

## ON THE SEMISIMPLICITY OF PURE SHEAVES

LEI FU

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ABSTRACT. We obtain a criteria for a pure sheaf to be semisimple. As a corollary, we prove the following: Let  $X_0$  and  $S_0$  be two schemes over a finite field  $\mathbf{F}_q$ , and let  $f_0 : X_0 \rightarrow S_0$  be a proper smooth morphism. Assume  $S_0$  is normal and geometrically connected, and assume there exists a closed point  $s$  in  $S_0$  such that the Frobenius automorphism  $F_s$  acts semisimply on  $H^i(X_{\bar{s}}, \overline{\mathbf{Q}}_l)$ , where  $X_{\bar{s}}$  is the geometric fiber of  $f_0$  at  $s$  (this last assumption is unnecessary if the semisimplicity conjecture is true). Then  $R^i f_{0*} \overline{\mathbf{Q}}_l$  is a semisimple sheaf on  $S_0$ . This verifies a conjecture of Grothendieck and Serre provided the semisimplicity conjecture holds. As an application, we prove that the Galois representations of function fields associated to the  $l$ -adic cohomologies of  $K3$  surfaces are semisimple. We also get a theorem of Zarhin about the semisimplicity of the Galois representations of function fields arising from abelian varieties.

The proof relies heavily on Deligne's work on Weil conjectures.

### 1. INTRODUCTION

Let  $\mathbf{F}_q$  be a finite field with  $q$  elements. Choose an algebraic closure  $\mathbf{F}$  of  $\mathbf{F}_q$ . Throughout this paper, schemes, morphisms and sheaves defined on the base field  $\mathbf{F}_q$  are denoted by letters with subscripts 0 and we indicate the base extension from  $\mathbf{F}_q$  to  $\mathbf{F}$  by dropping the subscripts 0. Schemes and morphisms are separated and of finite type.

Let  $X_0$  be a scheme over  $\mathbf{F}_q$ , let  $\mathcal{F}_0$  be a constructible  $\overline{\mathbf{Q}}_l$ -sheaf on  $X_0$ , and let  $\iota : \overline{\mathbf{Q}}_l \rightarrow \mathbf{C}$  be an isomorphism. Recall that  $\mathcal{F}_0$  is called  $\iota$ -pure with weight  $w$  if for every closed point  $x$  of  $X_0$  and for every eigenvalue  $\lambda$  of the (geometric) Frobenius automorphism  $F_x$  on  $\mathcal{F}_{\bar{x}}$ , the absolute value of  $\iota(\lambda)$  is  $N(x)^{w/2}$ , where  $N(x)$  is the number of elements of the residue field  $k(x)$ . Also recall that giving a lisse  $\overline{\mathbf{Q}}_l$ -sheaf on a connected scheme is the same as giving a  $\overline{\mathbf{Q}}_l$ -representation of the fundamental group of the scheme. So we can talk about the irreducibility and semisimplicity of a lisse sheaf.

If  $X_0$  is a normal geometrically connected scheme over  $\mathbf{F}_q$  and if  $\mathcal{F}_0$  is a  $\iota$ -pure lisse  $\overline{\mathbf{Q}}_l$ -sheaf on  $X_0$ , then by a theorem of Deligne ([D1], 3.4.1 (iii)),  $\mathcal{F}$  is a semisimple sheaf on  $X$ . In this paper, based on Deligne's work, we prove the following:

**Theorem.** *Let  $X_0$  be a normal geometrically connected scheme of finite type over  $\mathbf{F}_q$  and let  $\mathcal{F}_0$  be a  $\iota$ -pure lisse  $\overline{\mathbf{Q}}_l$ -sheaf on  $X_0$ . If there exists a closed point  $x$  of*

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$X_0$  such that the Frobenius automorphism  $F_x$  on  $\mathcal{F}_{\bar{x}}$  is semisimple, then  $\mathcal{F}_0$  is a semisimple sheaf on  $X_0$ .

Here a linear transformation on a vector space is said to be *semisimple* if the corresponding matrix is diagonalizable.

**Corollary.** *Let  $X_0$  and  $S_0$  be two schemes over the finite field  $\mathbf{F}_q$ , and let  $f_0 : X_0 \rightarrow S_0$  be a proper smooth morphism. Assume  $S_0$  is normal and geometrically connected, and assume there exists a closed point  $s$  in  $S_0$  such that the Frobenius automorphism  $F_s$  acts semisimply on  $H^i(X_{\bar{s}}, \overline{\mathbf{Q}}_l)$ , where  $X_{\bar{s}} = X_0 \otimes_{k(s)} \overline{k(s)}$  is the geometric fiber of  $f_0$  at  $\bar{s}$ . Then  $R^i f_{0*} \overline{\mathbf{Q}}_l$  is a semisimple sheaf on  $S_0$ .*

*Proof.* By the proper and smooth base change theorem, we know  $R^i f_{0*} \overline{\mathbf{Q}}_l$  is lisse. By Deligne’s theorem (i.e. Weil’s conjecture),  $R^i f_{0*} \overline{\mathbf{Q}}_l$  is pure. By assumption,  $F_s$  acts semisimply on  $(R^i f_{0*} \overline{\mathbf{Q}}_l)_{\bar{s}} = H^i(X_{\bar{s}}, \overline{\mathbf{Q}}_l)$ . So by the theorem,  $R^i f_{0*} \overline{\mathbf{Q}}_l$  is a semisimple sheaf on  $S_0$ .

The *semisimplicity conjecture* states that for any proper smooth scheme  $X_0$  over  $\mathbf{F}_q$ , the geometric Frobenius correspondence  $F$  acts semisimply on  $H^i(X, \overline{\mathbf{Q}}_l)$ . If this conjecture is true, then in the above corollary, we don’t need the assumption that  $F_s$  acts semisimply on  $H^i(X_{\bar{s}}, \overline{\mathbf{Q}}_l)$ . So provided the semisimplicity conjecture holds, the above corollary verifies for schemes over finite field a conjecture of Serre and Grothendieck stated in [T].

The semisimplicity conjecture is proved to be true for the following varieties over finite field:

- (a) smooth projective curves ([W]),
- (b) abelian varieties ([W]),
- (c)  $K3$  surfaces ([D2], [PS]).

Denote the function field of  $S_0$  by  $K$ . Let  $Y$  be a smooth projective variety defined over  $K$ . Assume  $Y$  is in one of the following families:

- (a) smooth projective curves,
- (b) abelian varieties,
- (c)  $K3$  surfaces.

Then by the above corollary the Galois representation

$$\text{Gal}(\overline{K}/K) \rightarrow H^i(Y \otimes_K \overline{K}, \overline{\mathbf{Q}}_l)$$

is semisimple. The semisimplicity of the Galois representations of function fields arising from abelian varieties was first proved by Zarhin ([Z]).

## 2. PROOF OF THE THEOREM

We first prove some lemmas.

**Lemma 1.** *Let  $G^0$  be a subgroup of  $G$  with finite index. Let  $G \rightarrow GL(V)$  be a finite dimensional representation of  $G$ . If the restriction  $G^0 \rightarrow GL(V)$  is semisimple, then  $G \rightarrow GL(V)$  is also semisimple.*

*Proof.* Let  $U$  be a  $G$ -stable subspace of  $V$ . We need to show  $U$  has a complement which is also  $G$ -stable. It is enough to show that there exists a homomorphism  $P : V \rightarrow U$  such that  $P|_U$  is identity and  $P$  is invariant under the action of  $G$ . Since  $V$  is a semisimple representation of  $G^0$ , we can find a homomorphism  $P_0$

such that  $P_0|_U$  is identity and that  $P_0$  is  $G^0$ -invariant. Let  $G^0g_1, \dots, G^0g_n$  be representatives of the right cosets of  $G^0$ , where  $n = [G : G^0]$ . Define

$$P = \frac{1}{n} \sum_{i=1}^n g_i^{-1} P_0 g_i.$$

One can check  $P|_U$  is identity. To see  $P$  is  $G$ -invariant, taking  $g \in G$ , then  $G^0g_1g, \dots, G^0g_ng$  are also representatives of the right cosets. So there is a permutation  $\sigma$  of  $\{1, \dots, n\}$  such that  $G^0g_ig = G^0g_{\sigma(i)}$ . Hence there exists  $g_i^0 \in G^0$  such that  $g_ig = g_i^0g_{\sigma(i)}$ . We have

$$\begin{aligned} Pg &= \frac{1}{n} \sum g_i^{-1} P_0 g_ig = \frac{1}{n} \sum g_i^{-1} P_0 g_i^0 g_{\sigma(i)} \\ &= \frac{1}{n} \sum g_i^{-1} g_i^0 P_0 g_{\sigma(i)} = \frac{1}{n} \sum gg_{\sigma(i)}^{-1} P_0 g_{\sigma(i)} = gP. \end{aligned}$$

So  $P$  is  $G$ -invariant.

**Lemma 2.** *Let  $G^0$  be a normal subgroup of  $G$ . Assume there exists an exact sequence*

$$0 \rightarrow G^0 \rightarrow G \xrightarrow{\text{deg}} \mathbf{Z} \rightarrow 0.$$

*We call the homomorphism  $G \xrightarrow{\text{deg}} \mathbf{Z}$  the degree homomorphism. Assume there exists an element in the center of  $G$  with nonzero degree. Let  $G \rightarrow GL(V)$  be a finite dimensional representation of  $G$ . If the restriction  $G^0 \rightarrow GL(V)$  is semisimple, and if there exists an element  $g \in G$  with nonzero degree such that matrix corresponding to  $g$  is diagonalizable, then  $G \rightarrow GL(V)$  is semisimple.*

*Proof.* Let  $G_d = \{x \in G : d|\text{deg}(x)\}$ , where  $d$  is the degree of an element in the center of  $G$  with nonzero degree. Then  $G_d$  is a subgroup of  $G$  and  $G/G_d$  is isomorphic to  $\mathbf{Z}/d\mathbf{Z}$ . By Lemma 1, to prove  $G \rightarrow GL(V)$  is semisimple, it is enough to prove the representation  $G_d \rightarrow GL(V)$  is semisimple. Replacing  $G$  by  $G_d$  and  $\text{deg}$  by  $\text{deg}/d$ , we may thus assume that there exists an element in the center of  $G$  with degree 1; that is, we may assume  $G = G^0 \times \mathbf{Z}$ .

By assumption, the restriction of  $G^0 \times \mathbf{Z} \rightarrow GL(V)$  to  $G^0$  is semisimple. So we have an isomorphism of  $G^0$ -representations:

$$\begin{aligned} \phi : \bigoplus_j (W_j \otimes \text{Hom}_{G^0}(W_j, V)) &\rightarrow V, \\ (w, f) &\mapsto f(w), \end{aligned}$$

where the direct sum on the left-hand side sums over all the irreducible representations  $W_j$  of  $G^0$ . If we let  $n \in \mathbf{Z}$  act on  $f \in \text{Hom}_{G^0}(W_j, V)$  by  $(nf)(w) = nf(w)$ , where  $nf(w)$  is the action of  $n \in \mathbf{Z} \subset G^0 \times \mathbf{Z}$  on  $f(w) \in V$ , then  $\phi$  is also an isomorphism of  $(G^0 \times \mathbf{Z})$ -representations. So we may assume  $V = \bigoplus_j (W_j \otimes U_j)$  as  $(G^0 \times \mathbf{Z})$ -representations, where  $W_j$  are irreducible representations of  $G^0$  and  $U_j$  are some representations of  $\mathbf{Z}$ . By assumption, there exists an element  $(g, n)$  with nonzero degree  $n$  such that it corresponds to a diagonalizable matrix in  $GL(V)$ . So  $(g, n)$  corresponds to a diagonalizable matrix in  $GL(W_j \otimes U_j)$  for each  $j$ . Then  $n$  corresponds to a diagonalizable matrix in  $GL(U_j)$  because of the fact that if the tensor product of two matrices is diagonalizable, then each is. Hence  $U_j$  is a semisimple representation of  $\mathbf{Z}$ . We can write  $U_j = \bigoplus_k U_{jk}$ , where each  $U_{jk}$  is a

one-dimensional representation of  $Z$ . We then have  $V = \bigoplus_{jk} (W_j \otimes U_{jk})$ . Obviously each  $W_j \otimes U_{jk}$  is an irreducible representation of  $G^0 \times \mathbf{Z}$ . So  $V$  is a semisimple representation of  $G^0 \times \mathbf{Z}$ .

Now let's prove the theorem. I learned the following proof (and the above Lemma 2) from P. Deligne.

We use the same notation as in [D1]. The lisse  $\overline{\mathbf{Q}}_l$ -sheaf  $\mathcal{F}_0$  gives rise to a  $\overline{\mathbf{Q}}_l$ -representation  $\pi_1(X_0, \bar{x}) \rightarrow GL(\mathcal{F}_{\bar{x}})$ . Applying the construction [D1] 1.3.7 to  $\mathcal{F}_0$ , we get

$$\begin{array}{ccccccc} 0 & \rightarrow & \pi_1(X, \bar{x}) & \rightarrow & W(X_0, \bar{x}) & \xrightarrow{\deg} & \mathbf{Z} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & G^0 & \rightarrow & G & \xrightarrow{\deg} & \mathbf{Z} \rightarrow 0 \\ & & & & \downarrow & & \\ & & & & GL(\mathcal{F}_{\bar{x}}) & & \end{array}$$

where  $G^0$  is the Zariski closure of the image of  $\pi_1(X, \bar{x}) \rightarrow GL(\mathcal{F}_{\bar{x}})$ , the group  $W(X_0, \bar{x})$  is the Weil group of  $X_0$ , and the second short exact sequence is obtained by pushing forward the first one using  $\pi_1(X, \bar{x}) \rightarrow G^0$ . By [D1] 3.4.1 (iii),  $\mathcal{F}$  is a semisimple sheaf on  $X$ , that is, the representation  $\pi_1(X, \bar{x}) \rightarrow GL(\mathcal{F}_{\bar{x}})$  is semisimple. By [D1] 1.3.9, the algebraic group  $G^0$  is an extension of a finite group by a semi-simple group, and by [D1] 1.3.11, if  $Z$  is the center of  $G$ , then the restriction of the degree map on  $Z$  has finite kernel and cokernel. ([D1] 1.3.9 and 1.3.11 hold if one just assumes  $\mathcal{F}$  is semisimple, as Deligne mentions in the proof of 1.3.9.) In particular, there exists an element in the center of  $G$  with nonzero degree.

Now assume the Frobenius automorphism  $F_x$  acts semisimply on  $\mathcal{F}_{\bar{x}}$ , and let's prove the representation  $\pi_1(X_0, \bar{x}) \rightarrow GL(\mathcal{F}_{\bar{x}})$  is semisimple. It is not hard to see the following statements are equivalent:

- (1)  $\pi_1(X_0, \bar{x}) \rightarrow GL(\mathcal{F}_{\bar{x}})$  is semisimple.
- (2)  $W(X_0, \bar{x}) \rightarrow GL(\mathcal{F}_{\bar{x}})$  is semisimple.
- (3)  $G \rightarrow GL(\mathcal{F}_{\bar{x}})$  is semisimple.

We will prove (3) is true. By [D1] 3.4.1 (iii), we know  $\pi_1(X, \bar{x}) \rightarrow GL(\mathcal{F}_{\bar{x}})$  is semisimple. It is not hard to see this implies that  $G^0 \rightarrow GL(\mathcal{F}_{\bar{x}})$  is semisimple. By assumption,  $F_x$  acts semisimply on  $\mathcal{F}_{\bar{x}}$ . Moreover there exists an element in the center of  $G$  with nonzero degree, and the degree of  $F_x$  is nonzero. So (3) holds by Lemma 2. This proves the theorem.

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INSTITUTE OF MATHEMATICS, NANKAI UNIVERSITY, TIANJIN, PEOPLE'S REPUBLIC OF CHINA  
*E-mail address*: `leifu@sun.nankai.edu.cn`