^*$. Hence $\overline{T}$ and $\Delta_\chi^{1/2}$ are both positive self-adjoint. Then we have

$$\overline{T}^2 = T^*\overline{T} = S_\chi^* J J \overline{S_\chi} = \Delta_\chi.$$

By the uniqueness of the positive square root, we have

$$\overline{T} = \Delta_\chi^{1/2}.$$

On the other hand, $\overline{T} = JJ_\chi\Delta_\chi^{1/2}$. Hence we have $J = J_\chi$. □

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Mathematical Institute, Tohoku University, Aza-Aoba, Aramaki, Aoba-ku, Sendai 980-8578, Japan
E-mail address: h-izumi@math.tohoku.ac.jp