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A CLASS OF 3-DIMENSIONAL MANIFOLDS WITH BOUNDED FIRST EIGENVALUE ON 1-FORMS

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Abstract. Let (P, g) be the framebundle over an oriented, C^∞ Riemannian surface S. Denote by $\lambda - 1, 1(P, g)$ the first nonzero eigenvalue of the Laplace operator acting on differential forms of degree 1. We prove that $\lambda_{1,1}(P, g) \leq c$ for all (P, g) with canonical metrics g of volume 1.

1. Introduction

Let (M^n, g) be a compact, connected Riemannian manifold of n dimensions. The Laplacian $\Delta_{g,p}$ acting on differential forms of degree p on M has discrete spectrum. Let $\lambda_{1,p}(g)$ denote the smallest positive eigenvalue of $\Delta_{g,p}$. For functions we set as usual $\lambda_1(g) = \lambda_{1,0}(g)$. Let Ω be the class of all metrics on an orientable closed surface S with given volume. Then we have

$$\lambda_1(g) \leq \frac{8\pi(\gamma + 1)}{\mathrm{Vol}(S, g)} \tag{1}$$

for any Riemannian metric $g \in \Omega$. This was proved by J. Hersch [7] (for S^2) and by P. Yang and S.-T. Yau [14] for surfaces of higher genius.

In connection with this result, M. Berger [1] and S. Tanno [11] asked whether there exists a constant $k(M)$ such that

$$\lambda_{1,p}(g) \leq \frac{k(M)}{\mathrm{Vol}(M^n, g)^{2/n}} \tag{2}$$

for any Riemannian metric g on M.

The answer is negative in the case of functions for $n \geq 3$ (Bleecker [3], Urakawa [12], Xu [13], Colbois and Dodziuk [5]), and also negative in the case of differential forms of degree $2 \leq p \leq n - 2$ and dimension $n \geq 4$ (Gentile and Pagliara [6]).

One can obtain positive answers if one restricts the class of manifolds (M, Ω). For Ω the class of all Kähler metrics on a complex manifold M, whose Kähler form represents a given cohomology class, an upper bound for $\lambda_1(M, g)$, i.e. functions, $g \in \Omega$ exists and is related to the algebraic-geometric properties of M. This was discovered by P. Li and S.-T. Yau [8] and by J.-P. Bourguignon, P. Li and S.-T. Yau [4]. Another positive answer for an (M, Ω) was given by L. Polterovich [10], when M is a closed symplectic manifold and Ω a special class of Kähler metrics.

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In the present paper we give a positive answer to the above problem on 1-forms in a special class of 3-dimensional manifolds. They are S^1-bundles P over 2-dimensional manifolds S with canonical metrics (see below). We estimate $\lambda_{1,1}(g)$ form above in terms of the fibrelength and the volume of the base-space S. In particular $\lambda_{1,1}(g)$ is bounded from above by a constant if we fix the volume of P, in contrast to the case of function, where D. Bleecker [3] showed, that $\lambda_1(g)$ cannot be bound by a constant in this class of manifolds.

2. The class of manifolds

Let P be the framebundle over an oriented closed Riemannian surface (S, h). This is a principal S^1-bundle. An S^1-invariant metric on the fibre and the metric h of the base space S induce a Riemannian metric g on P such that $\pi\colon (P, g) \to (S, h)$ is a Riemannian submersion with totally geodesic fibres and horizontal distribution associated with the Levi-Civita connection (A. Besse [2]).

Let ξ be the Killing vectorfield along the S^1-action and η its dual. Let $x = (X_1, X_2)$ be a frame at $u = \pi(x)$. Each $x \in P$ gives rise to a linear isomorphism $x\colon \mathbb{R}^2 \to T_u, u = \pi(x)$, $x(E_i) = X_i$ where (E_1, E_2) is the standard basis of $\mathbb{R}^2$. Let $V \in \mathbb{R}^2$, $x \in P$, $B(V)_x :=$ unique horizontal vector at x which projects on $x(V) \in T_u$. $B(V)$ is called a basic vectorfield (K. Nomizu [9]).

Lemma 1. i) $V \neq 0 \to B(V) \neq 0$.

ii) $(\xi, B(E_1), B(E_2))$ *gives rise to a complete parallelisation of* P.

iii) $[\xi, B(V)] = B(\xi \cdot V)$, *where the dot denotes standard representation of* $\xi \in LS^1$ *on* $\mathbb{R}^2$*:*

$$\xi \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The proof of this and the following lemma can be found in K. Nomizu [9, page 50–52]. Let (ξ, B_1, B_2) be an ONS for g, $B_i = B(E_i)$ and Ω the curvature form of the fibration.

Since the torsion vanishes we have:

Lemma 2. i) $[B_1, B_2] = -\Omega(B_1, B_2)$.

ii) $[\xi, B_1] = B(\xi \cdot E_1) = B(E_2) = B_2$.

iii) $[\xi, B_2] = B(\xi \cdot E_2) = B(-E_1) = -B_1$.

3. Results

Theorem 1. *Let* (P, g) *be as above. Then*

$$\lambda_{1,1}(g) \leq \min\left(\frac{1}{L^2}, \frac{8\pi(1+\gamma)}{\mathrm{Vol}(S, h)}\right)$$

where $2\pi L$ *is the length of the fibres. In particular if we fix* $\mathrm{Vol}(P, g) = 1$*, then*

$$\lambda_{1,1}(g) \leq [16\pi^2(1+\gamma)]^{2/3}.$$

Remarks. If $\mathrm{Vol}(P, g) = 1$, $\lambda_1(g) = \lambda_{1,0}(g)$ is not bounded as long as the curvature of (S, h) is nowhere 0 (D. Bleecker [3]).

Proof. The metrics on P are determined up to the length of the fibres from the metric on S and the condition imposed on P (A. Besse [2]). We want to examine the dependence of the eigenvalue on the length of the fibres. By rescaling the metric on P along the fibres while keeping it constant on the orthogonal complement (canonical variation), we obtain a 1-parameter family of metrics g_L on P for which $\pi\colon (P, g_L) \to (S, h)$ is a Riemannian submersion and the length of the fibres is $2\pi L$.

If (ξ, B_1, B_2) is an ONS for g and $\overline{\xi} = \frac{1}{L}\xi$, then $(\overline{\xi}, B_1, B_2)$ is an ONS for g_L. Let $(\overline{\eta}, \beta_1, \beta_2)$ be the dual base.

Lemma 3. *β_1 is an eigen 1-form of the Laplace operator on M*

$$\begin{aligned} d\beta_1 &= -\beta_1([\overline{\xi}, \beta_1])\overline{\eta} \wedge \beta_1 - \beta_1([\overline{\xi}, \beta_2])\overline{\eta} \wedge \beta_2 \\ &= -\frac{1}{L}\beta_1([\xi, \beta_2])\overline{\eta} \wedge \beta_2 \\ &= \frac{1}{L}\overline{\eta} \wedge \beta_2 \end{aligned}$$

where we have used Lemma 2.

$$\begin{aligned} \delta\beta_1 &= *d*\beta_1 \\ &= -*d(\overline{\eta} \wedge \beta_2) \\ &= -*(d\overline{\eta} \wedge \beta_2 - \overline{\eta} \wedge d\beta_2) \\ &= 0 \end{aligned}$$

again from Lemma 2. Therefore we get

$$\begin{aligned} \Delta\beta_1 &= (\delta d + d\delta)\beta_1 \\ &= *d*d\beta_1 \\ &= *d*\left(\frac{1}{L}\overline{\eta} \wedge \beta_2\right) \\ &= -\frac{1}{L}*d\beta_1 \\ &= \frac{1}{L^2}\beta_1. \end{aligned}$$

In the same way one finds that β_2 is also an eigen 1-form with eigenvalue $1/L^2$.

On the other hand, if Ψ is a nontrivial eigenfunction of (S, h), then, using the result of Yang and Yau, we have (1) and $\Delta \circ d = d \circ \Delta$

$$\Delta^S(d\Psi) \le \frac{8\pi(1+\gamma)}{\mathrm{Vol}(S, h)} d\Psi.$$

We lift the 1-form $d\Psi$ of S to P. This gives (since $\dim S = 2$)

$$\Delta^P(\pi^*(d\Psi)) = \pi^*(\Delta^S(d\Psi)) \le \frac{8\pi(1+\gamma)}{\mathrm{Vol}(S, h)}\pi^*(d\Psi).$$

Putting it together, we get

$$\lambda_{1,1}(g) \le \min\left(\frac{1}{L^2}, \frac{8\pi(1+\gamma)}{\mathrm{Vol}(S, h)}\right).$$

If we fix $\mathrm{Vol}(P, g) = 1$, then $\mathrm{Vol}(S, h) = \frac{1}{2\pi L}$ and the second assertion follows from

$$\min\left(\frac{1}{L^2}, 16\pi^2(1+\gamma)L\right) \le [16\pi^2(1+\gamma)]^{2/3} \quad \forall L > 0. \quad \square$$

Remarks. If one takes as class (P, Ω) the framebundles over manifolds M of arbitrary dimension n, with certain canonical metrics on $SO(N)$, one can still find an analogue of β_1, but there is no result which corresponds to the equality of Hersch-Yang-Yau (1). Also one does not know whether or not the first eigenvalue on functions is bounded in that class.

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