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SOME COROLLARIES OF FROBENIUS' NORMAL p-COMPLEMENT THEOREM

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ABSTRACT. For a prime divisor q of the order of a finite group G, we present the set of q-subgroups generating $\mathrm{O}^{q,q'}(G)$. In particular, we present the set of primary subgroups of G generating the last member of the lower central series of G. The proof is based on the Frobenius Normal p-Complement Theorem and basic properties of minimal nonnilpotent groups. Let G be a group and Θ a group-theoretic property inherited by subgroups and epimorphic images such that all minimal non-Θ-subgroups ($= \Theta_1$-subgroups) of G are not nilpotent. Then (see the lemma), if K is generated by all Θ_1-subgroups of G it follows that G/K is a Θ-group.

Let p, q be distinct primes, $\pi(G)$ the set of all prime divisors of the order of a finite group G (we consider only finite groups), π a set of primes, $\mathrm{Syl}_p(G)$ the set of Sylow p-subgroups of G, $\Phi(G)$, $\mathrm{F}(G)$, G' and $\mathrm{Z}(G)$ the Frattini subgroup, the Fitting subgroup, the commutator subgroup and the center of G, respectively. In what follows, Θ is a nonempty group-theoretic property inherited by subgroups and epimorphic images and such that there exists a non-Θ-group.

A group is said to be *minimal nonnilpotent* if it is not nilpotent but all its proper subgroups are nilpotent. Let G be minimal nonnilpotent. It is known (see [Hup], Satz 3.5.2 — this theorem is due to O. Yu. Schmidt [S] and Yu. A. Golfand [Gol]; L. Redei [R] gave the complete classification of such groups) that

(i) $\pi(G) = \{p, q\}$ and $G = PQ$, where $P \in \mathrm{Syl}_p(G)$, $G' = Q \in \mathrm{Syl}_q(G)$;

(ii) P is cyclic and $|P : P \cap \mathrm{Z}(G)| = p$;

(iii) $Q/Q \cap \mathrm{Z}(G)$ is a minimal normal subgroup of $G/Q \cap \mathrm{Z}(G)$, Q is special. (Recall that a q-group Q is special if it is elementary abelian or $Q' = \mathrm{Z}(Q) = \Phi(Q)$; in particular, the exponent of Q is at most q^2.) It is known that if $q > 2$, then $\exp(Q) = q$.

It is easy to check that a group possessing properties (i–iii) is minimal nonnilpotent. We call a group with this structure an $\mathrm{S}(p, q)$-group.

A group G is said to be *p-nilpotent* if it has a normal p-complement. By Frobenius' Normal p-Complement Theorem and Ito's remark (see [H], Theorem 14.4.7, [Hup], Satz 4.5.4, or [I]) G is q-nilpotent if and only if it has no $\mathrm{S}(p, q)$-subgroups. A group G is said to be *p-closed* if its Sylow p-subgroup is normal.

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Let $\mathrm{O}^p(G)$ be the subgroup generated by all p'-elements of G, and $\mathrm{O}^{p,p'}(G)$ the subgroup generated by all p-elements of $\mathrm{O}^p(G)$. Obviously, $G/\mathrm{O}^{p,p'}(G)$ is p-nilpotent but, for every normal subgroup N of G properly contained in $\mathrm{O}^{p,p'}(G)$, G/N is not p-nilpotent. Let $\mathrm{H}(G) = \bigcap_{p\in\pi(G)} \mathrm{O}^p(G)$; $\mathrm{H}(G)$ is the last member of the lower central series of G. It is clear that $\mathrm{O}^{p,p'}(G) \leq \mathrm{H}(G)$.

Suppose that $\Phi(G) \leq N \trianglelefteq G$. If $N/\Phi(G)$ has a normal π-Hall subgroup $K_1/\Phi(G)$, then N has a normal π-Hall subgroup as well. Indeed, K_1 has a π-Hall subgroup K and all such subgroups are conjugate in K_1, by the Schur-Zassenhaus Theorem. Therefore, $G = K_1\mathrm{N}_G(K) = K\Phi(G)\mathrm{N}_G(K) = \mathrm{N}_G(K)$, and our claim follows since K is a π-Hall subgroup of N. It follows from what has just been proved and the Schur-Zassenhaus Theorem that $\pi(G/\Phi(G)) = \pi(G)$.

Let G be an arbitrary group.

Definition 1. A group G is said to be a $\mathrm{B}(p,q)$-*group* if $G/\Phi(G)$ is an $\mathrm{S}(p,q)$-group for some primes p, q.

Definition 2. A group G is said to be a Θ_1-group (= minimal non-Θ-group) if it is not a Θ-group but all its proper subgroups are Θ-groups.

It is clear that G is a Θ-group if and only if it has no Θ_1-subgroups.

It follows from the remark preceding Definition 1 that a $\mathrm{B}(p,q)$-group G has the form PQ, where $P \in \mathrm{Syl}_p(G)$, $G' = Q \in \mathrm{Syl}_q(G)$, $|P : P \cap \mathrm{Z}(G)| = p$, P is cyclic and $Q/\Phi(Q)$ is a minimal normal subgroup of $G/\Phi(Q)$. It is clear that a nonnilpotent epimorphic image of a $\mathrm{B}(p,q)$-group is also a $\mathrm{B}(p,q)$-group. Let $Q > \{1\}$ be a q-group. There exists a $\mathrm{B}(p,q)$-group with Sylow subgroup Q if and only if Q possesses an automorphism of order p that acts irreducibly on $Q/\Phi(G)$. For any distinct p and q, there exists a $\mathrm{B}(p,q)$-group. For $p > 2$, $n \in \mathbb{N}$, the dihedral group of order $2p^n$ is a $\mathrm{B}(2,p)$-group.

In this and the following paragraph we shall define some characteristic subgroups of an arbitrary group. Let $\mathfrak{B}_q(G)$ be the subgroup generated by normal Sylow q-subgroups (= derived subgroups) of all $\mathrm{B}(p,q)$-subgroups in G ($p \in \pi(G) - \{q\}$). By the Frobenius Normal p-Complement Theorem, $\mathfrak{B}_q(G) = \{1\}$ if and only if G is q-nilpotent. Set $\mathfrak{B}(G) = \prod_{q\in\pi(G)} \mathfrak{B}_q(G)$.

Let $\Theta_1(G)$ denote the subgroup generated by all Θ_1-subgroups of G; $\Theta_1(G) = \{1\}$ if and only if G is a Θ-group. $\Theta_1(G)$ is characteristic in G. We shall show (see the lemma below) that $G/\Theta_1(G)$ is a Θ-group for some Θ's.

Let $G = A\Theta_1(G)$ be such that the subgroup A is as small as possible. Then $\Theta_1(A) \leq A \cap \Theta_1(G) \leq \Phi(A)$ is nilpotent. Hence, if $G/\Theta_1(G)$ is not a Θ-group, there exists in G a non-Θ-subgroup A such that $\Theta_1(A) \leq \Phi(A)$; in particular, all Θ_1-subgroups of A are nilpotent. Let, in addition, Θ = nilpotency. It follows from the remark above that then A is nilpotent so $G/\Theta_1(G)$ is nilpotent. We generalize this observation in the following

Lemma. *Suppose that all Θ_1-subgroups of a group G are not nilpotent. Then G has a Θ-subgroup A such that $G = A\Theta_1(G)$. In particular, $G/\Theta_1(G)$ is a Θ-group.*

Proof. Suppose that the lemma has been proved for all proper subgroups of G. We may assume that $\Theta_1(G) > \{1\}$ (otherwise, G is a Θ-group, and the lemma is obvious). Let A be a minimal subgroup of G such that $G = A\Theta_1(G)$. Since $\Theta_1(G)$ is nonnilpotent, it is not contained in $\Phi(G)$. It follows that $A < G$. By the induction hypothesis, $A/\Theta_1(A)$ is a Θ-group. Since $\Theta_1(A) \leq A \cap \Theta_1(G) \leq \Phi(A)$, we get $\Theta_1(A) = \{1\}$ since all Θ_1-subgroups of G are not nilpotent. This means that A is a Θ-group so $G/\Theta_1(G) \cong A/A \cap \Theta_1(G)$ is also a Θ-group, as desired. □

In particular, the lemma is true for Θ such that all nilpotent groups are Θ-groups, but the assumption in the lemma is weaker. In fact, consider the case when $\Theta =$ commutativity and G is a nonabelian group all of whose Sylow subgroups are abelian. Then all Θ_1-subgroups of G are nonnilpotent (however, there are nilpotent Θ_1-groups). By the lemma, $G/\Theta_1(G)$ is abelian.

It is easy to show that the lemma is not true for Θ's such that some Θ_1-groups are nilpotent. In fact, if Θ is such that only the identity group is a Θ-group, $\Theta_1(G)$ is the subgroup of G generated by the elements of prime orders; in that case, the structure of $G/\Theta_1(G)$ may be very complicated.

Remark 1. Suppose, in addition, that a tower of Θ-groups is a Θ-group and all subgroups of prime orders are Θ-groups (in that case, we will call Θ *strong*). We will prove that if G is a Θ_1-group, then $G/\Phi(G)$ is a nonabelian simple group. Indeed, let N be a maximal normal subgroup of G and H a maximal subgroup of G. Since HN is a Θ-group, we get $N \leq H$. It follows that $N = \Phi(G)$ and $G/\Phi(G)$ is simple. It remains to show that $G/\Phi(G)$ is nonabelian. Assume that this is not true. Then $|G : \Phi(G)| = p$, a prime, so $\Phi(G)$ is maximal in G. It follows that $\Phi(G)$ is a unique maximal subgroup of G so G is a cyclic p-group. By definition, $|G| > p$ and $\Phi(G)$ is a Θ-group. It follows that G, as a tower of Θ-groups, is a Θ-group, which is a contradiction. Note that solvability and π-separability are strong properties.

Remark 2. For strong properties Θ, one can say more than the lemma says. Indeed, in the lemma, let Θ be a strong property and let G not be a Θ-group. Then $G/\Theta_1(G)$ is a Θ-group, by the lemma. We claim that, in fact, G/K is not a Θ-group if a normal subgroup K of G is properly contained in $\Theta_1(G)$. Assume that this is not true. Then G has a Θ_1-subgroup L such that $L \not\leq K$. Since $L/L \cap K$ and $K \cap L$ are Θ-groups so is L, which is not the case. In particular, if $\Theta =$ solvability, then $\Theta_1(G)$ is the last member of the derived series of G.

Let $\mathfrak{S}(G)$ be the subgroup generated by all minimal nonnilpotent subgroups of G. By the lemma, $\mathrm{H}(G) \leq \mathfrak{S}(G)$. It follows that if all minimal nonnilpotent subgroups are normal in G, then $G/\mathrm{F}(G)$ is an extension of a direct product of elementary abelian groups by a nilpotent group. Indeed, if K is generated by normal maximal subgroups of all minimal nonnilpotent subgroups of G, then $K \leq \mathrm{F}(G)$ and $\mathfrak{S}(G)/K$ is generated by normal subgroups of prime orders. But we do not know whether the nilpotent length of G is bounded if all its minimal nonabelian subgroups are normal. In general, the inequality $\mathrm{H}(G) \neq \mathfrak{S}(G)$ is possible. In fact, let $G = \mathrm{S}_n$, $n > 3$. Then it is easy to check that $\mathfrak{S}(G) = G$: the general case follows from the case $n = 4$, for which our assertion is trivial. Since $\mathrm{H}(G) = \mathrm{A}_n$, our claim follows.

Let π be a set of primes. A group G is said to be π-*decomposable* if a π-Hall subgroup is a direct factor of G. Let K be the subgroup generated by all minimal nonnilpotent subgroups of G of orders divisible by a fixed prime p. We claim that G/K is p-decomposable. Indeed, let $\Theta = p$-decomposability. By [Hup], Satz 4.5.4, and basic properties of p-solvable groups, Θ_1-groups are minimal nonnilpotent of orders divisible by p. Therefore, $K = \Theta_1(G)$. By the lemma, G/K is a Θ-group (i.e., it is p-decomposable), as claimed. Similarly, if K is generated by all minimal non-π-decomposable subgroups of G, then, by the lemma, G/K is π-decomposable. (Note that the minimal non-π-decomposable groups are not classified.)

Our principal result is the following

Theorem. (a) $\mathfrak{B}_q(G) = O^{q,q'}(G)$, *i.e., commutator subgroups of all* $B(p,q)$-*subgroups of* G $(p \in \pi(G) - \{q\})$ *generate* $O^{q,q'}(G)$.

(b) $\mathfrak{B}(G) = H(G)$. *In other words, the subgroup, generated by commutator subgroups of all* $B(p,q)$-*subgroups of* G, *where* p, q *run over the set* $\pi(G)$, *coincides with the last member of the lower central series of* G.

Proof. (a) Assume that $\mathfrak{B}_q(G) \not\leq \mathrm{O}^{q,q'}(G)$. Then G has a $\mathrm{B}(p,q)$-subgroup $F = P \cdot Q$, where $p \in \pi(G) - \{q\}$, $P \in \mathrm{Syl}_p(F)$ and $F' = Q \in \mathrm{Syl}_q(F)$ such that $Q \not\leq \mathrm{O}^{q,q'}(G)$. Then $\overline{F} = F/F \cap \mathrm{O}^{q,q'}(G)$ $(\cong F\mathrm{O}^{q,q'}(G)/\mathrm{O}^{q,q'}(G))$ is of order divisible by q; therefore, $\bar{F}$ is not q-nilpotent (otherwise, F has a normal subgroup of index q which is not the case since $|F : F'|$ is a power of $p \neq q$). Since every epimorphic image of F is nilpotent or a $\mathrm{B}(p,q)$-group, it follows that $\bar{F}$ is a $\mathrm{B}(p,q)$-group. Thus, a non-q-nilpotent group $\bar{F}$ is isomorphic to a subgroup of the q-nilpotent group $G/\mathrm{O}^{q,q'}(G)$, which is a contradiction. Hence $\mathfrak{B}_q(G) \leq \mathrm{O}^{q,q'}(G)$. Recall that $\mathrm{O}^{q,q'}(G)$ is contained in every normal subgroup N of G such that G/N is q-nilpotent. Therefore, to prove the reverse inclusion, it is enough to show that $G/\mathfrak{B}_q(G)$ is q-nilpotent. Assume that $G/\mathfrak{B}_q(G)$ is not q-nilpotent. Then it has an $\mathrm{S}(p,q)$-subgroup $\bar{S} = S/\mathfrak{B}_q(G)$, by the Frobenius Normal q-Complement Theorem. Let A be a smallest subgroup such that $S = A\mathfrak{B}_q(G)$. Then $A \cap \mathfrak{B}_q(G) \leq \Phi(A)$ and $A/A \cap \mathfrak{B}_q(G)$ is an $\mathrm{S}(p,q)$-group since it is isomorphic to $\bar{S}$. It follows that $A/\Phi(A)$ is an $\mathrm{S}(p,q)$-group as a nonnilpotent epimorphic image of the $\mathrm{S}(p,q)$-group $\bar{S}$. Therefore, by Definition 1, A is a $\mathrm{B}(p,q)$-subgroup of G. Since q divides $|\bar{S}| = |A\mathfrak{B}_q(G)/\mathfrak{B}_q(G)|$, the Sylow q-subgroup of A is not contained in $\mathfrak{B}_q(G)$, contrary to the definition of the last subgroup. Thus, $G/\mathfrak{B}_q(G)$ is q-nilpotent so $\mathrm{O}^{q,q'}(G) \leq \mathfrak{B}_q(G)$, and (a) follows.

(b) Let us prove that $K = \prod_{q \in \pi(G)} \mathrm{O}^{q,q'}(G)$ is equal to $\mathrm{H}(G)$. Since G/K is q-nilpotent for all $q \in \pi(G)$, by (a), it is nilpotent, and so $\mathrm{H}(G) \leq K$. The reverse inclusion is evident since $\mathrm{O}^{q,q'}(G) \leq \mathrm{H}(G)$ for all $q \in \pi(G)$. Therefore, $K = \mathrm{H}(G)$. By (a),

$$\mathfrak{B}(G) = \prod_{p \in \pi(G)} \mathfrak{B}_p(G) = \prod_{p \in \pi(G)} \mathrm{O}^{p,p'}(G) = K = \mathrm{H}(G),$$

completing the proof of (b). □

If K is the subgroup generated by normal Sylow subgroups of all minimal non-nilpotent subgroups of G, then G/K is not necessarily nilpotent (let G be the dihedral group of order $2p^n$, $p > 2$, $n > 1$). Moreover, we cannot prove that, in the case under consideration, G/K is solvable.

In the following paragraph we will use the Baer-Suzuki Theorem (see [HB], Theorem 9.7.8):

(*) If x is a p-element of G, then $x \in \mathrm{O}_p(G)$ if and only if $\langle x, x^y \rangle$ is a p-subgroup for all $y \in G$.

The following result is a consequence of (*) (see [B], p. 27, where another proof is given):

(**) If G has no $\mathrm{S}(2,p)$-subgroups for all odd $p \in \pi(G)$, it is 2-closed.

Let us prove that if $\Theta =$ commutativity (or cyclicity), then $G/\Theta_1(G)$ is 2-closed. Let G be a counterexample of minimal order. Suppose that $\Theta_1(G)$ is not nilpotent. Then $\Theta_1(G)$ has a nonnormal Sylow subgroup P. By Frattini's Lemma, $G = \Theta_1(G)\mathrm{N}_G(P)$. Since $\Theta_1(\mathrm{N}_G(P)) \leq \Theta_1(G) \cap \mathrm{N}_G(P)$ and $\mathrm{N}_G(P) < G$, it follows by the induction hypothesis that $\mathrm{N}_G(P)/\Theta_1(\mathrm{N}_G(P))$ (and so its epimorphic image $\mathrm{N}_G(P)/\mathrm{N}_G(P) \cap \Theta_1(G))$ is 2-closed. Since $G/\Theta_1(G)$ is isomorphic to the last group, it is also 2-closed. Thus, $\Theta_1(G)$ is nilpotent so all Θ_1-subgroups of G are nilpotent. Since $\mathrm{S}(2,p)$-groups, $p > 2$, are Θ_1-groups (see [Hup], Satz 3.5.2), G has no $\mathrm{S}(2,p)$-subgroups for all odd $p \in \pi(G)$. It follows from (**) then that G is 2-closed, as desired.

If, as in the previous paragraph, $\Theta =$ commutativity, then all nonabelian Sylow subgroups are contained in $\Theta_1(G)$ (see added in proof).

In view of what has been said, it is interesting to study the groups whose minimal nonabelian subgroups are all nilpotent (such groups are 2-closed, by (**)).

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Added in proof

It is easy to deduce from this that G/K, where $K = \Theta_1(G)$, is abelian. Indeed, assume that G/K is nonabelian. Let S/K be a minimal nonabelian subgroup of G/K; then S/K is nonnilpotent. Let A be a minimal subgroup such that $S = AK$; then A is a $B(p,q)$-subgroup. Since Sylow subgroups of A are abelian, $Z(A)$ is a p-subgroup. It follows that $A = \Theta_1(A) \leq K$, a contradiction.

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