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# ON THE DIMENSION OF ALMOST $n$-DIMENSIONAL SPACES

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Abstract. Oversteegen and Tymchatyn proved that homeomorphism groups of positive dimensional Menger compacta are 1-dimensional by proving that almost 0-dimensional spaces are at most 1-dimensional. These homeomorphism groups are almost 0-dimensional and at least 1-dimensional by classical results of Brechner and Bestvina. In this note we prove that almost $n$-dimensional spaces for $n \geq 1$ are $n$-dimensional. As a corollary we answer in the affirmative an old question of R. Duda by proving that every hereditarily locally connected, non-degenerate, separable, metric space is 1-dimensional.

## 1. Introduction

We consider only separable metric spaces. A space $X$ is said to be almost $n$-dimensional if it has a basis $\{U_i\}$ such that if $\operatorname{cl} U_i \cap \operatorname{cl} U_j = \emptyset$, then $X = G \cup H$ where $G$ and $H$ are closed sets, $U_i \subset G \setminus H$, $U_j \subset H \setminus G$ and $\dim G \cap H \leq n-1$ and $n$ is the smallest natural number such that such a basis exists for $n$. It is clear that $n$-dimensional spaces are at most almost $n$-dimensional. We shall prove that for $n \geq 1$ the converse is also true. We shall also prove that if $X = X_1 \cup X_2$ where $X_1$ is almost 0-dimensional and $X_2$ is 0-dimensional, then $\dim X \leq 1$.

A property equivalent to almost 0-dimensionality was first considered in [7]. The Erdös space of irrational sequences in Hilbert space is known to be a universal almost 0-dimensional space [5]. Erdös space is 1-dimensional.

In [7] Erdös space was used to construct a hereditarily locally connected space (i.e., a connected space all of whose connected subsets are locally connected) which is not rim-countable. In [8] it was proved that hereditarily locally connected spaces are at most 2-dimensional. This was a partial solution to a question of R. Duda. In this paper we answer Duda's question in the affirmative by proving that hereditarily locally connected spaces are at most 1-dimensional.

A subset $X$ of a compactum $K$ is $L$-embedded in $K$ if for every open cover $\mathcal{U}$ of $K$ there is a neighbourhood $U$ of $X$ in $K$ such that the continua in $U$ refine $\mathcal{U}$. An almost 0-dimensional space is $L$-embeddable in a compactum [6] and

**Theorem 1.1** (Levin-Pol, [6]). *If a space $X$ is $L$-embeddable in a compactum $K$, then* $\dim X \leq 1$.

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## 2. Almost $n$-dimensional spaces

Almost 0-dimensional spaces are at most 1-dimensional and the 1-dimensionality cannot be improved. Our first result shows that this interesting behavior does not occur in higher dimensions.

**Theorem 2.1.** *If $X$ is almost $n$-dimensional, $n \geq 1$, then $X$ is $n$-dimensional.*

*Proof.* Let $\mathcal{U} = \{U_i\}$ be a basis of open sets for $X$ which witnesses the almost $n$-dimensionality of $X$, i.e. if $\operatorname{cl} U_i \cap \operatorname{cl} U_j = \emptyset$ and $i < j$, then $X = G_{ij} \cup H_{ij}$ where $G_{ij}$ and $H_{ij}$ are closed sets, $U_i \subset G_{ij} \setminus H_{ij}$, $U_j \subset H_{ij} \setminus G_{ij}$ and $\dim G_{ij} \cap H_{ij} \leq n-1$. Let $X'$ be a metric compactification of $X$. Index $\{(U_i, U_j) : i < j \text{ and } \operatorname{cl} U_i \cap \operatorname{cl} U_j = \emptyset\}$ by $\{A_k\}_{k=1}^{\infty}$. If $A_k = (U_i, U_j)$, let $B_k = \operatorname{cl}_{X'} G_{ij}$ and $C_k = \operatorname{cl}_{X'} H_{ij}$. Form an inverse sequence as follows:

$X_0 = X'$,

$X_1 = B_1 \times \{0\} \cup C_1 \times \{1\} \subset X_0 \times 2$ and let $\pi_0^1 : X_1 \longrightarrow X_0$ be the natural projection.

If spaces $X_j$, $j = 0, ..., n$, and maps $\pi_j^i : X_i \longrightarrow X_j$ are defined for $j \leq i \leq n$, let $X_{n+1} = ((\pi_0^n)^{-1}(B_n) \times \{0\}) \cup ((\pi_0^n)^{-1}(C_n) \times \{1\}) \subset X_n \times 2 \subset X_0 \times 2^{n+1}$. Let $\pi_n^{n+1} : X_{n+1} \longrightarrow X_n$ be the natural projection and for $0 \leq j < n$ let $\pi_j^{n+1} : X_{n+1} \longrightarrow X_j$ be the map $\pi_j^n \circ \pi_n^{n+1}$.

Let $\hat{X}' = \varprojlim (X_n, \pi_j^n) \subset X' \times 2^{\omega}$. $\hat{X}'$ is a compactum. Let $\pi_i : \hat{X}' \longrightarrow X_i$ be the projection. Then $\pi_0 : \hat{X}' \longrightarrow X_0 = X'$ is 0-dimensional and onto. Let $\psi : \hat{X}' \longrightarrow 2^{\omega}$ be the natural projection.

Let $\hat{X} = \pi_0^{-1}(X) \subset \hat{X}' \subset X' \times 2^{\omega}$. We show $\hat{X}$ is $L$-embedded in $\hat{X}'$.

For each positive integer $n$ let $G_n = \bigcup\{U_j' = X' \setminus \operatorname{cl}_{X'}(X \setminus U_j) : U_j \in \mathcal{U}$ and $\operatorname{diam} U_j \leq 1/n\}$ where $\operatorname{diam} U_j$ is determined with respect to a metric in $X'$.

Then $G_n$ is open in $X'$ and $X \subset G_n$. Let $C$ be a continuum in $\pi_0^{-1}(G_n)$. Then $\psi(C)$ is a singleton and, hence, $\operatorname{diam} C = \operatorname{diam} \pi_0(C)$ for the product metric in $X' \times 2^{\omega}$. If $\operatorname{diam} \pi_0(C) > 3/n$, then there exist $U_i, U_j \in \mathcal{U}$ with $\operatorname{diam} U_i$, $\operatorname{diam} U_j < 1/n$, $i < j$, $\pi_0(C) \cap U_i' \neq \emptyset$, $\pi_0(C) \cap U_j' \neq \emptyset$ and $\operatorname{cl} U_i \cap \operatorname{cl} U_j = \emptyset$. Set $A_k = (U_i, U_j)$. Then $\pi_k(C)$ meets $X_{k-1} \times \{0\}$ and $X_{k-1} \times \{1\}$ since

$\pi_k(\pi_0^{-1}(U_i')) \subset \pi_k(\pi_0^{-1}(B_k \setminus C_k)) \subset X_{k-1} \times \{0\}$ and

$\pi_k(\pi_0^{-1}(U_j')) \subset \pi_k(\pi_0^{-1}(C_k \setminus B_k)) \subset X_{k-1} \times \{1\}$.

This is a contradiction as $\pi_k(C)$ is connected. Hence each continuum in $\pi_0^{-1}(G_n)$ has diam $\leq 3/n$ and by Theorem 1.1 $\hat{X}$ is at most 1-dimensional.

Let $K = \bigcup_k (B_k \cap C_k \cap X)$. Clearly $\dim K \leq n-1$. It is easy to see that for every $x \in X \setminus K$, $\pi_0^{-1}(x)$ is a singleton. Note that $\pi_0|_{\hat{X}} : \hat{X} \longrightarrow X$ is closed, 0-dimensional and onto. Hence by Vainstein's second theorem ([3], p. 245, Theorem 4.3.10) $\dim X \leq n$. Clearly $\dim X \geq n$ and we have $\dim X = n$. □

**Corollary 2.2.** *If $X$ is a hereditarily locally connected, non-degenerate space, then* $\dim X = 1$.

*Proof.* By [7], Theorem 7.4.1, each pair of disjoint, closed, connected subsets of $X$ can be separated by a closed countable subset of $X$. Hence each basis for $X$ of open connected sets witnesses the almost 1-dimensionality of $X$. By Theorem 2.1 $X$ is 1-dimensional. □

**Theorem 2.3.** *Let $X = X_1 \cup X_2$ where $X_1$ is almost* 0*-dimensional and $X_2$ is* 0*-dimensional. Then* $\dim X \leq 1$.

*Proof.* We may assume that $X_1$ is dense in $X$. Let $\mathcal{U} = \{U_i\}$ be a collection of open sets in $X$ such that $\{U_i \cap X_1\}$ is a basis of $X_1$ which witnesses the almost 0-dimensionality of $X_1$. Since $X_2$ is 0-dimensional each pair $(U_i, U_j)$ of $\mathcal{U}$ with $\operatorname{cl} U_i \cap \operatorname{cl} U_j = \emptyset$ can be separated by a 0-dimensional closed subset. We use the same notation and construction as in the proof of Theorem 2.1. The difference between our case and the proof of Theorem 2.1 is that $\mathcal{U}$ is not a basis of $X$. Therefore we need a more subtle approach to show that $\hat{X}$ is $L$-embedded in $\hat{X}'$.

$G_n$ covers $X_1$. Take a cover $\mathcal{V}_n$ of $X_2$ by open disjoint subsets of $X'$ with $\operatorname{diam} < 1/n$ and let $V_n = \bigcup\{V : V \in \mathcal{V}_n\}$. Let $C$ be a continuum in $\pi_0^{-1}(G_n \cup V_n)$.

If $\pi_0(C) \cap G_n = \emptyset$, then $\pi_0(C)$ is a subset of $V_n$ and clearly $\operatorname{diam} \pi_0(C) < 1/n$.

If $\pi_0(C) \cap G_n \neq \emptyset$, then by the reasoning of the proof of Theorem 2.1 we get that $\operatorname{diam} \pi_0(C) \cap G_n \leq 3/n$. As $V_n$ is the union of disjoint open sets $\pi_0(C) \subset O = (\bigcup\{V : V \in \mathcal{V}_n, V \cap \pi_0(C) \cap G_n \neq \emptyset\}) \cup (\pi_0(C) \cap G_n)$. Clearly $\operatorname{diam} O < 3/n + 2/n$. Thus, $\hat{X}$ is $L$-embedded in $\hat{X}'$. □

*Remark.* Note that the union of two almost 0-dimensional spaces fails to be of $\dim \leq 1$. Indeed, let $Y$ be 1-dimensional and almost 0-dimensional, let $M$ be a 1-dimensional compactum and let $M = M_1 \cup M_2$, $\dim M_1 = \dim M_2 = 0$. Then $X_1 = Y \times M_1$ and $X_2 = Y \times M_2$ are almost 0-dimensional, and by a theorem of Hurewicz [4] (see also [3], p. 78, 1.9.E(b)) $X = X_1 \cup X_2 = Y \times M$ is 2-dimensional.

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