

SERRE'S CONDITION R_k FOR ASSOCIATED GRADED RINGS

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ABSTRACT. A criterion is given for when the associated graded ring of an ideal satisfies Serre's condition R_k . As an application, the integrality and quasi-Gorensteinness of such rings is investigated.

1. INTRODUCTION

This note grew out of an attempt to better understand what it means for an ideal I to have an associated graded ring $gr_I(R)$ satisfying Serre's condition R_k . We were inspired by a result of Huneke which essentially says that this condition holds if the local analytic spreads of I are "sufficiently small" ([7]). In our main result we are going to prove that at least for ideals of finite projective dimension, Huneke's upper bounds for the local analytic spreads of I are actually implied by the R_k property of $gr_I(R)$ (Theorem 2.4). Combining both facts one obtains a characterization for when the associated graded ring satisfies R_k (Corollary 2.6). Another immediate consequence of our main theorem is a rather surprising result by Huneke, Simis, and Vasconcelos to the effect that if the associated graded ring of a prime ideal of finite projective dimension is reduced, then it is already a domain ([8]) (Corollary 3.1). We use this fact in turn to investigate the connection between the reducedness and the quasi-Gorenstein property of associated graded rings (Theorem 3.2), a theme that goes back to earlier work by Hochster and by Herzog, Simis, and Vasconcelos ([6], [4], [5]).

2. THE PROPERTY R_k

If S is a positively graded ring, we write $S_+ = \bigoplus_{i>0} S_i$ and denote by $\text{Proj}(S)$ the set of all homogeneous prime ideals of S not containing S_+ . We say that S satisfies Serre's condition R_k on a subset X of $\text{Proj}(S)$ if for every $p \in X$ with $\dim S_p \leq k$, S_p or, equivalently, the homogeneous localization $S_{(p)}$ (as in [1, p. 30]) is a regular ring.

Let R be a Noetherian ring, I an R -ideal, and t a variable. We will consider the Rees algebra $\mathcal{R} = R[It]$ and the extended Rees algebra $T = R[It, t^{-1}]$ of I , as well as the associated graded ring $G = gr_I(R)$. Notice that $G \cong \mathcal{R}/I\mathcal{R} \cong T/(t^{-1})$ with t^{-1} a homogeneous T -regular element. If (R, m) is local and $I \neq R$, then the *analytic spread* $\ell(I)$ is defined to be the dimension of $G/mG \cong \mathcal{R}/m\mathcal{R}$. One always

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has $ht I \leq \ell(I) \leq \dim R$. In case R/m is infinite, $\ell(I)$ gives the minimal number of generators of any minimal reduction of I .

The next result, though easy to show, provides the crucial step in the proof of the main theorem.

Proposition 2.1. *Let (R, m) be a Noetherian local ring and let I be an R -ideal with $ht I > 0$ and $I \subset m^2$. If $G = gr_I(R)$ satisfies R_k on $\text{Proj}(G) \cap V(mG)$, then $\ell(I) \leq \dim R - k - 1$.*

Proof. Write $d = \dim R$. Since $\dim G/mG = \ell(I) \geq ht I > 0$, there exists a homogeneous prime ideal q of G with $mG \subset q$ and $\dim G_q \leq d - \ell(I) < d$. Thus, $q \in \text{Proj}(G) \cap V(mG)$ with $\dim G_q \leq d - \ell(I)$. Now suppose that the asserted inequality does not hold, then $d - \ell(I) \leq k$ and hence G_q is regular, say of dimension s .

Let Q be the preimage of q in $\mathcal{R} = R[It]$, and let x_1, \dots, x_s be elements of $Q\mathcal{R}_Q$ whose images in G_q form a regular system of parameters. Then $Q\mathcal{R}_Q = (I, x_1, \dots, x_s)\mathcal{R}_Q$. But $I \subset m^2 \subset Q^2\mathcal{R}_Q$, and hence $Q\mathcal{R}_Q = (x_1, \dots, x_s)\mathcal{R}_Q$ by Nakayama's Lemma. Thus, \mathcal{R}_Q would be a regular local ring of dimension s as well. But then $I\mathcal{R}_Q = 0$, and hence I would be contained in a minimal prime of \mathcal{R} , which is impossible since $ht I > 0$ ([10, p. 121]). \square

To continue we need two lemmas which are essentially well known.

Lemma 2.2. *Let R be a Noetherian ring, I an R -ideal, $G = gr_I(R)$, x a superficial element for I , $x' = x + I^2 \in [G]_1$, and $\bar{I} = I/(x) \subset \bar{R} = R/(x)$. Then the natural epimorphism of graded G -modules*

$$G/(x') \twoheadrightarrow gr_{\bar{I}}(\bar{R})$$

is an isomorphism on $\text{Proj}(G)$.

Proof. By the Artin-Rees Lemma and the definition of superficial elements ([11, p. 72]), there is an integer k so that for $n \gg 0$,

$$I^n \cap (x) = I^n \cap xI^{n-k} = x((I^n : (x)) \cap I^{n-k}) = xI^{n-1}.$$

Thus, $\ker(G/(x') \twoheadrightarrow gr_{\bar{I}}(\bar{R}))$ is concentrated in finitely many degrees, and hence is annihilated by some power of G_+ . \square

Lemma 2.3 ([11, 27.5], [9, p. 130]). *Let (R, m) be a Noetherian local ring, let $x \notin m^2$ be an R -regular element, and let M be a finitely generated R -module annihilated by x . If M has finite projective dimension over R , then M has finite projective dimension over $R/(x)$.*

Proof. By passing to a sufficiently high syzygy module of M over $\bar{R} = R/(x)$, we may assume that $\text{projdim}_R M = 1$. Now $M \cong F/H$, where $H \subset F$ are free R -modules of rank n with $xF \subset H \subset mF$. There exists a basis $\{e_1, \dots, e_n\}$ of F so that $\{xe_1, \dots, xe_k\}$ form part of a basis of H and $\{xe_{k+1}, \dots, xe_n\} \subset mH \subset m^2F$, for some integer k with $0 \leq k \leq n$. Now $k = n$ since otherwise $x \in m^2$. Hence, $H = xF$, which means that M is a free \bar{R} -module. \square

We are now ready to prove the main result.

Theorem 2.4. *Let R be a Noetherian ring, let I be an R -ideal of finite projective dimension, and write $G = gr_I(R)$. If G satisfies R_k on $\text{Proj}(G)$, then $\ell(I_p) \leq \max\{ht I_p, \dim R_p - k - 1\}$ for every $p \in V(I)$.*

Proof. Localizing at $p \in V(I)$ we may assume that (R, m) is local. We are going to prove by induction on $g = \text{grade } I$ that $\ell(I) \leq \max\{ht I, \dim R - k - 1\}$.

If $g = 0$, then $I = 0$ since $\text{proj dim}_R R/I < \infty$, and the assertion follows. So let $g > 0$. By Proposition 2.1 we may assume that $I \not\subset m^2$. Choose generators f_1, \dots, f_n of I , let Z_1, \dots, Z_n be variables, and write $\tilde{R} = R(Z_1, \dots, Z_n)$, $\tilde{m} = m\tilde{R}$, $\tilde{I} = I\tilde{R}$, and $\tilde{G} = G \otimes_R \tilde{R}$. Further, set $x = \sum_{i=1}^n Z_i f_i$, $x' = x + \tilde{I}^2 \in [\tilde{G}]_1$, and $\bar{I} = \tilde{I}/(x) \subset \bar{R} = \tilde{R}/(x)$. Since $I \not\subset m^2$ and $g > 0$, it follows that $x \notin \tilde{m}^2$ is \tilde{R} -regular, x being a generic element for I . Thus, $\text{proj dim}_{\bar{R}} \bar{R}/\bar{I} < \infty$ by Lemma 2.3. On the other hand, x' is a generic element for G_+ . Therefore, x is a superficial element of \tilde{I} , and $\tilde{G}/(x')$ still satisfies R_k on $\text{Proj}(\tilde{G}/(x'))$ since for every $q \in \text{Proj}(\tilde{G}/(x'))$, $(\tilde{G}/(x'))_q$ is the localization of a polynomial ring over G . Hence, by Lemma 2.2, $gr_{\bar{I}}(\bar{R})$ satisfies R_k on $\text{Proj}(gr_{\bar{I}}(\bar{R}))$. Now the induction hypothesis yields $\ell(\bar{I}) \leq \max\{ht \bar{I}, \dim \bar{R} - k - 1\} = \max\{ht I, \dim R - k - 1\} - 1$.

It remains to show the inequality $\ell(I) - 1 \leq \ell(\bar{I})$. Indeed, writing $K = \tilde{R}/\tilde{m}$ and using the convention $\dim \emptyset = -1$, one concludes from Lemma 2.2 that

$$\begin{aligned} \ell(\bar{I}) &= \dim \text{Proj}(gr_{\bar{I}}(\bar{R}) \otimes_{\bar{R}} K) + 1 = \dim \text{Proj}(\tilde{G}/(x') \otimes_{\tilde{R}} K) + 1 \\ &= \dim(\tilde{G} \otimes_{\tilde{R}} K)/(x') \geq \ell(I) - 1. \end{aligned}$$

□

The assumption of I having finite projective dimension cannot be deleted in the above theorem even if the ideal is generically a complete intersection. This can be seen by taking $R = k[X, Y, U, V]/(XU - YV)$, k a field, and $I = (X, Y)R$; notice that $gr_I(R) \cong R$ satisfies R_2 , whereas $\ell(I_m) = 2$, with m denoting the irrelevant maximal ideal.

The inequalities that appear in Theorem 2.4 have an easy dimension theoretic interpretation (see also [7, the proof of Proposition 2.1]).

Remark 2.5. Let R be a Noetherian ring that is locally equidimensional and universally catenary, let I be an R -ideal, and write $G = gr_I(R)$. The following are equivalent:

- (a) $\ell(I_p) \leq \dim R_p - k - 1$ for every $p \in V(I)$ with $\dim(R/I)_p \geq k + 1$.
- (b) For every $q \in \text{Spec}(G)$ with $\dim G_q \leq k$ one has $\dim(R/I)_p \leq k$, where p is the contraction of q .

Proof. Localizing at $p \in V(I)$ we may assume that (R, m) is local with $\dim R/I \geq k + 1$. It suffices to show that $\ell(I) \leq \dim R - k - 1$ if and only if $\dim G_q \geq k + 1$ for every $q \in V(mG)$. This amounts to proving that $\dim G/mG \leq \dim G - k - 1$ if and only if $ht mG \geq k + 1$. But indeed, $\dim G = ht mG + \dim G/mG$, because $G \cong T/(t^{-1})$ with t^{-1} a homogeneous regular element and $T = R[It, t^{-1}]$ a graded ring that has a unique maximal homogeneous ideal and is equidimensional and catenary ([10, pp. 121–122]). □

In [7], Huneke proved the following result: Let R be a homomorphic image of a regular domain, let I be a prime ideal that is generically a complete intersection, and assume that either R is a Cohen-Macaulay Nagata domain or that $G = gr_I(R)$ is Cohen-Macaulay. Further, suppose that $\ell(I_p) \leq \max\{ht I, \dim R_p - k - 1\}$ for every $p \in V(I)$. Then G satisfies R_k if and only if R/I does.

Combining his method of proof with Theorem 2.4 one obtains the following characterization:

Corollary 2.6. *Let R be a Cohen-Macaulay ring and let I be an R -ideal of finite projective dimension that is a complete intersection locally at each of its minimal primes. Then $gr_I(R)$ satisfies R_k if and only if R/I satisfies R_k and $\ell(I_p) \leq \max\{ht I_p, \dim R_p - k - 1\}$ for every $p \in V(I)$.*

Proof. By Theorem 2.4 we may assume that in either case,

$$\ell(I_p) \leq \max\{ht I_p, \dim R_p - k - 1\}$$

for every $p \in V(I)$. It remains to show that $G = gr_I(R)$ satisfies R_k if and only if R/I has this property. By Remark 2.5, every prime q of G with $\dim G_q \leq k$ contracts to a prime $p \in V(I)$ with $\dim(R/I)_p \leq k$. Thus, localizing at $p \in V(I)$, we may assume that R is a local ring with $\dim R/I \leq k$. Now by our assumption on the local analytic spreads, $\ell(I) = ht I$. Hence, I is a complete intersection, because R is Cohen-Macaulay and I is a complete intersection locally at each of its minimal primes ([2]). Therefore, G is a polynomial ring over R/I and thus, satisfies R_k if and only if R/I does. \square

3. INTEGRALITY AND GORENSTEINNESS

Another immediate consequence of our main result is the following theorem of Huneke, Simis, and Vasconcelos about the minimal primes of associated graded rings ([8, 1.2]):

Corollary 3.1. *Let R be a Noetherian ring that is locally equidimensional and universally catenary, let I be an R -ideal of finite projective dimension that is a complete intersection locally at each of its minimal primes, and write $G = gr_I(R)$. If G satisfies R_0 on $\text{Proj}(G)$, then the natural map $\text{Spec}(G) \rightarrow \text{Spec}(R/I)$ yields a one-to-one correspondence between the sets of minimal primes $\text{Min}(G)$ and $\text{Min}(R/I)$.*

Proof. By Theorem 2.4 and Remark 2.5, every minimal prime of G contracts to a minimal prime of R/I , thus giving a well-defined map $\text{Min}(G) \rightarrow \text{Min}(R/I)$. This map is bijective since for every $p \in \text{Min}(R/I)$, $G \otimes_R k(p)$ is a polynomial ring over the residue field $k(p)$. \square

Corollary 3.1 implies that if I is an ideal of finite projective dimension in a ring R as above and if the associated graded ring $G = gr_I(R)$ is reduced, then G is torsionfree as a module over R/I ; in particular, G is a domain in case I is prime ([8, 1.1]).

We now turn to the question of whether the reducedness of the associated graded ring implies that this ring or the extended Rees algebra is (quasi)-Gorenstein (cf. also [6], [4], [5]). Recall that a Noetherian ring S is said to be quasi-Gorenstein if for every maximal ideal m of S , S_m is the canonical module of S_m . In case S happens to be a Noetherian graded ring with a unique maximal homogeneous ideal, then S is quasi-Gorenstein if and only if the graded canonical module ω_S of S exists and is of the form $\omega_S \cong S(-n)$ for some integer n (see [1, Section 3.6] for information about graded canonical modules). The next result has been shown by Herzog, Simis, and Vasconcelos under the additional assumptions that R is a normal Cohen-Macaulay ring, that $ht I \geq 2$, and that the Rees algebra of I is Cohen-Macaulay ([5, 4.2.3]). In their setting, the quasi-Gorensteinness of the extended Rees algebra simply means that the associated graded ring is Gorenstein.

Theorem 3.2. *Let R be a quasi-Gorenstein ring that is an epimorphic image of a local Gorenstein ring. Let I be a proper R -ideal of finite projective dimension and assume that $gr_I(R)$ is reduced. Then $R[It, t^{-1}]$ is quasi-Gorenstein if and only if I is unmixed.*

Proof. Write $G = gr_I(R)$, $T = R[It, t^{-1}]$, and $\omega = \omega_T$. Notice that T is a graded ring with a unique maximal homogeneous ideal and that ω is a graded T -module. Furthermore, the T -regular element t^{-1} is homogeneous of degree -1 , and $T/(t^{-1}) \cong G$. This yields a homogeneous embedding $\omega \otimes_T G \hookrightarrow \omega_G(1)$, which is an isomorphism whenever T or, equivalently, G is Cohen-Macaulay. The local ring R is universally catenary, and, being quasi-Gorenstein, it is S_2 , hence equidimensional ([3, 2.4.1]). Thus, T and G are equidimensional, and therefore ω and ω_G localize. Furthermore, if p is a minimal prime of I of height g , then $IR_p = pR_p$ is a complete intersection, since I is reduced and has finite projective dimension. Hence, $G \otimes_R R_p$ is a polynomial ring in g variables over a field. Thus, $\omega_{G \otimes_R R_p} \cong G \otimes_R R_p(-g)$ and then, by the graded Nakayama Lemma, $\omega \otimes_R R_p \cong \omega_{T \otimes_R R_p} \cong T \otimes_R R_p(-g+1)$ is generated in degree $g-1$.

Now if T is quasi-Gorenstein, then $\omega \cong T(-n)$ for some integer n . Hence, for every minimal prime p of I of height g , $T \otimes_R R_p(-g+1) \cong \omega \otimes_R R_p \cong T \otimes_R R_p(-n)$. But the maximal homogeneous ideal of T is a maximal ideal; thus $-g+1 = -n$ ([1, 1.5.16]), showing that $ht\ p$ does not depend on the choice of p .

Conversely, assume that I is unmixed of height g . Notice that $T_{t^{-1}} \cong R[t, t^{-1}]$ is a graded quasi-Gorenstein ring with a unique maximal homogeneous ideal, and thus $\omega \otimes_T T_{t^{-1}} \cong \omega_{R[t, t^{-1}]} \cong R[t, t^{-1}](-1) \cong R[t, t^{-1}]$. We make the identifications $\omega \subset \omega \otimes_T T_{t^{-1}} = R[t, t^{-1}]$. Hence, $[\omega]_i = Rt^i$ for every integer $i \ll 0$, and whenever $[\omega]_i = Rt^i$, then $[\omega]_i$ generates $\omega \otimes_T T_{t^{-1}}$ as a module over $T_{t^{-1}}$.

Now write $A = R/I$ and $K = \text{Quot}(A)$. By Corollary 3.1 and the remark following it, G is torsionfree over A , hence ω_G has the same property, and thus $\omega_G \hookrightarrow \omega_G \otimes_A K \cong \omega_{G \otimes_A K}$. By the above, $\omega_{G \otimes_A K}$ is concentrated in degrees $\geq g$, and therefore the same holds for $\omega \otimes_T G(-1) \hookrightarrow \omega_G$. Hence, $[\omega]_{i-1} = [\omega]_i t^{-1}$ whenever $i \leq g-1$, showing that $[\omega]_{g-1} = Rt^{g-1}$ generates $\omega \otimes_T T_{t^{-1}}$ as a module over $T_{t^{-1}}$.

We claim that $[\omega]_{g-1} = Rt^{g-1}$ generates ω as a T -module. Indeed, as a quasi-Gorenstein ring, $T_{t^{-1}}$ satisfies S_2 . On the other hand, $T/(t^{-1}) \cong G$ is S_1 by reducedness. Therefore, T satisfies S_2 ([1, 2.2.33]). Consequently, it suffices to check the equality $[\omega]_{g-1}T = \omega$ locally at every $q \in \text{Spec}(T)$ with $\dim T_q \leq 1$. Since $[\omega]_{g-1}T_{t^{-1}} = \omega \otimes_T T_{t^{-1}}$, we may assume that q contains t^{-1} . But then $q/(t^{-1})$ is a minimal prime of G , and so by Corollary 3.1, $p = q \cap R$ is a minimal prime of I . Thus, $\omega \otimes_R R_p$ is generated in degree $g-1$, which yields the asserted equality $[\omega]_{g-1}T_q = \omega \otimes_T T_q$. \square

As before, Corollary 3.1 and Theorem 3.2 do not hold without the assumption of I having finite projective dimension ([8, Example 1.2] and [4, 1.2]).

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