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A NON-METRIZABLE COMPACT LINEARLY ORDERED TOPOLOGICAL SPACE, EVERY SUBSPACE OF WHICH HAS A σ-MINIMAL BASE

WEI-XUE SHI

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Abstract. A collection $\mathcal{D}$ of subsets of a space is minimal if each element of $\mathcal{D}$ contains a point which is not contained in any other element of $\mathcal{D}$. A base of a topological space is σ-minimal if it can be written as a union of countably many minimal collections. We will construct a compact linearly ordered space X satisfying that X is not metrizable and every subspace of X has a σ-minimal base for its relative topology. This answers a problem of Bennett and Lutzer in the negative.

1. Introduction

The concept of σ-minimal bases was introduced by Aull in [1] and it was pointed out that every quasi-developable space has a σ-minimal base. Bennett and others proved that a space X with a σ-minimal base need not be quasi-developable even if every subspace of X has a σ-minimal base (cf. [2] and [3]). On the other hand, the condition compactness forces a quasi-developable space to be metrizable, but a compact space with a σ-minimal base need not be metrizable even if the space is a linearly ordered topological space (LOTS) [3]. It is observed that the space constructed in [3] has a subspace which has no σ-minimal base. Recently Bennett and Lutzer constructed a non-metrizable LOTS such that every subspace of it has a σ-minimal base for its relative topology, but the LOTS is not itself compact [5]. So the following question posed by Bennett and Lutzer (cf. [3], [4], [6] and [9]) becomes more interesting.

Problem 1. Suppose that X is a compact linearly ordered topological space and suppose that every subspace of X has a σ-minimal base for its relative topology. Must X be metrizable?

In this paper, we will answer this problem negatively by constructing a non-metrizable compact LOTS X such that every subspace of X has a σ-minimal base for its relative topology.

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Recall that a *LOTS* (a *linearly ordered topological space*) is a triple $\langle X, \lambda, \leqslant\rangle$, where $\langle X, \leqslant\rangle$ is an ordered set and λ is the interval topology on $\langle X, \leqslant\rangle$ and a *GO-space* (a *generalized ordered space*) is a triple $\langle X, \tau \leqslant\rangle$, where τ is a topology on $\langle X, \leqslant\rangle$ which is T_1 and has a local base consisting of ordered convex sets at every point of X. A collection $\mathcal{D}$ of subsets of a space is said to be *minimal* if for every proper subcollection $\mathcal{D}'$ of $\mathcal{D}$, $\bigcup \mathcal{D}' \subsetneqq \bigcup \mathcal{D}$. A base $\mathcal{B}$ of a space is called a *σ-minimal base* if $\mathcal{B} = \bigcup\{\mathcal{B}_n \mid n \in \omega_0\}$, where for each $n \in \omega_0$, $\mathcal{B}_n$ is minimal.

We use an Aronszajn tree to construct our LOTS satisfying the required conditions. Now we review some related definitions and results.

A *tree* is a partially ordered set $\langle T, \leqslant_T\rangle$, simply written as T, such that for every $t \in T$ the set $(\cdot, t)_T = \{s \in T \mid s <_T t\}$ is well-ordered. The *height* $\mathrm{ht}_T(t)$ of t in $\langle T, \leqslant_T\rangle$ is the order type of $(\cdot, t)_T$. The *αth level* of T is the set $T_\alpha = \{t \in T \mid \mathrm{ht}_T(t) = \alpha\}$. The *height* $\mathrm{ht}(T)$ of T is the ordinal $\min\{\alpha \mid T_\alpha = \emptyset\}$. A *chain* of a tree T is a totally ordered subset of T. A *branch* of a tree T is a maximal chain of T. If x is a branch of a tree, then we denote the order type of x by $\mathrm{bht}(x)$. An *antichain* of a tree T is a set of pairwise incomparable points of T. A *path* p of T is a chain such that for each $t \in p$, $(\cdot, t)_T \subset p$. We use $\mathrm{ht}(p)$ to denote the order type of a path p. A *node* of a tree T is any equivalence class of the relation $\sim$ defined on T by $s \sim t$ if and only if $(\cdot, s)_T = (\cdot, t)_T$. Obviously each level is an antichain and a disjoint union of nodes. Especially T_0 is a node since for any $t, s \in T_0$, $(\cdot, s)_T = (\cdot, t)_T = \emptyset$.

Let T be a tree and let $\mathcal{N}(T)$ be the set of all nodes of T. If p is a bounded path of T, let N_p be the first level of the tree

$$\{t \in T \mid s <_T t \text{ for every } s \in p\}.$$

Then $N_p \in \mathcal{N}(T)$.

Let B_T be the set of all branches of T. Suppose that each $N \in \mathcal{N}(T)$ is endowed with a linear ordering $\leqslant_N$. Then the lexicographical ordering $\preccurlyeq$ on B_T induced by $\{\leqslant_N \mid N \in \mathcal{N}(T)\}$ is defined by

$$l \preccurlyeq m \text{ if and only if } l_N \leqslant_N m_N,$$

where $N = N_{l\cap m}$, $\{l_N\} = l \cap N$ and $\{m_N\} = m \cap N$. Then $\preccurlyeq$ is a linear ordering on B_T. For every $t \in T$, let $B_t = \{m \in B_T \mid t \in m\}$. Then it is easy to see that B_t is a convex set in $\langle B_T, \preccurlyeq\rangle$.

It is known that, if $\langle N, \leqslant_N\rangle$ is a complete linear ordering for each $N \in \mathcal{N}(T)$, then $\langle B_T, \preccurlyeq\rangle$ is also complete (see [10, Proposition 2.5]). For an ordinal α, by α^+ we mean the successor of α, for a subset Y of B_T, let $Y\lceil\alpha$ denote the set $\{x \in Y \mid \mathrm{bht}(x) < \alpha\}$, and for an $x \in B_T$, let $x\lceil\alpha$ denote the set $\{t \in x \mid \mathrm{ht}_T(t) < \alpha\}$. A tree T is called an *Aronszajn tree* if $\mathrm{ht}\,(T) = \omega_1$ and each branch and each level of it are countable. A tree T is said to be *special* if T is the union of countably many antichains. It is well-known that there is a special Aronszajn tree in ZFC (see [10, Theorem 5.2]).

For a space X and its subset Y, the interior and closure of Y in X are denoted by $\mathrm{int}_X\, Y$ and $\mathrm{cl}_X\, Y$ respectively. For an ordered space X and a subspace Y of X, if $a, b \in Y$ and $a < b$, by $(a, b)_Y$ we mean the open interval taken in Y. Define the intervals $[a, b]_Y$, $(a, b]_Y$, $[a, b)_Y$ analogously. For undefined terminology we refer to [7] and [10].

2. A construction of a LOTS and some lemmas

Let T be the Aronszajn tree constructed as in [10, Theorem 5.2]. T is special since T is $\mathbb{Q}$-embeddable. Thus $T = \bigcup\{A_n \mid n \in \omega_0\}$, where each A_n is an antichain of T. From the construction of T it is easy to see that the following facts are true.

Fact 1. For each $N \in \mathcal{N}(T)$ with $N \subset T_\alpha$, $|N| = \omega_0$ if α is a successor ordinal or $\alpha = 0$, and $|N| = 1$ if α is a limit ordinal and $\alpha \neq 0$.

Fact 2. No branch of T has a maximum element.

Suppose $N \in \mathcal{N}(T)$. If $N \subset T_\alpha$ for some successor ordinal α or $\alpha = 0$, define a linear ordering $\leqslant_N$ on N such that $\leqslant_N$ well-orders N and $\langle N, \leqslant_N\rangle$ has the order type $\omega_0 + 1$. Then we may write $N = \{a(N)_n \mid n \in \omega_0 + 1\}$ such that

$$a(N)_0 <_N a(N)_1 <_N \ldots <_N a(N)_n <_N \ldots <_N a(N)_{\omega_0}.$$

If $N \subset T_\alpha$ for some limit ordinal $\alpha > 0$, let $\leqslant_N$ be the trivial ordering on N. Then for every $N \in \mathcal{N}(T)$, $\langle N, \leqslant_N\rangle$ is complete. Let $\preccurlyeq$ be the lexicographical ordering on B_T induced by $\{\leqslant_N \mid N \in \mathcal{N}(T)\}$ and let λ be the interval topology on $\langle B_T, \preccurlyeq\rangle$. Then $\langle B_T, \lambda, \preccurlyeq\rangle$ is a compact LOTS since it is complete and has maximum and minimum points. We will simply denote $\langle B_T, \lambda, \preccurlyeq\rangle$ by B_T. For an $x \in B_T$, by Fact 2, $\mathrm{bht}(x)$ is a limit ordinal. For $\alpha < \mathrm{bht}(x)$, let $N(x, \alpha) = N_{x \restriction \alpha}$. Put

$$\begin{aligned} B_{T0} = \big\{x \in B_T \mid {} & \text{there is } \beta < \mathrm{bht}(x) \text{ such that, for each } \alpha \\ & \text{with } \beta < \alpha^+ < \mathrm{bht}(x),\ x \cap N(x, \alpha^+) = \{a\big(N(x, \alpha^+)\big)_0\}\big\}, \\ B_{T1} = \big\{x \in B_T \mid {} & \text{there is } \beta < \mathrm{bht}(x) \text{ such that, for each } \alpha \\ & \text{with } \beta < \alpha^+ < \mathrm{bht}(x),\ x \cap N(x, \alpha^+) = \{a\big(N(x, \alpha^+)\big)_{\omega_0}\}\big\}, \end{aligned}$$

and

$$B_{T2} = B_T - (B_{T1} \cup B_{T0}).$$

That is, each branch $x \in B_{T1}$ always picks up the maximum point in each node which x meets at levels $> \beta$ and each branch $x \in B_{T0}$ always picks up the minimum point in each node which x meets at levels $> \beta$. For $x \in B_{Ti},\ i = 0, 1$, let

$$\eta_x = \min\big\{\beta < \mathrm{bht}(x) \mid \text{for each } \alpha \text{ with } \beta < \alpha^+ < \mathrm{bht}(x),\ x \cap N(x, \alpha^+) = \{a\big(N(x, \alpha^+)\big)_{\rho(i)}\}\big\}$$

where $\rho(i) = \begin{cases} 0, & \text{if } i = 0, \\ \omega_0, & \text{if } i = 1, \end{cases}$

and put

$$\begin{aligned} B_{Ti}^{suc} &= \{x \in B_{Ti} \mid \eta_x \text{ is a successor ordinal or } \eta_x = 0\}, \\ B_{Ti}^{lim} &= \{x \in B_{Ti} \mid \eta_x \text{ is a limit ordinal and } \eta_x \neq 0\}, \end{aligned}$$

and

$$B_{T0}^{suc1} = \big\{x \in B_{T0}^{suc} \mid x \cap N(x, \eta_x) = \{a\big(N(x, \eta_x)\big)_{\omega_0}\}\big\}.$$

Thus each branch $x \in B_{T0}^{suc1}$ picks up the maximum point in the node which x meets at the η_xth level and always picks up the minimum point in the nodes which x meets above the η_xth level.

Since $\mathrm{ht}(T) = \omega_1$, it is easy to see the following fact.

Fact 3. For $i = 0, 1$, $|B_{Ti}| = \omega_1$ and if $\alpha < \omega_1$ $|B_{Ti} \restriction \alpha| = \omega_0$.

Suppose $t \in T_\alpha$. By the definition of the ordering $\preccurlyeq$, the point x in $B_t \cap B_{T1}$ with $\eta_x \leqslant \alpha$ is the maximum point of B_t and the point x in $B_t \cap B_{T0}$ with $\eta_x \leqslant \alpha$ is the minimum point of B_t. So we have

Fact 4. For each $t \in T$, B_t is a closed interval in B_T, the maximum point of B_t is in B_{T1} and the minimum point of B_t is in B_{T0}.

The following is an extension of the proof of Theorem 5.1 in [10].

Lemma 1. *$\langle B_T, \lambda, \preccurlyeq\rangle$ is a first countable space.*

Proof. Suppose that B_T is not first countable. Then there is an $x \in B_T$ such that x has no countable neighborhood base. So there is an increasing sequence $\{a_\alpha \mid \alpha \in \omega_1\}$ homeomorphic to ω_1 or decreasing sequence $\{b_\alpha \mid \alpha \in \omega_1\}$ homeomorphic to the converse of ω_1 in B_T. For instance, there is an increasing sequence $A = \{a_\alpha \mid \alpha \in \omega_1\}$. Then for each $\alpha \in \omega_1$, there is a $t_\alpha \in T_\alpha$ such that $\{a \in A \mid t_\alpha \in a\}$ is uncountable since T_α is countable. If for $\alpha < \beta < \omega_1$, $t_\alpha \not<_T t_\beta$, then t_α and t_β are incomparable. Hence $B_{t_\alpha} \cap B_{t_\beta} = \emptyset$. Then both $B_{t_\alpha} \cap A$ and $B_{t_\beta} \cap A$ are uncountable subsequences of A. Since B_{t_α} and B_{t_β} are disjoint convex sets in B_T, A cannot be increasing, a contradiction. Thus $\{t_\alpha \mid \alpha \in \omega_1\}$ is an uncountable chain in T. This is impossible because T is an Aronszajn tree. □

Lemma 2. *Let X be a subspace of $\langle B_T, \lambda, \preccurlyeq\rangle$. The following conditions are equivalent.*

(1) *X is separable.*
(2) $\{\mathrm{bht}(x) \mid x \in X\}$ *is not cofinal in* ω_1.
(3) *There is a countable collection $\mathcal{P}$ of open sets in $\langle B_T, \lambda, \preccurlyeq\rangle$ such that for any $x \in X$ and any open neighborhood U of x in B_T, there is a $V \in \mathcal{P}$ such that $x \in V \subset U$.*
(4) *X has a countable base.*

Proof. (1)⇒(2). Let Y be a countable dense subset of X. Then $\{\mathrm{bht}(x) \mid x \in Y\}$ has an upper bound α, and clearly we may take α as a limit ordinal. So $Y \subset B_T \restriction \alpha^+$. Notice that $B_T - B_T \restriction \alpha^+ = \bigcup\{B_t \mid t \in T_\alpha\}$ and each B_t is a closed interval in B_T. Hence among the elements of B_T with the order type large than α, only the endpoints of B_t's where $t \in T_\alpha$ possibly belong to $X \subset \mathrm{cl}_{B_T} Y$. The set of all endpoints of B_t's with $t \in T_\alpha$ is countable since $|T_\alpha| = \omega_0$. It follows that $\{\mathrm{bht}(x) \mid x \in X \subset \mathrm{cl}_{B_T} Y\}$ is not cofinal in ω_1.

(2)⇒(3). Let α be an upper bound of $\{\mathrm{bht}(x) \mid x \in X\}$ and assume that α is a limit ordinal. Then $X \subset B_T \restriction \alpha^+$. The topology on $\mathrm{cl}_{B_T}(B_T \restriction \alpha^+)$ as a subspace of B_T coincides with the topology on $\mathrm{cl}_{B_T}(B_T \restriction \alpha^+)$ as a LOTS since $\mathrm{cl}_{B_T}(B_T \restriction \alpha^+)$ is compact as a closed subspace of B_T. Let $\mathcal{I}$ be the collection of the convex components of $B_T - \mathrm{cl}_{B_T}(B_T \restriction \alpha^+)$. Then $|\mathcal{I}| \leqslant \omega_0$ since each element I of $\mathcal{I}$ contains an element of $\{\mathrm{int}_{B_T} B_t \mid t \in T_\alpha\}$ and $|T_\alpha| = \omega_0$. Let $T^* = \bigcup\{x \mid x \in B_T \restriction \alpha^+\} = \bigcup\{T_\beta \mid \beta < \alpha\}$. It follows from the definition of T that for each $t \in T^*$, $|B_t \restriction \alpha^+| > \omega_0$. Hence $\mathrm{int}_{B_T} B_t \cap B_T \restriction \alpha^+ \neq \emptyset$. Fix a point

$x_t \in \mathrm{int}_{B_T} B_t \cap B_T{\restriction}\alpha^+$ and let $Y = \{x_t \mid t \in T^*\}$. Then $|Y| = \omega_0$ since $|T^*| = \omega_0$. For any $x \in B_T \restriction \alpha^+$ and a convex neighborhood $(x_1, x_2)_{B_T}$ of x in B_T, let $\beta = \max\{\mathrm{ht}(x_1 \cap x),\ \mathrm{ht}(x \cap x_2)\}$ and $\{t\} = x \cap N(x, \beta^+)$. Then $t \in T^*$ and $B_t \subset (x_1, x_2)_{B_T}$. Hence $(x_1, x_2)_{B_T} \cap Y \neq \emptyset$. Therefore Y is a dense subset of $\mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+)$. Recall that a point of a linearly ordered set is a jump point if the point has an immediate successor. It is easy to show that if $x \in B_T{\restriction}\alpha^+ \cap B_{T2}$, then for any $y \in B_T$ with $x \prec y$, $(x, y)_{B_T} \cap B_T{\restriction}\alpha^+ \neq \emptyset$. So no element of $B_T{\restriction}\alpha^+ \cap B_{T2}$ is a jump point of $\mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+)$. Notice that

$$\mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+) - B_T{\restriction}\alpha^+ \subset \{e(t) \mid e(t) \text{ is an endpoint of } B_t,\ \ t \in T_\alpha\}.$$

Therefore the set of all jump points of $\mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+)$ is a subset of the following countable set

$$(B_{T0} \cup B_{T1}){\restriction}\alpha^+ \cup \{e(t) \mid e(t) \text{ is an endpoint of } B_t,\ \ t \in T_\alpha\}.$$

It is known that a LOTS has a countable base if the LOTS is separable and the set of its jump points is countable (see the insert of [8]). So $\mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+)$ has a countable base $\mathcal{C}$ consisting of open intervals in $\mathrm{cl}_{B_T}(B_T \restriction \alpha^+)$. By Lemma 1, B_T is first countable. For each endpoint x of $I \in \mathcal{I}$, let $\mathcal{V}(x)$ be the countable neighborhood base at x in B_T. For each $C \in \mathcal{C}$, let J_C be the open interval in B_T having the same endpoints with C. Put

$$\mathcal{P} = \{J_C \mid C \in \mathcal{C}\} \cup \Big(\bigcup\{\mathcal{V}(x) \mid x \text{ is an endpoint of } I,\ I \in \mathcal{I}\}\Big).$$

Then $\mathcal{P}$ is a countable collection.

We prove that $\mathcal{P}$ is the required collection. Take any $x \in X$ and any open neighborhood U of x in B_T. If x is an endpoint of $I \in \mathcal{I}$, an element $V \in \mathcal{V}(x) \subset \mathcal{P}$ is contained in U. Next suppose that x is not an endpoint of I for any $I \in \mathcal{I}$ and U is a neighborhood of x in B_T. We may assume that $U = (u_0, u_1)_{B_T}$. If u_0 (or u_1) is not in $\mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+)$, then u_0 (or u_1) is in I for some $I \in \mathcal{I}$. Let $I = (x_0, x_1)_{B_T}$. We have $x_1 \prec x$ (or $x \prec x_0$) and $x_1 \in \mathrm{cl}_{B_T}(B_T \restriction \alpha^+)$ (or $x_0 \in \mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+)$). Let $u_0' = x_1$ (or $u_1' = x_0$). Then u_0' (or u_1')$\in \mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+)$ and $u_0 \prec u_0' \prec x$ (or $x \prec u_1' \prec u_1$). So we always can choose $u_0', u_1' \in \mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+)$ such that $x \in (u_0', u_1')_{B_T} \subset U$. Since $\mathcal{C}$ is the base of $\mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+)$, there is a $C \in \mathcal{C}$ such that $x \in C \subset (u_0', u_1')_{B_T} \cap \mathrm{cl}_{B_T}(B_T{\restriction}\alpha^+)$. It follows that $x \in J_C \subset (u_0', u_1')_{B_T} \subset U$. Thus (3) is true.

(3)⇒(4)⇒(1) is obvious. □

Lemma 3. *Let X be a subspace of $\langle B_T, \lambda, \preccurlyeq\rangle$. If X is not separable, then there is a collection of disjoint open subsets of X of the cardinal ω_1.*

Proof. Let $X_1 = (B_{T1} \cup B_{T0}) \cap X$, $X_0 = X - X_1$ and let

$$T' = \bigcup\{x \mid x \in X_0\} \cup \Big(\bigcup\{x{\restriction}\eta_x \mid x \in X_1\}\Big).$$

If $|X_1| > \omega_0$, then the set $\{\eta_x \mid x \in X_1\}$ is cofinal in ω_1 since each T_α is countable. If $|X_1| \leqslant \omega_0$, then X_0 is not separable. In any case, it follows from Lemma 2 that $\mathrm{ht}\, T' = \omega_1$. So T' is also a special Aronszajn tree. Hence $T' = \bigcup\{A_n' \mid n \in \omega_0\}$, where $A_n' = T' \cap A_n$. Then there is an $n_0 \in \omega_0$ such that $|A_{n_0}'| = \omega_1$ since $|T'| = \omega_1$. Suppose that $t \in A_{n_0}'$. Then there is an $x \in X$ such that $t \in x$. By the definition of T', x is not an endpoint of B_t. Hence $x \in \mathrm{int}_{B_T} B_t \cap X$. Since A_{n_0}' is an antichain of T, $\{\mathrm{int}_{B_T} B_t \cap X \mid t \in A_{n_0}'\}$ is a collection of disjoint open sets of X with the cardinal ω_1. □

3. Theorem

Theorem 4. *$\langle B_T, \lambda, \preccurlyeq \rangle$ is a non-metrizable compact LOTS such that every subspace has a σ-minimal base for its relative topology.*

Proof. As was mentioned at the begining of Section 2, B_T is compact. By Lemma 2, B_T is not separable. Therefore B_T is not metrizable. We only need to show that every subspace of B_T has a σ-minimal base for its relative topology. Let X be a subspace of B_T. If X is separable, then by Lemma 2, X has a countable base which clearly is a σ-minimal base.

In the following we assume that X is not separable. By Lemma 2, $\{\mathrm{bht}(x) \mid x \in X\}$ is cofinal in ω_1. Put

$$T(X) = \bigcup\{x \mid x \in X\}.$$

Then $T(X)$ is also a special Aronszajn tree since

$$T(X) = \bigcup\{A_n(X) \mid n \in \omega_0\},$$

where $A_n(X) = A_n \cap T(X)$.

It is obvious that $\{B_t \cap X \mid t \in A_n(X)\}$ is a disjoint collection. Hence $\{\mathrm{int}_X(B_t \cap X) \mid t \in A_n(X)\}$ is a disjoint collection of open sets in X, and so it is trivially minimal. Thus the collection $\{\mathrm{int}_X(B_t \cap X) \mid t \in T(X)\}$ is a σ-minimal collection of open sets in X. Of course, it is probably not a base for X in general. But for quite large subsets of X, the collection serves as a base. In the rest of the proof, we will "refine" the collection $\{B_t \cap X \mid t \in T(X)\}$ to produce a base of X keeping the σ-minimality. For this purpose, put

$$\begin{aligned}
&E = B_{T2} \cap X,\\
&D_0 = B_{T0}^{suc1} \cap X,\\
&D_1 = \big(B_{T1}^{suc} \cup (B_{T0}^{suc} - B_{T0}^{suc1})\big) \cap X = \big((B_{T1}^{suc} \cup B_{T0}^{suc}) \cap X\big) - D_0,\\
&G_1 = B_{T1}^{lim} \cap X, \quad \text{and}\\
&G_0 = B_{T0}^{lim} \cap X.
\end{aligned}$$

Then $X = (B_{T1} \cup B_{T0} \cup B_{T2}) \cap X = D_0 \cup D_1 \cup E \cup G_0 \cup G_1$.

In fact, for every point in E and D_1, $\{\mathrm{int}_X(B_t \cap X) \mid t \in T(X)\}$ contains a neighborhood base of the point (see Cases 1 and 3 below). Now we consider the points in D_0, G_0 and G_1. Suppose that $t \in T(X)$. By Fact 4, we may write B_t as $[b_0(t), b_1(t)]_{B_T}$.

(i) If there is an $x \in D_0$ such that $t \in x \cap T_\alpha$ and $\eta_x = \alpha^+$, then $x \in \mathrm{int}_{B_T} B_t$. Let $\mathcal{H}(t) = \{H(t,k) \mid k \in \omega_0\}$ be a countable neighborhood base at x in X such that each $H(t,k) \subset B_t$.

(ii) Let $s_1(t)$ be the minimum point in B_t such that $\big[s_1(t), b_1(t)\big]_{B_T} \cap G_1$ is separable. Because of the compactness and first countability of B_T, $s_1(t)$ exists. For $\big[s_1(t), b_1(t)\big)_{B_T} \cap G_1$, if it is not empty, by Lemma 2, there is a countable collection $\mathcal{S}_1^r(t) = \{S^r(t,k) \mid k \in \omega_0\}$ in X which contains neighborhood bases of all points in $\big[s_1(t), b_1(t)\big)_{B_T} \cap G_1$ since it is separable. Clearly we may assume that each $S^r(t,k) \subset B_t$.

If $\big(b_0(t), s_1(t)\big)_{B_T} \cap G_1$ is not empty, then it is not separable. By Lemma 1, B_T is first countable, so we may take an increasing sequence $\{d_k(t)\}$ in $\big[b_0(t), s_1(t)\big)_{B_T}$

such that $\{d_k(t)\}$ converges to $s_1(t)$. Also there is a $k \in \omega_0$ such that $\big[b_0(t), d_k(t)\big]_{B_T} \cap G_1$ is not separable, hence we may assume that $b_0(t) \prec d_0(t)$ and $\big(b_0(t), d_0(t)\big)_{B_T} \cap G_1$ is not separable. Let $G_1(t,k) = \big(b_0(t), d_k(t)\big)_{B_T} \cap G_1$. For each $x \in G_1(t,k)$, let $\{g_1(t,k,x,j) \mid j \in \omega_0\}$ be an increasing sequence in $\big(b_0(t), d_k(t)\big)_{B_T}$ converging to x. Let

$$K_1(t,k) = \{g_1(t,k,x,j) \mid j \in \omega_0,\ x \in G_1(t,k)\}.$$

Then $|K_1(t,k)| = \omega_1$. Observe that the separability is a hereditary property in GO-spaces. Since $\big(d_k(t), s_1(t)\big)_{B_T} \cap G_1$ is not separable by the minimality of $s_1(t)$, so is $\big(d_k(t), s_1(t)\big)_{B_T} \cap X$. By Lemma 3, there is a collection $\mathcal{O}_1(t,k)$ of disjoint open sets in X contained in $\big(d_k(t), s_1(t)\big)_{B_T}$ such that $|\mathcal{O}_1(t,k)| = \omega_1$. Let

$$\phi_{1,t,k} :\ K_1(t,k) \to \mathcal{O}_1(t,k)$$

be a bijection. Put

$$\mathcal{G}_1(t,k) = \Big\{ \Big(\big(g_1(t,k,x,j), d_k(t)\big)_{B_T} \cap X \Big) \cup \phi_{1,t,k}\big(g_1(t,k,x,j)\big) \mid j \in \omega_0,\ x \in G_1(t,k) \Big\}.$$

Notice that $\phi_{1,t,k}\big(g_1(t,k,x,j)\big) \subset \big(d_k(t), s_1(t)\big)_{B_T}$ and $g_1(t,k,x,j) \in K_1(t,k) \subset \big[b_0(t), d_k(t)\big)_{B_T}$. Since $\mathcal{O}_1(t,k)$ is a disjoint collection, it follows that $\mathcal{G}_1(t,k)$ is a minimal collection of open sets in X contained in B_t.

(iii) Similar to (ii), for G_0, we may define $s_0(t)$, $\{c_k(t)\}$, $\{g_0(t,k,x,j) \mid j \in \omega_0\}$, $\phi_{0,t,k}$, $\mathcal{G}_0(t,k)$, and $\{S^l(t,k) \mid k \in \omega_0\}$ corresponding to $s_1(t)$, $\{d_k(t)\}$, $\{g_1(t,k,x,j) \mid j \in \omega_0\}$, $\phi_{1,t,k}$, $\mathcal{G}_1(t,k)$, and $\{S^r(t,k) \mid k \in \omega_0\}$ in (ii) respectively by replacing minimum and increasing by maximum and decreasing respectively.

Thus each of the following collections, if it is defined for t, is a minimal collection of open sets in X contained in B_t.

(1) $\{\mathrm{int}_X(B_t \cap X)\}$;
(2) $\{H(t,k)\}$ for $k \in \omega_0$;
(3) $\mathcal{G}_0(t,k)$ for $k \in \omega_0$;
(4) $\mathcal{G}_1(t,k)$ for $k \in \omega_0$;
(5) $\{S^r(t,k)\}$ for $k \in \omega_0$;
(6) $\{S^l(t,k)\}$ for $k \in \omega_0$.

For each $t \in T(X)$, enumerate all these minimal collections in (1) to (6) above which are defined for t as $\{\mathcal{B}(t,k) \mid k \in \omega_0\}$.

For $t_1, t_2 \in A_n(X)$, $n \in \omega_0$, if $t_1 \neq t_2$, then $B_{t_1} \cap B_{t_2} = \emptyset$ since $A_n(X)$ is an antichain of $T(X)$. Each element of $\mathcal{B}(t,k)$ is a subset of B_t. It follows that the collection $\mathcal{B}_{k,n} = \bigcup\{\mathcal{B}(t,k) \mid t \in A_n(X)\}$ is also a minimal collection for each pair $k, n \in \omega_0$.

There are three special points we should consider if they are in X: $z_0 \in D_0$ satisfying $\eta_{z_0} = 0$, the maximum point z_1 of B_T, and the minimum point z_2 of B_T. Let $\mathcal{W}(z_i) = \{W_{i,n} \mid n \in \omega_0\}$ be a countable neighborhood base at z_i in B_T for $i = 0, 1, 2$ respectively. Hence

$$\mathcal{B} = \bigcup\{\mathcal{B}_{k,n} \mid k, n \in \omega_0\} \cup \{W_{i,n} \cap X \mid n \in \omega_0,\ i = 0, 1, 2\}$$

is a σ-minimal collection in X.

Now we prove that $\mathcal{B}$ is a base of X. Trivially $\mathcal{B}$ contains the neighborhood bases at z_i for $i = 0, 1, 2$. So suppose that $x \in X$ with $x \neq z_i$ where $i = 0, 1, 2$ and U is a neighborhood of x in X. Then there is an open interval $(a, b)_{B_T}$ of B_T such that $x \in (a, b)_{B_T} \cap X \subset U$. Then by the definition of the ordering on B_T,

$$(*) \qquad a_{N_{a\cap x}} <_{N_{a\cap x}} x_{N_{a\cap x}} \ \text{ and } \ x_{N_{x\cap b}} <_{N_{x\cap b}} b_{N_{x\cap b}}.$$

Case 1. $x \in E$. Let

$$\alpha = \max\{\mathrm{ht}(a \cap x), \mathrm{ht}(x \cap b)\} \ \text{ and } \ \{t\} = x \cap N(x, \alpha^+).$$

Then $t \in T(X)$. It follows from $(*)$ that $B_t \subset (a, b)_{B_T}$ since for any $y \in B_t$, $a \cap y = a \cap x$, $x \cap b = y \cap b$, and $y_{N_{a\cap y}} = x_{N_{a\cap x}}$, $y_{N_{y\cap b}} = x_{N_{x\cap b}}$. It is obvious that $x \in B_t$. Moreover by Fact 4, x cannot be the maximum point or minimum point of B_t since $x \in E$. Hence $x \in \mathrm{int}_X(B_t \cap X) \subset (a, b)_{B_T} \cap X \subset U$ and $\mathrm{int}_X(B_t \cap X) \in \mathcal{B}$.

Case 2. $x \in D_0$. Since $x \neq z_0$, $\eta_x = \alpha^+$ for some α. Let $\{t\} = x \cap T_\alpha$. It follows from (i) that for some $k_0 \in \omega_0$, $x \in H(t, k_0) \subset U$ and $H(t, k_0) \in \mathcal{B}$.

Case 3. $x \in D_1$. If $x \in B^{suc}_{T1}$, since x is not the maximum point of B_T, $x \cap N(x, \eta_x) \neq \{a\big(N(x, \eta_x)\big)_{\omega_0}\}$ even if $\eta_x = 0$. So there is a $y \in B^{suc}_{T0}$ such that $\eta_y = \eta_x$, $N(y, \eta_y) = N(x, \eta_x)$ and if $x \cap N(x, \eta_x) = \{a\big(N(x, \eta_x)\big)_i\}$, then $y \cap N(x, \eta_x) = \{a\big(N(x, \eta_x)\big)_{i+1}\}$, where $i \in \omega_0$. It is easy to check that y is an immediate successor of x in B_T. Remember $x \in (a, b)_{B_T}$. Let

$$\alpha = \max\{\mathrm{ht}(a \cap x), \eta_x\} \ \text{and} \ \{t\} = x \cap N(x, \alpha^+).$$

Then x is the maximum point of B_t and $x \in \mathrm{int}_X(B_t \cap X)$ since x has an immediate successor in B_T. Since for any $y \in B_t$, $a \cap y = a \cap x$ and $y_{N_{a\cap y}} = x_{N_{a\cap x}}$, by $(*)$, we have $x \in \mathrm{int}_X(B_t \cap X) \subset (a, b)_{B_T} \cap X \subset U$ and $\mathrm{int}_X(B_t \cap X) \in \mathcal{B}$. Similarly, if $x \in B^{suc}_{T0}$, then x has an immediate predecessor in B_T since $x \notin D_0$. It follows that for some $t \in T(X)$, $x \in \mathrm{int}_X(B_t \cap X) \subset (a, b)_{B_T} \cap X \subset U$ and $\mathrm{int}_X(B_t \cap X) \in \mathcal{B}$.

Case 4. $x \in G_1$. If $\mathrm{ht}(a \cap x) < \eta_x$, let $\alpha = \max\{\mathrm{ht}(a \cap x), \mathrm{ht}(x \cap b)\}$. Then $\alpha^+ < \eta_x$ since $x \in B^{lim}_{T1}$. Let $\{t\} = x \cap N(x, \alpha^+)$. Similar to Case 1, $x \in \mathrm{int}_X(B_t \cap X) \subset U$ and $\mathrm{int}_X(B_t \cap X) \in \mathcal{B}$.

If $\mathrm{ht}(a \cap x) > \eta_x$, let $\alpha = \mathrm{ht}(x \cap b)$. Then $\alpha^+ < \eta_x$. Let $\{t\} = x \cap N(x, \alpha^+)$. Then $x \in \big(b_0(t), b_1(t)\big)_{B_T}$ and for any $y \in B_t$, $y \prec b$. If $s_1(t) \preccurlyeq x$, by (ii), there is some $k_0 \in \omega_0$ such that $x \in S^r(t, k_0) \subset U$ and $S^r(t, k_0) \in \mathcal{B}$. If $x \prec s_1(t)$, let $\{d_k(t)\}$, $\{g_1(t, k, x, j) \mid j \in \omega_0\}$ and $\mathcal{O}_1(t, k)$ be the sequences and collection in (ii). Since $x \prec s_1(t)$, there is a $k_0 \in \omega_0$ such that $x \prec d_{k_0} \prec s_1(t)$ and there is some $j_0 \in \omega_0$ such that $a \prec g_1(t, k_0, x, j_0) \prec x$. Let

$$V = \Big(\big(g_1(t, k_0, x, j_0), d_{k_0}\big)_{B_T} \cap X\Big) \cup \phi_{1,t,k_0}\big(g_1(t, k_0, x, j_0)\big).$$

Then $x \in V$ and $V \subset (a, b)_{B_T} \cap X \subset U$ and $V \in \mathcal{B}$.

Case 5. $x \in G_0$. Similar to Case 4, we can choose a $V \in \mathcal{B}$ such that $x \in V \subset U$.

Thus we have proved that $\mathcal{B}$ is a σ-minimal base of X, and this completes the proof. □

Remark 1. Suppose that X is a compact LOTS. Insert a copy of the usual interval $(0, 1)$ of real line into each jump (A, B) in X. Then the resulting LOTS X' will be a continuum. If X is not metrizable and satisfies the conditions in Problem 1, so does X' since $X' - X$ is a union of disjoint open intervals in X' and each of those open intervals is homeomorphic to $(0,1)$. Then it is easy to check that X' satisfies first countability, non-separability and that the closure of any countable subset of X' is second countable. This means that X' is an Aronszajn continuum. By Theorem 4, we may claim that there is an Aronszajn continuum with the interval topology which is a counterexample for Problem 1 and that any counterexample for Problem 1 must be a subspace of an Aronszajn continuum with the interval topology.

Remark 2. Since a compact quasi-developable space is metrizable, Theorem 4 also negatively answers the problem about whether a LOTS which has σ-minimal bases hereditarily is quasi-developable (see [6]).

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Department of Mathematics, Changchun Teachers College, Changchun 130032, China

Current address: Institute of Mathematics, University of Tsukuba, Tsukuba-shi, Ibaraki 305, Japan

E-mail address: shi@abel.math.tsukuba.ac.jp