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ON THE SEMISIMPLICITY OF PURE SHEAVES

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Abstract. We obtain a criteria for a pure sheaf to be semisimple. As a corollary, we prove the following: Let X_0 and S_0 be two schemes over a finite field $\mathbf{F}_q$, and let $f_0 : X_0 \to S_0$ be a proper smooth morphism. Assume S_0 is normal and geometrically connected, and assume there exists a closed point s in S_0 such that the Frobenius automorphism F_s acts semisimply on $H^i(X_{\bar{s}}, \overline{\mathbf{Q}_l})$, where $X_{\bar{s}}$ is the geometric fiber of f_0 at s (this last assumption is unnecessary if the semisimplicity conjecture is true). Then $R^i f_{0*}\overline{\mathbf{Q}_l}$ is a semisimple sheaf on S_0. This verifies a conjecture of Grothendieck and Serre provided the semisimplicity conjecture holds. As an application, we prove that the galois representations of function fields associated to the l-adic cohomologies of $K3$ surfaces are semisimple. We also get a theorem of Zarhin about the semisimplicity of the Galois representations of function fields arising from abelian varieties.

The proof relies heavily on Deligne's work on Weil conjectures.

1. Introduction

Let $\mathbf{F}_q$ be a finite field with q elements. Choose an algebraic closure $\mathbf{F}$ of $\mathbf{F}_q$. Throughout this paper, schemes, morphisms and sheaves defined on the base field $\mathbf{F}_q$ are denoted by letters with subscripts 0 and we indicate the base extension from $\mathbf{F}_q$ to $\mathbf{F}$ by dropping the subscripts 0. Schemes and morphisms are separated and of finite type.

Let X_0 be a scheme over $\mathbf{F}_q$, let $\mathcal{F}_0$ be a constructible $\overline{\mathbf{Q}_l}$-sheaf on X_0, and let $\iota : \overline{\mathbf{Q}_l} \to \mathbf{C}$ be an isomorphism. Recall that $\mathcal{F}_0$ is called *ι-pure* with *weight w* if for every closed point x of X_0 and for every eigenvalue λ of the (geometric) Frobenius automorphism F_x on $\mathcal{F}_{\bar{x}}$, the absolute value of $\iota(\lambda)$ is $N(x)^{w/2}$, where $N(x)$ is the number of elements of the residue field $k(x)$. Also recall that giving a lisse $\overline{\mathbf{Q}_l}$-sheaf on a connected scheme is the same as giving a $\overline{\mathbf{Q}_l}$-representation of the fundamental group of the scheme. So we can talk about the irreducibility and semisimplicity of a lisse sheaf.

If X_0 is a normal geometrically connected scheme over $\mathbf{F}_q$ and if $\mathcal{F}_0$ is a ι-pure lisse $\overline{\mathbf{Q}_l}$-sheaf on X_0, then by a theorem of Deligne ([D1], 3.4.1 (iii)), $\mathcal{F}$ is a semisimple sheaf on X. In this paper, based on Deligne's work, we prove the following:

Theorem. *Let X_0 be a normal geometrically connected scheme of finite type over $\mathbf{F}_q$ and let $\mathcal{F}_0$ be a ι-pure lisse $\overline{\mathbf{Q}_l}$-sheaf on X_0. If there exists a closed point x of*

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X_0 such that the Frobenius automorphism F_x on $\mathcal{F}_{\bar{x}}$ is semisimple, then $\mathcal{F}_0$ is a semisimple sheaf on X_0.

Here a linear transformation on a vector space is said to be *semisimple* if the corresponding matrix is diagonalizable.

Corollary. *Let X_0 and S_0 be two schemes over the finite field $\mathbf{F}_q$, and let $f_0 : X_0 \to S_0$ be a proper smooth morphism. Assume S_0 is normal and geometrically connected, and assume there exists a closed point s in S_0 such that the Frobenius automorphism F_s acts semisimply on $H^i(X_{\bar{s}}, \overline{\mathbf{Q}_l})$, where $X_{\bar{s}} = X_0 \otimes_{k(s)} \overline{k(s)}$ is the geometric fiber of f_0 at $\bar{s}$. Then $R^i f_{0*}\overline{\mathbf{Q}}_l$ is a semisimple sheaf on S_0.*

Proof. By the proper and smooth base change theorem, we know $R^i f_{0*}\overline{\mathbf{Q}_l}$ is lisse. By Deligne's theorem (i.e. Weil's conjecture), $R^i f_{0*}\overline{\mathbf{Q}_l}$ is pure. By assumption, F_s acts semisimply on $(R^i f_*\overline{\mathbf{Q}_l})_{\bar{s}} = H^i(X_{\bar{s}}, \overline{\mathbf{Q}_l})$. So by the theorem, $R^i f_{0*}\overline{\mathbf{Q}_l}$ is a semisimple sheaf on S_0.

The *semisimplicity conjecture* states that for any proper smooth scheme X_0 over $\mathbf{F}_q$, the geometric Frobenius correspondence F acts semisimply on $H^i(X, \overline{\mathbf{Q}_l})$. If this conjecture is true, then in the above corollary, we don't need the assumption that F_s acts semisimply on $H^i(X_{\bar{s}}, \overline{\mathbf{Q}_l})$. So provided the semisimplicity conjecture holds, the above corollary verifies for schemes over finite field a conjecture of Serre and Grothendieck stated in [T].

The semisimplicity conjecture is proved to be true for the following varieties over finite field:
(a) smooth projective curves ([W]),
(b) abelian varieties ([W]),
(c) $K3$ surfaces ([D2], [PS]).
Denote the function field of S_0 by K. Let Y be a smooth projective variety defined over K. Assume Y is in one of the following families:
(a) smooth projective curves,
(b) abelian varieties,
(c) $K3$ surfaces.
Then by the above corollary the Galois representation

$$\mathrm{Gal}(\overline{K}/K) \to H^i(Y \otimes_K \overline{K}, \overline{\mathbf{Q}_l})$$

is semisimple. The semisimplicity of the Galois representations of function fields arising from abelian varieties was first proved by Zarhin ([Z]).

2. Proof of the Theorem

We first prove some lemmas.

Lemma 1. *Let G^0 be a subgroup of G with finite index. Let $G \to GL(V)$ be a finite dimensional representation of G. If the restriction $G^0 \to GL(V)$ is semisimple, then $G \to GL(V)$ is also semisimple.*

Proof. Let U be a G-stable subspace of V. We need to show U has a complement which is also G-stable. It is enough to show that there exists a homomorphism $P : V \to U$ such that $P|_U$ is identity and P is invariant under the action of G. Since V is a semisimple representation of G^0, we can find a homomorphism P_0

such that $P_0|_U$ is identity and that P_0 is G^0-invariant. Let $G^0g_1, \dots, G^0g_n$ be representatives of the right cosets of G^0, where $n = [G : G^0]$. Define

$$P = \frac{1}{n}\sum_{i=1}^{n} g_i^{-1}P_0g_i.$$

One can check $P|_U$ is identity. To see P is G-invariant, taking $g \in G$, then $G^0g_1g, \dots, G^0g_ng$ are also representatives of the right cosets. So there is a permutation σ of $\{1, \dots, n\}$ such that $G^0g_ig = G^0g_{\sigma(i)}$. Hence there exists $g_i^0 \in G^0$ such that $g_ig = g_i^0g_{\sigma(i)}$. We have

$$Pg = \frac{1}{n}\sum g_i^{-1}P_0g_ig = \frac{1}{n}\sum g_i^{-1}P_0g_i^0g_{\sigma(i)}$$

$$= \frac{1}{n}\sum g_i^{-1}g_i^0P_0g_{\sigma(i)} = \frac{1}{n}\sum gg_{\sigma(i)}^{-1}P_0g_{\sigma(i)} = gP.$$

So P is G-invariant.

Lemma 2. *Let G^0 be a normal subgroup of G. Assume there exists an exact sequence*

$$0 \to G^0 \to G \stackrel{deg}{\to} \mathbf{Z} \to 0.$$

We call the homomorphism $G \stackrel{deg}{\to} \mathbf{Z}$ the degree homomorphism. Assume there exists an element in the center of G with nonzero degree. Let $G \to GL(V)$ be a finite dimensional representation of G. If the restriction $G^0 \to GL(V)$ is semisimple, and if there exists an element $g \in G$ with nonzero degree such that matrix corresponding to g is diagonalizable, then $G \to GL(V)$ is semisimple.

Proof. Let $G_d = \{x \in G : d|deg(x)\}$, where d is the degree of an element in the center of G with nonzero degree. Then G_d is a subgroup of G and G/G_d is isomorphic to $\mathbf{Z}/d\mathbf{Z}$. By Lemma 1, to prove $G \to GL(V)$ is semisimple, it is enough to prove the representation $G_d \to GL(V)$ is semisimple. Replacing G by G_d and deg by deg/d, we may thus assume that there exists an elment in the center of G with degree 1; that is, we may assume $G = G^0 \times \mathbf{Z}$.

By assumption, the restriction of $G^0 \times \mathbf{Z} \to GL(V)$ to G^0 is semisimple. So we have an isomorphism of G^0-representations:

$$\phi : \bigoplus_j \big(W_j \otimes \mathrm{Hom}_{G^0}(W_j, V)\big) \to V,$$

$$(w, f) \mapsto f(w),$$

where the direct sum on the left-hand side sums over all the irreducible representations W_j of G^0. If we let $n \in \mathbf{Z}$ act on $f \in \mathrm{Hom}_{G^0}(W_j, V)$ by $(nf)(w) = nf(w)$, where $nf(w)$ is the action of $n \in \mathbf{Z} \subset G^0 \times \mathbf{Z}$ on $f(w) \in V$, then ϕ is also an isomorphism of $(G^0 \times \mathbf{Z})$-representations. So we may assume $V = \bigoplus_j (W_j \otimes U_j)$ as $(G^0 \times \mathbf{Z})$-representations, where W_j are irreducible representations of G^0 and U_j are some representations of $\mathbf{Z}$. By assumption, there exists an element (g, n) with nonzero degree n such that it corresponds to a diagonalizable matrix in $GL(V)$. So (g, n) corresponds to a diagonalizable matrix in $GL(W_j \otimes U_j)$ for each j. Then n corresponds to a diagonalizable matrix in $GL(U_j)$ because of the fact that if the tensor product of two matrices is diagonalizable, then each is. Hence U_j is a semisimple representation of $\mathbf{Z}$. We can write $U_j = \bigoplus_k U_{jk}$, where each U_{jk} is a one-dimensional representation of Z. We then have $V = \bigoplus_{jk}(W_j \otimes U_{jk})$. Obviously each $W_j \otimes U_{jk}$ is an irreducible representation of $G^0 \times \mathbf{Z}$. So V is a semisimple representation of $G^0 \times \mathbf{Z}$.

Now let's prove the theorem. I learned the following proof (and the above Lemma 2) from P. Deligne.

We use the same notation as in [D1]. The lisse $\overline{\mathbf{Q}_l}$-sheaf $\mathcal{F}_0$ gives rise to a $\overline{\mathbf{Q}}_l$-representation $\pi_1(X_0, \bar{x}) \to GL(\mathcal{F}_{\bar{x}})$. Applying the construction [D1] 1.3.7 to $\mathcal{F}_0$, we get

$$
\begin{array}{ccccccccc}
0 & \to & \pi_1(X, \bar{x}) & \to & W(X_0, \bar{x}) & \stackrel{deg}{\to} & \mathbf{Z} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & G^0 & \to & G & \stackrel{deg}{\to} & \mathbf{Z} & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & GL(\mathcal{F}_{\bar{x}}) & & & &
\end{array}
$$

where G^0 is the Zariski closure of the image of $\pi_1(X, \bar{x}) \to GL(\mathcal{F}_{\bar{x}})$, the group $W(X_0, \bar{x})$ is the Weil group of X_0, and the second short exact sequence is obtained by pushing forward the first one using $\pi_1(X, \bar{x}) \to G^0$. By [D1] 3.4.1 (iii), $\mathcal{F}$ is a semisimple sheaf on X, that is, the representation $\pi_1(X, \bar{x}) \to GL(\mathcal{F}_{\bar{x}})$ is semisimple. By [D1] 1.3.9, the algebraic group G^0 is an extension of a finite group by a semi-simple group, and by [D1] 1.3.11, if Z is the center of G, then the restriction of the degree map on Z has finite kernel and cokernel. ([D1] 1.3.9 and 1.3.11 hold if one just assumes $\mathcal{F}$ is semisimple, as Deligne mentions in the proof of 1.3.9.) In particular, there exists an element in the center of G with nonzero degree.

Now assume the Frobenius automorphism F_x acts semisimply on $\mathcal{F}_{\bar{x}}$, and let's prove the representation $\pi_1(X_0, \bar{x}) \to GL(\mathcal{F}_{\bar{x}})$ is semisimple. It is not hard to see the following statements are equivalent:

(1) $\pi_1(X_0, \bar{x}) \to GL(\mathcal{F}_{\bar{x}})$ is semisimple.
(2) $W(X_0, \bar{x}) \to GL(\mathcal{F}_{\bar{x}})$ is semisimple.
(3) $G \to GL(\mathcal{F}_{\bar{x}})$ is semisimple.

We will prove (3) is true. By [D1] 3.4.1 (iii), we know $\pi_1(X, \bar{x}) \to GL(\mathcal{F}_{\bar{x}})$ is semisimple. It is not hard to see this implies that $G^0 \to GL(\mathcal{F}_{\bar{x}})$ is semisimple. By assumption, F_x acts semisimply on $\mathcal{F}_{\bar{x}}$. Moreover there exists an element in the center of G with nonzero degree, and the degree of F_x is nonzero. So (3) holds by Lemma 2. This proves the theorem.

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