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THE LENGTH AND THICKNESS OF WORDS IN A FREE GROUP

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Abstract. In this paper we generalize the notion of a cut point of a graph. We assign to each graph a non-negative integer, called its thickness, so that a graph has thickness 0 if and only if it has a cut point. We then apply a method of J. H. C. Whitehead to show that if the coinitial graph of a given word has thickness t, then any word equivalent to it in a free group of rank n has length at least $2nt$. We also define what it means for a word in a free group to be separable and we show that there is an algorithm to decide whether or not a given word is separable.

1. Introduction

Let F be a free group of rank $n \geq 2$ with basis $\{x_1, x_2, ..., x_n\}$. We shall regard each element x of F as a reduced word in the letters $x_1, x_1^{-1}, x_2, x_2^{-1}, ..., x_n, x_n^{-1}$. Following the usual convention, the symbol $|x|$ denotes the length of x as a word in our alphabet.

Definition 1. The symbol $[x]$ represents the set of all words in F equivalent to x, i.e. $\{\Phi(x) : \Phi \in Aut(F)\}$. The length of the smallest element equivalent to x is denoted as $\|x\|$.

It is clear that all cyclic conjugates of x are in $[x]$ and that if $y \in [x]$, then $[x] = [y]$. By a result of J. H. C. Whitehead, [5], $\|x\|$ can be effectively computed. Now $\|x\| = 1$ if and only if x is primitive, i.e. part of a basis for F. It is not difficult to prove that $\|x^m\| = |m|\|x\|$, however in general $\|\|$ behaves rather unpredictably with respect to multiplication in F even when the multiplication does not involve cancellation. For example in the free group F_2 of rank 2 generated by a and b, $\|ab^2a\| = 4$ and $\|b^3\| = 3$ yet $\|ab^2ab^3\| = 1$ since $\{ab^2ab^3, ab^2\}$ is a basis for F_2.

We would like to introduce another integer, $t(x)$, which we call the thickness of $x \in F$. This integer will behave "better" under multiplication than $||$.

Definition 2. Let Γ be a finite graph without loops but with possibly multiple edges between vertices. By a path in Γ between v and w we mean a sequence of vertices $v_1, v_2, ..., v_m$ where $v = v_1, w = v_m$ and there exists at least one edge between consecutive vertices. The thickness of a path is the largest integer k, such that there exist at least k edges between consecutive vertices. We define $t(\Gamma)$, the

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thickness of Γ, to be the largest integer k, such that for all triples of distinct vertices $\{u, v, w\}$, there is a path between u and v of thickness k which misses w.

We see that $t(\Gamma)$ is in a certain way a generalization of a cut point of a graph. By that we mean $t(\Gamma) = 0$ if and only if Γ is disconnected or has a cut point.

Definition 3. Let x be a cyclically reduced word in F; the thickness of x, $t(x)$, is equal to $t(\Gamma_x)$ where Γ_x is the coinitial(star) graph of the word x.

Let $y_1, y_2, ..., y_m$ be reduced words in our alphabet. The expression $z \equiv y_1 y_2 ... y_m$ means that word $y_1 y_2 ... y_m$ is reduced and that $z = y_1 y_2 ... y_m$. According to [4], if $t(x) > 0$, then x is not primitive.

Several trivial facts about $t(x)$ are as follows:

(1) $t(x^k) = |k| t(x)$.

(2) If $y \equiv uxv$, then $t(y) \geq t(x) - 1$.

J. H. C. Whitehead [5] introduced geometric methods to study the relationship between x and $\Phi(x)$. In this paper we will apply his methods to give a lower bound on $[x]$ in terms of $t(x)$.

We will now introduce rather briefly the method of J. H. C. Whitehead. The terminology employed will be that used in [1]; in particular the more descriptive term "endcap" introduced in the latter is used in place of Whitehead's "two element".

If D is a closed 3-ball smoothly embedded in S^3, $int(D)$ will denote the interior of D and $bd(D)$ will denote its boundary. Now suppose that $D_1, D_1', D_2, D_2', ..., D_n, D_n'$ are disjoint 3-balls smoothly embedded in S^3, define $\tilde{M}$ to be

$$S^3 \setminus \bigcup_{i=1}^{n} (int(D_i) \cup int(D_i'))$$

and let M be the quotient space of $\tilde{M}$ obtained when $bd(D_i)$ is identified to $bd(D_i')$ via an orientation reversing homeomorphism. M is easily seen to be a connected sum of n copies of $S^1 \times S^2$ and thus $\Pi_1(M) \cong F$. From now on the 3-balls $D_1, D_1', D_2, D_2', ..., D_n, D_n'$ will be fixed and shall be referred to as the standard elements. Let S_i denote the image in M of $bd(D_i)$. In S^3, let each $bd(D_i)$ be given a normal direction by the vector which points away from $int(D_i)$. The image of this vector is a normal direction to each S_i. The collection of 2-spheres $\{S_1, S_2, ..., S_n\}$, together with their outward normals is called the standard sphere basis of M and is denoted as S. Let $\{\Sigma_1, \Sigma_2, ..., \Sigma_n\}$ be another family of disjoint 2-spheres embedded in M, where each Σ_i has some predefined normal direction. If we denote this family as Σ, then we say that Σ is a sphere basis of M if there exists a homeomorphism of M onto itself which sends S_i onto Σ_i for all i and which preserves the outward normal directions. It is an easily proved fact that if Σ is a sphere basis, then any choice of outward normals to the Σ_i also gives a sphere basis.

Let T be a smoothly embedded 2-sphere in M which is transverse to each S_i and let $\tilde{T}$ be its preimage in $\tilde{M}$. Either $\tilde{T}$ is connected and is thus a 2-sphere or else it is disconnected and consists of endcaps (2-cells) and 2-cells with holes. Let us assume that the latter is true and let A be some component of $\tilde{T}$. The boundary of A consists of disjoint circles each one of which lies on one of the 2-spheres $bd(D_i)$ or $bd(D_i')$. If each component of the boundary of A lies on a different 2-sphere, we shall say that A is *normal* with respect to S. We say that T is *normal* with respect to S if either $\tilde{T}$ is connected or else each component of $\tilde{T}$ is *normal* with respect to S.

Definition 4. Let E be an endcap of $\tilde{T}$ and assume that the boundary C of E lies on $bd(D_i)$. Now C separates $bd(D_i)$ into two discs. The proper disc, E_p, is the disc so that, in S^3, $E \cup E_p$ separates $int(D_i)$ from $int(D'_1)$. Exactly one of the discs is proper. The other disc is called the improper disc and is denoted as E_q.

2. Some preliminary results

Proposition 1. *Let T be a 2-sphere smoothly embedded in M which is transverse to S; then exactly one of the following holds:*

(1) *T is trivial, i.e. T bounds a three ball in M.*
(2) *T is primitive, i.e. there exists a homeomorphism of M onto itself which sends T to some S_i.*
(3) *T is separant, i.e. $M \setminus T$ consists of two components, each homeomorphic to either $S^1 \times S^2$ with a closed 3-ball removed or a non-trivial connected sum of $S^1 \times S^2$'s with a closed 3-ball removed.*

Proof. Our proof follows the method in [5]. Let k be the minimum number of circles of intersection of T with Σ as Σ ranges over all sphere bases for M which are transverse to T. Whitehead's method is by induction on the number k. We first assume that $k = 0$. In this situation we can find a homeomorphism of M onto itself which sends Σ onto S. The image of T, which we also denote as T, does not intersect S. The lift of T in S^3, which we also call T, is a 2-sphere in S^3 and therefore either

(1) T bounds a 3-ball which contains none of the balls D_i or D'_i.
(2) T separates some D_i from D'_i.
(3) T is nontrivial and no D_i is separated from D'_i by T, i.e. T is separant.

According to Singer [3], if we are in (2), then there exists a homeomorphism of M which sends S_i onto T and which maps each S_j, $i \neq j$, onto itself. Thus when $k = 0$ our lemma is proven.

We now assume that the lemma is proven for all $k \leq m - 1$ and we may assume, by the argument above, that $T \cap S$ consists of m circles. Let E be an endcap of T and assume that the boundary of E lies on $bd(D_i)$. Now $E \cup E_p$ is a 2-sphere embedded in S^3 which separates $int(D_i)$ from $int(D'_i)$. Slightly push this 2-sphere into the component which contains $int(D'_i)$ and call this new 2-sphere E^*. Now $T \cap E^*$ has fewer components than $T \cap S_i$ and E^* separates D_i from D'_i. Now replace S_i by E^* and call the new family Σ. Σ is also a sphere basis for M and T intersects Σ in fewer than m circles. Applying our inductive hypothesis concludes the proof of the lemma. $\square$

We should point out that Proposition 1 is valid if S is replaced by an arbitrary sphere basis. The advantage, which we shall exploit in the next section, of replacing some sphere basis with S is that it allows arguments to take place in S^3.

3. Main results

Definition 5. Let α be a directed simple closed curve, with base point, embedded in M. If Σ is a sphere basis which is transverse to α, the word obtained by transversing α, see e.g. [1], is denoted as $\alpha(\Sigma)$.

According to Whitehead [5], as Σ ranges over all sphere bases for M the set of elements obtained is just $[x]$ where $x = \alpha(S)$. We shall thus denote this set of elements as $[\alpha]$. If T is a 2-sphere embedded in M which is transverse to α, we shall use the notation $\alpha \cdot T$ to denote the number of points of intersection of α and T.

Theorem 1. *Let T be a non-trivial 2-sphere smoothly embedded in M and let x be a cyclically reduced word in F such that $t(x) \geq 1$. If α is a simple closed curve such that $x \in [\alpha]$ and α is transverse to T, then $\alpha \cdot T \geq 2t(x)$.*

Proof. For each curve α, which is transverse to T and $x \in [\alpha]$ let Σ_α be a sphere basis with the following properties:

(1) $\alpha(\Sigma_\alpha) = x$.
(2) Σ_α is transverse to T.
(3) The number of circles of intersection of Σ_α and T is minimal with respect to (1) and (2).

Our proof is by induction on k, the number of circles of intersection of T with Σ_α, where α varies over all curves with the property that $x \in [\alpha]$.

We first assume that $k = 0$. In this case we have a closed curve α, a sphere basis Σ_α such that $\alpha(\Sigma_\alpha) = x$ and $T \cap \Sigma_\alpha = \emptyset$. Thus by taking a homeomorphism we may assume that $T \cap S = \emptyset$ and $\alpha(S) = x$. Let $\tilde{\alpha}$ be the lift of α to S^3. $\tilde{\alpha}$ consists of disjoint arcs with endpoints on the boundary of the D_i and D_i' which miss the interior of these balls. The graph obtained by collapsing each D_i and D_i' to individual vertices and naming these vertices x_i and x_i^{-1} is isomorphic to Γ_x. Since $k = 0$, T can be regarded as a 2-sphere in S^3. T separates S^3 into two 3-balls, B and B'. Both B and B' must contain some D_i or D_i' since T is assumed to be non-trivial. Let the corresponding vertices in Γ_x be v and v' respectively. Now there exists a path P in Γ_x between v and v' whose thickness is at least $t(x)$. Thus there exist 2 consecutive vertices in P, such that the corresponding standard elements, D and D', have the property that D lies in B and D' lies in B'. The edges connecting these vertices correspond injectively to a collection of edges in $\tilde{\alpha}$ which meet T, each of which meets T at least once. Since $n \geq 2$, there must be another standard element inside B or B'. Without loss of generality, assume this standard element, denoted D'', lies in B. If w denotes the vertex in Γ_x which corresponds to D'', then there is a path P' between w and v' which misses v and whose thickness is greater than or equal to $t(x)$. This implies that there is a second pair of standard elements, one of which lies in B and the other in B', which is not identically equal to D and D' and that there are at least $t(x)$ edges between them. These edges are disjoint from the first set. Hence we have at least $2t(x)$ subarcs of α which meet T at least once. Our theorem is therefore proven in the case that $T \cap \Sigma_\alpha$ is empty.

We now assume that the theorem is valid for all α such that T intersects Σ_α in $m - 1$ or fewer circles. Let us assume that the number of circles of intersection of T and Σ_α is m. Again we may assume that Σ_α is the standard basis S. Our argument again takes place in S^3. Let E be an endcap of T. Now E meets a unique standard element in some circle; without loss of generality we shall assume it meets the standard element labelled D_i. We will denote the circle of intersection as C. Now let us consider $E \cup E_q$, which we regard as a 2-sphere in S^3. This 2-sphere separates S^3 into two 3-balls, B and B', where B' refers to the component which contains both D_i and D_i'.

Suppose that B does not contain any standard element. We first isotope, if necessary, α, to another closed curve called β, in the following manner. If a component of $\tilde{\alpha}$ has neither endpoint on E_q, it is isotoped, keeping its endpoints fixed, to an arc which does not meet T. If a component has one of its endpoints on E_q (since x is assumed to be cyclically reduced, it can have at most one endpoint on E_q), it is isotoped, keeping its endpoints fixed, to an arc which meets T exactly once. Choosing the basepoint and direction of β appropriately it is clear that $\beta(S) = x$ and that $\beta \cdot T \leq \alpha \cdot T$. We now replace S_i by E^* and call this new sphere basis Σ. Since T intersects Σ in fewer than m circles, we have by the induction hypothesis that $\beta \cdot T \geq 2t(x)$; hence $\alpha \cdot T \geq 2t(x)$.

We thus may assume that, for each endcap of T, the corresponding 3-ball B contains some standard element. Let E be an endcap of T and let D denote a standard element contained in B. In Γ_x, we denote the vertices which correspond to the distinct standard elements D_i, D_i' and D as v, v' and w respectively. There is a path P in Γ_x of thickness $t(x)$ from w to v' which misses v. This implies that there are two standard elements, one in B and the other in B' neither of which is D_i, and at least $t(x)$ components of $\tilde{\alpha}$ which have their endpoints on these elements. Each of these components must meet E. Since T must have at least two endcaps, our theorem is proven. □

Corollary 1. *Suppose that $x \in F$ is cyclically reduced and that $y \in [x]$. The number of times that x_i or x_i^{-1} appears in y is greater than or equal to $2t(x)$.*

Proof. Let $y = \Phi(x)$ for some automorphism Φ. According to [5], there are a sphere basis Σ, which is normal to S, and a directed, based simple closed curve α such that the $\alpha(S) \equiv x$ and $\alpha(\Sigma) \equiv y$. If Σ_i is a 2-sphere in Σ, then Σ_i is non-trivial and thus meets α at least $2t(x)$ times. This implies that x_i appears at least $2t(x)$ times. □

Corollary 2. *If x is cyclically reduced, then $\|x\| \geq 2nt(x)$.*

We shall now list some easily proven facts:

(1) $t(x_1^2 x_2^2 ... x_n^2) = 1$.
(2) If n is even, then $t([x_1, x_2][x_3, x_4]...[x_{n-1}, x_n]) = 1$.
(3) $t(x^m) = mt(x)$ when x is cyclically reduced.
(4) If $x \equiv uv$, then $t(x) \geq t(u) - 1$.
(5) If $t(x) \geq 1$, then $t(x^m y)$ goes to infinity as m goes to infinity for y fixed. We do not require that the word xy is reduced.

A proper factorization of F is a pair of proper subgroups, F_1 and F_2, so that the inclusions induce an isomorphism from $F_1 * F_2$ onto F. We say that x is separable if there exists a proper factorization of F and elements $u_i \in F_i$ such that $x = u_1 u_2$.

Corollary 3. *Let $x \in F$; if $t(x) \geq 2$, then x is not separable.*

Proof. Suppose x were separable. Choose a basis $\{y_1, y_2, ..., y_m\}$ for F_1 and choose a basis for F_2, say $\{z_1, z_2, ..., z_{n-m}\}$. We can find a sphere basis

$$\Sigma = \Sigma_1, \Sigma_2, ..., \Sigma_m, \Sigma_1', \Sigma_2', ..., \Sigma_{n-m}'$$

so that if α is a closed curve with the property that $\alpha(S) = x$, then $\alpha(\Sigma) =$ the representation of x with respect to the basis

$$\{y_1, y_2, ..., y_m, z_1, z_2, ..., z_{n-m}\}.$$

Let T be 2-sphere which separates the Σ-spheres from the Σ'-spheres. Since x was assumed to be separable, α can be isotoped to meet T at most twice: this is a contradiction since $t(x) \geq 2$ implies it meets T at least 4-times. □

As an application we note that if $x = x_1^{m_1} x_2^{m_2} ... x_n^{m_n}$ where $m_k \geq 2$, then x is *separable* but x^p is not *separable* if $p \geq 2$.

Theorem 2. *Let $x \in F$. If x is separable, then there exists $y \in [x]$ such that $|y| = \|x\|$ and $y \equiv uv$ where u is a word in the first m letters and v is a word in the last $n - m$ letters for some $1 \leq m < n$.*

Proof. Let us assume that $|x| = \|x\|$. If x were in a proper free factor, then the result follows from a theorem of A. Shenitzer [2]. Thus we may assume there exists an automorphism Φ such that $\Phi(x) = u_1 u_2$, where u_1 is a non-trivial word in the first m-letters and u_2 is a non-trivial word in the last $n - m$-letters. Using Whitehead's method we can find a sphere basis Σ and a directed simple closed curve with base point, α, such that $\alpha(S) \equiv x$ and $\alpha(\Sigma) \equiv u_1 u_2$. Thus there exists a separant sphere T such that α intersects T twice. Our proof is by induction on the number of circles, k, in $S \cap T$. If $k = 0$, then by a change of basepoints (cyclic permutation of x), the theorem is proven.

Our inductive step is the following assumption. Suppose that x is a cyclically reduced word and α is a simple closed curve in M so that $\alpha(S) = x$. Suppose that T is a separant sphere such that $\alpha \cdot T = 2$ and $T \cap S$ consists of $p - 1$ or fewer circles. Then there exists an automorphism Φ so that $|\Phi(x)| \leq |x|$ and $\Phi(x) \equiv u_1 u_2$ where u_1 is a word in the first m letters and u_2 is a word in the last $n - m$ letters. We now assume that an α and T are given as above and that $T \cap S$ consists of p circles. Let E be an endcap of T. If $\alpha \cap E$ is empty, then replacing S_i by E^* as in the previous arguments proves the theorem.

We thus may assume that E intersects α exactly once, since there are least 2 endcaps and T intersects α exactly twice. Now B either contains some standard element D (D') but not D' (D) or the standard elements in B come in pairs D, D', etc. If the latter is true, then α must meet E_q, since the only places α can enter or leave B is at E or E_q. By our previous arguments if we replace S_i by E^*, we obtain our result. If the former is true and α meets E_q, then we use the same argument as in the line above. If α does not meet E_q, then it meets the 2-sphere $\Sigma_1' = E \cup E_q$ transversely in one point. This 2-sphere cannot separate M since if it did any simple closed curve would have to meet it transversely an even number of times. Thus according to Proposition 1, Σ_1' is primitive. Let Σ' be a sphere basis in which Σ_1' is the first sphere. The letter x_1 appears exactly once in the word $\Phi'(x)$ where Φ' is the automorphism of F determined by Σ'. This implies that $\Phi'(x)$ and hence x is primitive. Our result clearly follows in this case and our theorem is proven. □

Since it is a decidable problem to list all the words of minimal length in $[x]$, it is therefore decidable whether or not x is *separable*. Unlike the Shenitzer result not all separable words of minimal length can be factored even after taking a cyclic permutation. For example no cyclic permutation of $x = aba^{-1}b$ can be factored. The coinitial graph of x has thickness 1 and thus x is of minimal length. However x is separable since the automorphism which sends a to a and b to ab sends x to a^2b^2.

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