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WEIGHTED INEQUALITIES FOR ITERATED CONVOLUTIONS

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ABSTRACT. Given a fixed exponent p, $1 \le p < \infty$, and suitable nonnegative weight functions v_j, $j = 1, \dots, m$, an optimal associated weight function ω_m is constructed for which the iterated convolution product satisfies

$$\int_0^\infty \left| \left[\prod_{j=1}^m *F_j \right](x) \right|^p \frac{dx}{\omega_m(x)} \le \prod_{j=1}^m \int_0^\infty |F_j(t)|^p \frac{dt}{v_j(t)}$$

for all complex valued measurable functions F_j with $\int_0^\infty |F_j(t)|^p \, dt/v_j(t) < \infty$. Here $[\prod_{j=1}^2 *F_j](x) = [F_1 * F_2](x) = \int_0^x F_1(t)F_2(x-t)\,dt$ and for each $m > 2$, $\prod_{j=1}^m *F_j = \left[\prod_{j=1}^{m-1} *F_j \right] * F_m$. Analogous results are given when $R^+ = (0, \infty)$ is replaced by R^n and also when the convolution $F_1 * F_2$ on R^+ is taken instead to be $\int_0^\infty F(t)G(x/t)\,dt/t$. The extremal functions are also discussed.

1. INTRODUCTION

It is well known that the convolution $(F_1 * F_2)(x) = \int_0^x F_1(t)F_2(x-t)\,dt$ of Lebesgue integrable functions $F_1, F_2 \in L(R^+)$ belongs to $L(R^+)$, and more generally, if $1 \le p < \infty$ the convolution of $F_1 \in L(R^+)$ and $F_2 \in L^p(R^+)$ is also an $L^p(R^+)$ function. Indeed, Young's Theorem asserts that

$$||F_1 * F_2||_{L^p(R^+)} \le ||F_1||_{L(R^+)} ||F_2||_{L^p(R^+)}. \tag{1.1}$$

However, the convolution of two $L^p(R^+)$ functions need not belong to $L^p(R^+)$. If the ambient space $\Omega = R^+ = (0, \infty)$ is replaced by R^n, the convolution $(F_1 * F_2)(x) = \int_{R^n} F_1(t)F_2(x-t)\,dt$ satisfies the analogue of (1.1), but for $F_1, F_2 \in L^p(R^n)$ the convolution need not exist, let alone belong to $L^p(R^n)$. The purpose of this paper is to determine, when $\Omega = R^+$ or R^n, conditions on the measures μ_j, $j = 1, 2$, on Ω which ensure that $F_1 * F_2$ exists whenever $F_j \in L^p(\Omega, \mu_j)$ and to construct an associated measure ν on Ω so that $F_1 * F_2 \in L^p(\Omega, \nu)$ and satisfies a weighted substitute for (1.1), namely

$$||F_1 * F_2||_{L^p(\Omega,\nu)} \le ||F_1||_{L^p(\Omega,\mu_1)} ||F_2||_{L^p(\Omega,\mu_2)}.$$

In fact, we consider more general inequalities for the iterated convolution products $\prod_{j=1}^m *F_j$ defined by $\prod_{j=1}^2 *F_j = F_1 * F_2$ and $\prod_{j=1}^m *F_j = \left[\prod_{j=1}^{m-1} *F_j \right] * F_m$ for $m > 2$.

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Our main result is the following theorem which generalizes a recent result of Cwikel and Kerman [2] and certain earlier results of Saitoh [4] and Burbea [1].

Theorem 1. *Let $1 \le p < \infty$, $1/p + 1/p' = 1$, and let $m \ge 2$ be an integer. Suppose v_j, $j = 1, \dots, m$, are nonnegative measurable functions on R^+ with $v_j^{1/p} \in L^{p'}(0, R)$ for every $R > 0$. Define $\omega_1 = v_1$, and set*

$$\omega_j(x) = \begin{cases} [\omega_{j-1}^{p'/p} * v_j^{p'/p}]^{p/p'}(x), & \text{if } 1 < p < \infty, \\ \operatorname{ess.\,sup}_{0<t<x} \omega_{j-1}(t) v_j(x-t), & \text{if } p = 1 \end{cases}$$

for $j = 2, \dots, m$. Then

$$\int_{R^+} \left| \left[\prod_{j=1}^m *F_j \right](x) \right|^p \frac{dx}{\omega_m(x)} \le \prod_{j=1}^m \int_{R^+} |F_j(t)|^p \frac{dt}{v_j(t)} \tag{1.2}$$

for all complex valued measurable functions F_j with $\int_{R^+} |F_j(t)|^p \frac{dt}{v_j(t)} < \infty$, $j = 1, \dots, m$.

If $1 < p < \infty$, equality holds in (1.2) if

$$F_j(t) = c_j e^{\alpha t + i\beta t} v_j^{p'/p}(t) \qquad \text{a.e.} \tag{1.3}$$

for complex constants c_j and real numbers α, β such that $\int_{R^+} e^{p\alpha t} v_j^{p'/p}(t)\, dt < \infty$, $j = 1, \dots, m$. Conversely, if $v_j(t) > 0$ on an open interval I_j and $v_j(t) = 0$ on $R^+ \setminus I_j$, $j = 1, \dots, m$, and equality holds in (1.2), then F_j, $j = 1, \dots, m$, is given by (1.3) unless $F_j(t) = 0$ a.e. for some j.

For $r, s > 0$ the Beta function $B(r, s) = \int_0^1 y^{r-1}(1-y)^{s-1}\, dy = \Gamma(r)\Gamma(s)/\Gamma(r+s)$ shows that

$$[t^{r-1} * t^{s-1}](x) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} x^{r+s-1}, \qquad x > 0,$$

and hence, if $1 < p < \infty$, $q_j > 0$ and $v_j(t) = t^{(q_j - 1)(p-1)}$, then

$$\omega_m(x) = \left[\frac{\Gamma(q_1) \cdots \Gamma(q_m)}{\Gamma(q_1 + \cdots + q_m)} \right]^{p-1} x^{(q_1 + \cdots + q_m - 1)(p-1)}.$$

On the other hand, for $p = 1$, $q_j \ge 0$ and $v_j(t) = t^{q_j}$, a simple calculation yields

$$\omega_m(x) = \frac{q_1^{q_1} \cdots q_m^{q_m}}{(q_1 + \cdots + q_m)^{q_1 + \cdots + q_m}} x^{q_1 + \cdots + q_m}$$

where q^q is taken to be 1 whenever $q = 0$. Thus, we obtain the following result for power weights on R^+.

Corollary 1. *If $q_j \ge 0$, $j = 1, \dots, m$, then*

$$\int_{R^+} x^{-(q_1 + \cdots + q_m)} \left| \left[\prod_{j=1}^m *F_j \right](x) \right| dx \le \left[\frac{q_1^{q_1} \cdots q_m^{q_m}}{(q_1 + \cdots + q_m)^{q_1 + \cdots + q_m}} \right] \prod_{j=1}^m \int_{R^+} |F_j(t)| t^{-q_j}\, dt$$

for all complex valued measurable functions F_j with $\int_{R^+} |F_j(t)| t^{-q_j}\, dt < \infty$, $j = 1, \dots, m$.

If $1 < p < \infty$ *and* $q_j > 0$, $j = 1, \dots, m$, *then*

$$\int_{R^+} x^{-(q_1+\cdots+q_m-1)(p-1)} \left| \left[\prod_{j=1}^{m} *F_j \right](x) \right|^p dx \leq \left[\frac{\Gamma(q_1)\cdots\Gamma(q_m)}{\Gamma(q_1+\cdots+q_m)} \right]^{p-1} \prod_{j=1}^{m} \int_{R^+} |F_j(t)|^p t^{(1-q_j)(p-1)}\, dt \tag{1.4}$$

for all complex valued measurable functions F_j *with* $\int_{R^+} |F_j(t)|^p t^{(1-q_j)(p-1)}\, dt < \infty$, $j = 1, \dots, m$. *Unless* $F_j = 0$ a.e. *for some* j, *equality holds in* (1.4) *if and only if* $F_j(t) = c_j e^{-\alpha t + i\beta t} t^{q_j - 1}$ a.e. *for complex constants* $c_j \neq 0$, $j = 1, \dots, m$, *and real constants* α, β *with* $\alpha > 0$.

Special cases of Corollary 1 are known. The case $p = 2$, $q_j = 1$ for all j, and m even was obtained by Saitoh [4] and this was extended to $q_j > 0$ and arbitrary m by Burbea [1]. For $1 < p < \infty$ Cwikel and Kerman [2] obtained Corollary 1 for the case that $q_j = 1$ for each j.

The analogue of Theorem 1 for R^n is as follows. As usual, for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ we write $x \cdot y = x_1 y_1 + \cdots + x_n y_n$.

Theorem 2. *Let* $1 \leq p < \infty$, $1/p + 1/p' = 1$, *and let* $m \geq 2$ *be an integer. Suppose* v_j, $j = 1, \dots, m$, *are nonnegative measurable functions on* R^n *with* $v_j^{1/p} \in L^{p'}(R^n)$. *Define* $\omega_1 = v_1$, *and set*

$$\omega_j(x) = \begin{cases} [\omega_{j-1}^{p'/p} * v_j^{p'/p}]^{p/p'}(x), & \text{if } 1 < p < \infty, \\ \operatorname{ess.sup}_{t \in R^n} \omega_{j-1}(t) v_j(x - t), & \text{if } p = 1 \end{cases}$$

for $j = 2, \dots, m$. *Then*

$$\int_{R^n} \left| \left[\prod_{j=1}^{m} *F_j \right](x) \right|^p \frac{dx}{\omega_m(x)} \leq \prod_{j=1}^{m} \int_{R^n} |F_j(t)|^p \frac{dt}{v_j(t)} \tag{1.5}$$

for all complex valued measurable functions F_j *with* $\int_{R^n} |F_j(t)|^p \dfrac{dt}{v_j(t)} < \infty$, $j = 1, \dots, m$.

If $1 < p < \infty$, *equality holds in* (1.5) *if*

$$F_j(t) = c_j e^{\alpha \cdot t + i\beta \cdot t} v_j^{p'/p}(t) \qquad \text{a.e.} \tag{1.6}$$

for complex constants c_j *and fixed* $\alpha, \beta \in R^n$ *such that* $\int_{R^n} e^{p(\alpha \cdot t)} v_j^{p'/p}(t)\, dt < \infty$, $j = 1, \dots, m$. *Conversely, if* $v_j(t) > 0$ *on an open path connected set* I_j *and* $v_j(t) = 0$ *on* $R^n \backslash I_j$, $j = 1, \dots, m$, *and equality holds in* (1.5), *then* F_j, $j = 1, \dots, m$, *is given by* (1.6) *unless* $F_j(t) = 0$ a.e. *for some* j.

For $n = 1$ and $v_1(t) = v_2(t) = \chi_{\{t:|t|<r\}}(t)$ for fixed $r > 0$ we have $\omega_2(x) = (2r - |x|)^{p-1} \chi_{\{x:|x|<2r\}}(x)$. Thus we obtain the following result. The case $p = 2$ was obtained by Saitoh [5, p. 54] (note that the result is misstated there).

Corollary 2. *If* $1 \leq p < \infty$, F_1 *and* F_2 *belong to* $L^p(R)$ *and are supported in the interval* $\{t : |t| < r\}$, *then*

$$\int_{\{x:|x|<2r\}} \frac{|(F_1 * F_2)(x)|^p}{(2r - |x|)^{p-1}}\, dx \leq \int_{\{t:|t|<r\}} |F_1(t)|^p\, dt \int_{\{t:|t|<r\}} |F_2(t)|^p\, dt.$$

The Poisson kernel $P_b(t)$ given by

$$P_b(t) = \frac{1}{\pi} \frac{b}{[|t|^2 + b^2]^{(n+1)/2}}, \qquad t \in R^n, \ b > 0,$$

satisfies $P_b(at) = a^{-n} P_{b/a}(t)$ and the semigroup property $P_{b_1} * P_{b_2} = P_{b_1+b_2}$. Thus, taking $v_j(t) = [P_{b_j}(a_j t)]^{p/p'}$ for $1 < p < \infty$, $a_j > 0$, $b_j > 0$ yields

$$\omega_m^{p'/p}(x) = \frac{\left(\prod_{j=1}^m a_j\right)^{-n} \sum_{j=1}^m \dfrac{b_j}{a_j}}{\pi\left[|x|^2 + \left(\sum_{j=1}^m \dfrac{b_j}{a_j}\right)^2\right]^{(n+1)/2}}$$

and we have the following corollary which was obtained in the case $n = 1$, $m = p = 2$ by Saitoh [6, p. 516].

Corollary 3. *If $1 < p < \infty$, $a_j > 0$ and $b_j > 0$ for $j = 1, \ldots, m$, then*

$$\int_{R^n} \left|\left[\prod_{j=1}^m *F_j\right](x)\right|^p \left[\left(\prod_{j=1}^m a_j\right)^2 |x|^2 + \left(\prod_{j=1}^m a_j \sum_{j=1}^m \frac{b_j}{a_j}\right)^2\right]^{(n+1)(p-1)/2} dx$$
$$\le \left[\pi^{m-1}\left(\prod_{j=1}^m \frac{a_j}{b_j}\right) \sum_{j=1}^m \frac{b_j}{a_j}\right]^{p-1} \prod_{j=1}^m \int_{R^n} |F_j(t)|^p [a_j^2|t|^2 + b_j^2]^{(n+1)(p-1)/2}\, dt$$

for all complex valued functions F_j satisfying

$$\int_{R^n} |F_j(t)|^p [a_j^2|t|^2 + b_j^2]^{(n+1)(p-1)/2}\, dt < \infty, \qquad j = 1, \ldots, m.$$

Unless $F_j = 0$ a.e. for some j, equality holds if and only if

$$F_j(t) = c_j e^{i\beta\cdot t}\left[a_j^2|t|^2 + b_j^2\right]^{-(n+1)/2} \qquad \text{a.e.}$$

for complex constants $c_j \ne 0$, $j = 1, \ldots, m$, and fixed $\beta \in R^n$.

For power weights on R^n we have the following result, where we have set $\gamma(\alpha) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2)/\Gamma\big((n-\alpha)/2\big)$ for $0 < \alpha < n$.

Corollary 4. *If $1 < p < \infty$, $m \ge 2$ an integer and $q_j > 0$ with $\sum_{j=1}^m q_j < n$, then*

$$\int_{R^n} |x|^{-(q_1+\cdots+q_m-n)(p-1)} \left|\left[\prod_{j=1}^m *F_j\right](x)\right|^p dx$$
$$\le \left[\frac{\gamma(q_1)\cdots\gamma(q_m)}{\gamma(q_1+\cdots+q_m)}\right]^{p-1} \prod_{j=1}^m \int_{R^n} |F_j(t)|^p |t|^{(n-q_j)(p-1)}\, dt$$

for all complex valued measurable functions F_j with $\int_{R^n} |F_j(t)|^p t^{(n-q_j)(p-1)}\, dt < \infty$, $j = 1, \ldots, m$. Equality holds if and only if $F_j(t) = 0$ a.e. for some j.

The convolution defined on R^+ by $(f_1 \star f_2)(x) = \int_{R^+} f_1(t) f_2(x/t)\, dt/t$ may be expressed as $(F_1 * F_2)(\xi)$ where $x = e^\xi$ and $F_j(\xi) = f_j(x)$, $j = 1, 2$. Thus Theorem 2 yields the following corollary.

Corollary 5. *Let $1 \le p < \infty$, $1/p + 1/p' = 1$, and let $m \ge 2$ be an integer. Suppose v_j, $j = 1, \ldots, m$, are nonnegative measurable functions on R^+ with $v_j^{1/p} \in L^{p'}(R^+, dt/t)$. Define $\omega_1 = v_1$, and set*

$$\omega_j(x) = \begin{cases} [\omega_{j-1}^{p'/p} \star v_j^{p'/p}]^{p/p'}(x), & \text{if } 1 < p < \infty, \\ \operatorname{ess.\,sup}_{t \in R^+} \omega_{j-1}(t) v_j(x/t), & \text{if } p = 1 \end{cases}$$

for $j = 2, \ldots, m$. Then

$$\int_{R^+} \left| \left[\prod_{j=1}^{m} \star f_j \right](x) \right|^p \frac{dx}{x\omega_m(x)} \le \prod_{j=1}^{m} \int_{R^+} |f_j(t)|^p \frac{dt}{tv_j(t)} \tag{1.7}$$

for all complex valued measurable functions f_j with $\int_{R^+} |f_j(t)|^p \dfrac{dt}{tv_j(t)} < \infty$, $j = 1, \ldots, m$.

If $1 < p < \infty$, equality holds in (1.7) if

$$f_j(t) = c_j t^{\alpha + i\beta} v_j^{p'/p}(t) \qquad \text{a.e.} \tag{1.8}$$

for complex constants c_j and real numbers α, β such that $\int_{R^+} t^{p\alpha} v_j^{p'/p}(t)\, dt/t < \infty$, $j = 1, \ldots, m$. Conversely, if $v_j(t) > 0$ on an open interval I_j and $v_j(t) = 0$ on $R^+ \setminus I_j$, $j = 1, \ldots, m$, and equality holds in (1.7), then f_j, $j = 1, \ldots, m$, is given by (1.8) unless $f_j(t) = 0$ a.e. for some j.

2. Proofs

We write $|E|$ for the Lebesgue measure of a measurable $E \subset R^n$, $E_1 + E_2 = \{t_1 + t_2 : t_1 \in E_1, t_2 \in E_2\}$ for $E_1, E_2 \subset R^n$ and $rE = \{rt : t \in E\}$ for $E \subset R^n$ and $r > 0$.

We will need the following Lemma.

Lemma. *Let $I_j \subset R^n$ be open and path connected and suppose f_j is complex valued and measurable on I_j, $j = 1, 2$. If h is measurable on $I_1 + I_2$ and satisfies*

$$f_1(x_1) f_2(x_2) = h(x_1 + x_2) \quad \text{a.e.} \quad x_1 \in I_1, \quad x_2 \in I_2, \tag{2.1}$$

then there are $\alpha, \beta \in R^n$ and complex constants $c_j \ne 0$ such that

$$f_j(t) = c_j e^{\alpha \cdot t + i\beta \cdot t} \quad \text{a.e.} \quad t \in I_j, \quad j = 1, 2,$$

unless $f_j = 0$ a.e. on I_j for some j.

Proof of Lemma. It suffices to prove the Lemma for the case that I_1 and I_2 are open balls of the same radius. For if this has been proved, then if f_j is not zero almost everywhere on I_j, $j = 1, 2$, there are balls $B_j \subset I_j$ of the same radius r such that f_j is not zero almost everywhere on B_j and hence

$$f_j(t) = c_j e^{\alpha \cdot t + i\beta \cdot t} \quad \text{a.e.} \quad t \in B_j, \quad j = 1, 2. \tag{2.2}$$

But then, since I_1 is open and path connected, if $x \in I_1 \setminus B_1$ there is a finite sequence of overlapping balls B_k, $k = 3, \ldots, m$, of equal radius not exceeding r such that $|B_1 \cap B_3| > 0$, $|B_k \cap B_{k+1}| > 0$, $k = 3, \ldots, m-1$, and $x \in B_m$. Choosing a ball $B_2' \subset B_2$ with radius equal to that of B_3 and noting that (2.2) shows $|f_1| > 0$ a.e. on $B_3 \cap B_1$ and $|f_2| > 0$ a.e. on B_2', the special case of the Lemma applied successively to the balls $B_k \subset I_1$, $k = 3, \ldots, m$, $B_2' \subset I_2$ shows that $f_1(t) = c_1 e^{\alpha \cdot t + i\beta \cdot t}$ a.e. on

B_m. Thus $f_1(t) = c_1 e^{\alpha \cdot t + i\beta \cdot t}$ a.e. on a neighbourhood of x and hence also a.e. on I_1. A similar argument shows that $f_2(t) = c_2 e^{\alpha \cdot t + i\beta \cdot t}$ a.e. on I_2.

Suppose then that I_1 and I_2 are balls of the same radius, and that f_j is not zero almost everywhere on I_j, $j = 1, 2$. By considering $F_j(x_j) = f_j(x_j - a_j)$ and $H(x) = h(x - a_1 - a_2)$ if necessary, we may further assume that $I_1 = I_2 = B$ where B has center at the origin.

We first prove that $|f_j| > 0$ a.e. on B, $j = 1, 2$. Let $A_j = \{x \in B : f_j(x) = 0\}$ and suppose, to derive a contradiction, that $|A_1| + |A_2| > 0$. If $|A_1| > 0$, then (2.1) and Fubini's Theorem (for nonnegative but not necessarily integrable functions) shows that

$$0 = \int_{A_1} |f_1(x_1)|\, dx_1 \int_B |f_2(x_2)|\, dx_2 = \iint_{A_1 \times B} |h(x_1 + x_2)|\, dx_1 dx_2$$
$$= \int_{A_1} dx_1 \int_B |h(x_1 + x_2)|\, dx_2$$

and hence, for almost all $x_1 \in A_1$, $h(t) = 0$ a.e. $t \in x_1 + B$. A similar argument holds with the role of f_1 and f_2 reversed. Thus, for a.e. $x \in A_1 \cup A_2$,

$$h(t) = 0 \quad \text{a.e.} \quad t \in x + B \tag{2.3}$$

and in particular, since $|A_1 \cup A_2| > 0$, there is $x_0 \in B$ such that $h = 0$ a.e. on $x_0 + r_0 B$ with $r_0 = 1$. For $m \geq 0$, set $x_{m+1} = x_m/2$ and $r_{m+1} = r_m/2 + 1$. Now, suppose it has been verified that $h = 0$ a.e. on $x_m + r_m B$ for some m, and let $E = x_m/2 + (r_m/2)B$. Then $E + E = x_m + r_m B$ and hence (2.1) shows

$$\int_E |f_1(x_1)|\, dx_1 \int_E |f_2(x_2)|\, dx_2 = \iint_{E \times E} |h(x_1 + x_2)|\, dx_1 dx_2 = 0 \tag{2.4}$$

so that, for some j, $f_j = 0$ a.e. on E. Hence (2.3) shows that $h = 0$ a.e. on $E + B = x_{m+1} + r_{m+1}B$. Thus $h = 0$ a.e. on $x_m + r_m B$ for all $m \geq 0$. Clearly, $x_m \to 0$ and an easy induction shows that $r_m \uparrow 2$. Thus, $h = 0$ a.e. on $2B$, and taking $E = B$ in (2.4) we find $f_j = 0$ a.e. on B for some j, the desired contradiction.

Now, since $|f_j| > 0$ a.e. on B, $j = 1, 2$, (2.1) shows that

$$\frac{f_2(x_2)}{f_1(x_2)} = \frac{f_2(x_1)}{f_1(x_1)} \quad \text{a.e.} \quad x_j \in B$$

and hence $f_2 = cf_1$ a.e. on B for a complex constant $c \neq 0$, and hence also

$$f_j(x_1) f_j(x_2) = f_j^2\Big(\frac{x_1 + x_2}{2}\Big) \quad \text{a.e.} \quad x_j \in B, \quad j = 1, 2. \tag{2.5}$$

Writing $f_j(t) = e^{i\theta_j(t)}|f_j(t)|$ where $\theta_j(t) \in (-\pi, \pi]$, (2.5) shows that $\log|f_j(t)|$ and $\theta_j(t)$, $j = 1, 2$, satisfy the functional equation

$$\phi\Big(\frac{x_1 + x_2}{2}\Big) = \frac{\phi(x_1) + \phi(x_2)}{2} \quad \text{a.e.} \quad x_j \in B.$$

Since the solutions of this equation [7, p. 231] are of the form $\phi(t) = \alpha \cdot t + \gamma$ a.e. for fixed $\alpha \in R^n$ and $\gamma \in R$, this shows that there are $\alpha, \beta \in R^n$ and complex constants $c_j \neq 0$, $j = 1, 2$, such that $f_j(t) = c_j e^{\alpha \cdot t + i\beta \cdot t}$ a.e. on B. This completes the proof of the Lemma.

Proof of Theorem 1. Observe first that ω_j is well defined, nonnegative, and $\omega_j^{1/p} \in L^{p'}(0,R)$ for every $R>0$. Thus ω_j satisfies the same hypothesis satisfied by each weight function v_j. For $j=2$ this follows from

$$\begin{aligned}\int_0^R \omega_2^{p'/p}(x)\,dx &= \int_0^R dx \int_0^x v_1^{p'/p}(t) v_2^{p'/p}(x-t)\,dt \\ &= \int_0^R v_1^{p'/p}(t)\,dt \int_t^R v_2^{p'/p}(x-t)\,dx \\ &\le \int_0^R v_1^{p'/p}(t)\,dt \int_0^R v_2^{p'/p}(x)\,dx\end{aligned}$$

if $1<p<\infty$ and from

$$\begin{aligned}\operatorname*{ess.\,sup}_{0<x<R} w_2(x) &= \operatorname*{ess.\,sup}_{0<x<R} \operatorname*{ess.\,sup}_{0<t<x} v_1(t) v_2(x-t) \\ &\le \operatorname*{ess.\,sup}_{0<t<R} v_1(t) \operatorname*{ess.\,sup}_{0<x<R} v_2(x)\end{aligned}$$

if $p=1$, and an induction yields the same conclusion for $j>2$.

Now let $E_j = \{t : v_j(t) > 0\}$ and $G_j = \{t : \omega_j(t) > 0\}$. We prove (1.2) by induction. With $\prod_{j=1}^{1} *F_j = F_1$, (1.2) holds for $m=1$, so we suppose that $k \ge 2$, $\int_{R^+} |F_j(t)|^p\,dt/v_j(t) < \infty$, $j=1,\dots,k$, and that (1.2) has been verified for $m=k-1$. Set $H_{k-1}(t) = \big[\prod_{j=1}^{k-1} *F_j\big](t)$. Then $\int_{R^+} |H_{k-1}(t)|^p\,dt/\omega_{k-1}(t) < \infty$ shows that $H_{k-1}=0$ a.e. on $R^+ \setminus G_{k-1}$ so that

$$\begin{aligned}&\left|\int_0^x H_{k-1}(t) F_k(x-t)\,dt\right|^p \\ &\quad= \left|\int_0^x \frac{H_{k-1}(t)\chi_{G_{k-1}}(t)}{\omega_{k-1}^{1/p}(t)} \frac{F_k(x-t)\chi_{E_k}(x-t)}{v_k^{1/p}(x-t)} [\omega_{k-1}(t) v_k(x-t)]^{1/p}\,dt\right|^p \\ &\quad\le \left[\int_0^x \frac{|H_{k-1}(t)|^p}{\omega_{k-1}(t)} \frac{|F_k(x-t)|^p}{v_k(x-t)}\,dt\right]\omega_k(x)\end{aligned} \tag{2.6}$$

by Hölder's inequality. Fubini's Theorem shows that the last integral is finite a.e. since

$$\begin{aligned}&\int_{R^+}\int_0^x \frac{|H_{k-1}(t)|^p}{\omega_{k-1}(t)} \frac{|F_k(x-t)|^p}{v_k(x-t)}\,dt\,dx \\ &\quad= \int_{R^+}\int_t^\infty \frac{|H_{k-1}(t)|^p}{\omega_{k-1}(t)} \frac{|F_k(x-t)|^p}{v_k(x-t)}\,dx\,dt \\ &\quad= \int_{R^+} |H_{k-1}(t)|^p \frac{dt}{\omega_{k-1}(t)} \int_{R^+} |F_k(x)|^p \frac{dx}{v_k(x)} < \infty.\end{aligned} \tag{2.7}$$

Thus, (2.6) shows that $(H_{k-1} * F_k)(x)$ exists a.e. and $H_{k-1} * F_k = 0$ a.e. on $R^+ \setminus G_k$. Combining (2.6) and (2.7) yields

$$\int_{R^+} \left|\left[\prod_{j=1}^{k} *F_j\right](x)\right|^p \frac{dx}{\omega_k(x)} \le \left[\prod_{j=1}^{k-1} \int_{R^+} |F_j(t)|^p \frac{dt}{v_j(t)}\right]\left[\int_{R^+} |F_k(t)|^p \frac{dt}{v_k(t)}\right]$$

which is (1.2) for $m=k$. This completes the induction proof of (1.2).

If $1<p<\infty$, a straightforward calculation shows that equality holds in (1.2) if F_j is given by (1.3). Conversely, suppose $E_j = I_j$ is an open interval and equality holds in (1.2). A simple induction shows that $G_j = G_{j-1} + I_j$ and hence G_j is also an open interval. It follows from (2.6) and the induction argument that, apart from the cases in which $F_j = 0$ a.e. for some j, equality in (1.2) requires equality in Hölder's inequality for each k, $k = 2, \ldots, m$. Thus [3, p. 39] there is a complex valued function h_k such that for almost all $x \in R^+$

$$H_{k-1}(t)F_k(x-t) = h_k(x)\omega_{k-1}^{p'/p}(t)v_k^{p'/p}(x-t) \quad \text{a.e.} \quad t \in G_{k-1} \cap (x - I_k)$$

which we write as

$$\text{(2.8)} \qquad \text{for a.e. } x \in R^+, \quad f_{k1}(t)f_{k2}(x-t) = h_k(x) \quad \text{a.e.} \quad t \in G_{k-1} \cap (x - I_k),$$

where

$$f_{k1}(t) = H_{k-1}(t)\omega_{k-1}^{-p'/p}(t), \quad t \in G_{k-1},$$
$$f_{k2}(t) = F_k(t)v_k^{-p'/p}(t), \quad t \in I_k.$$

We first show that h_k is measurable on G_k. Let $U(x)$ be positive and integrable on R^+, say $U(x) = e^{-x}$, and let $V(t)$ be real valued, continuous, bounded and strictly increasing on R, say $V(t) = \arctan t$. Then

$$\Psi(x,t) = \frac{U(x)\chi_{G_k}(x)}{\omega_k^{p'/p}(x)} V\big(\mathrm{Re}[f_{k1}(t)f_{k2}(x-t)]\big)\omega_{k-1}^{p'/p}(t)\chi_{G_{k-1}}(t)v_k^{p'/p}(x-t)\chi_{I_k}(x-t)$$

is integrable on $R^+ \times R^+$ and hence Fubini's Theorem shows that

$$\Psi(x) = \int_{R^+} \Psi(x,t)\,dt$$

is measurable. In view of (2.8), $\Psi(x) = U(x)V\big(\mathrm{Re}[h_k(x)]\big)\chi_{G_k}(x)$ for a.e. $x \in R^+$ and hence $\mathrm{Re}[h_k(x)] = V^{-1}\big(\Psi(x)/U(x)\big)$ a.e. on G_k. Thus, $\mathrm{Re}[h_k]$ is measurable on G_k. A similar argument applies to $\mathrm{Im}[h_k(x)]$, and hence h_k is measurable on G_k.

Since H_k is measurable on G_k, $\phi(x_1, x) = |f_{k1}(x_1)f_{k2}(x - x_1) - h_k(x)|\chi_{I_k}(x - x_1)$ is measurable for $(x_1, x) \in G_{k-1} \times G_k$ and in view of (2.8) and Fubini's Theorem we have

$$\begin{aligned} 0 &= \int_{G_k} dx \int_{G_{k-1}} \phi(x_1, x)\,dx_1 = \iint_{G_k \times G_{k-1}} \phi(x_1, x)\,dx dx_1 \\ &= \int_{G_{k-1}} dx_1 \int_{G_k} \phi(x_1, x)\,dx \\ &= \int_{G_{k-1}} dx_1 \int_{[G_{k-1}+I_k]-x_1} |f_{k1}(x_1)f_{k2}(x_2) - h_k(x_2 + x_1)|\chi_{I_k}(x_2)\,dx_2. \end{aligned}$$

Since $x_1 \in G_{k-1}$ implies $I_k \subset [G_{k-1} + I_k] - x_1$, this shows that

$$f_{k1}(x_1)f_{k2}(x_2) = h_k(x_1 + x_2) \quad \text{a.e.} \quad x_1 \in G_{k-1}, \quad x_2 \in I_k.$$

Thus, the Lemma now shows that (2.8) requires

$$\text{(2.9)} \qquad f_{kj}(x_j) = c_j e^{\alpha_k x_j + i\beta_k x_j} \quad \text{a.e.} \quad x_1 \in G_{k-1}, \quad x_2 \in I_k$$

for real constants α_k, β_k and complex constants $c_j \neq 0$, $j = 1, 2$, unless $f_{kj} = 0$ a.e. for some j.

The necessity of (1.3) may now be proved by induction. If $F_j = 0$ a.e. fails for each j, $j = 1, \ldots, m$, then taking $k = 2$ in (2.8) shows, in view of (2.9), that (1.3) holds for $j = 1, 2$, and to complete the induction, suppose (1.3) has been verified for $j = 1, \dots, k-1$. Then $H_{k-1}(t) = \big[\prod_{j=1}^{k-1} *F_j\big](t) = \big(\prod_{j=1}^{k-1} c_j\big) e^{\alpha t + i\beta t}\omega_{k-1}^{p'/p}(t)$ a.e. on G_{k-1} and hence (2.8) and (2.9) show that $F_k(t) = c_k e^{\alpha t+i\beta t} v_k^{p'/p}(t)$ a.e. on I_k. Thus (1.3) holds for $j = k$, and the proof is complete.

Proof of Theorem 2 and Corollary 4. The proof of Theorem 2 is similar to that of Theorem 1 so the details are omitted. Note however that the local integrability hypothesis imposed on v_j in Theorem 1 was used only to ensure that the ω_j are well defined and satisfy the same hypothesis as the weights v_j. In R^n, local integrability is not sufficient to provide this assurance, and hence the stronger hypothesis of Theorem 2.

On the other hand, any hypothesis ensuring that the ω_j are well defined is sufficient to validate the remaining arguments used in the proof. Thus, while the power functions $v_j(t) = |t|^{(q_j-n)(p-1)}$ do not satisfy the hypothesis of Theorem 2, the corresponding ω_j are well defined under the hypothesis of Corollary 4 as we now show.

For $x \neq 0$ and $e_1 = (1, 0, \dots, 0) \in R^n$,

$$\begin{aligned}(|t|^{q_1-n} * |t|^{q_2-n})(x) &= |x|^{q_1+q_2-n}\int_{R^n} |t|^{q_1-n}|e_1 - t|^{q_2-n}\,dt \\ &= B_n(q_1, q_2)|x|^{q_1+q_2-n}\end{aligned}$$

and the Beta integral $B_n(q_1, q_2)$ is finite if $\min[q_1, q_2] > 0$ and $q_1 + q_2 < n$ with value $\gamma(q_1)\gamma(q_2)/\gamma(q_1 + q_2)$ where

$$\gamma(\alpha) = \pi^{n/2}2^{\alpha}\Gamma(\alpha/2)/\Gamma\big((n-\alpha)/2\big).$$

Thus, with $v_j = |t|^{(q_j-n)(p-1)}$, $q_j > 0$ and $q_1 + \cdots + q_m < n$, an induction shows we have

$$\omega_m(x) = \left[\frac{\gamma(q_1)\cdots\gamma(q_m)}{\gamma(q_1+\cdots+q_m)}\right]^{p-1}|x|^{(q_1+\cdots+q_m-n)(p-1)}.$$

Thus, in view of the remarks above, we have the inequality of Corollary 4, and since $\int_{R^n} e^{p(\alpha\cdot t)}|t|^{(q_j-n)(p-1)}\,dt = \infty$ for every $\alpha \in R^n$, equality requires $F_j = 0$ a.e. for some j.

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