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AN APPLICATION OF THE REGULARIZED SIEGEL-WEIL FORMULA ON UNITARY GROUPS TO A THETA LIFTING PROBLEM

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ABSTRACT. Let $U(2)$ and $U(2,1)$ be the pair of unitary groups over a global field F and π an irreducible cuspidal representation of $U(2)$ which satisfies a certain L-function condition. By using a regularized Siegel-Weil formula, we can show that the global theta lifting of π in $U(2,1)$ is non-trivial if every local factor π_υ of π has a local theta lifting (Howe lifting) in $U(2,1)(F_\upsilon)$.

1. INTRODUCTION

In this paper, we will apply a regularized Siegel-Weil formula [Tan2] to a theta lifting problem for the dual pair of unitary groups $U(2)$ and $U(2,1)$ defined over a global field F. The motivation of this work come from [KR] and [KRS] which proved the regularized Siegel-Weil formula for the symplectic-orthogonal dual pair $Sp(n), O(m)$ and applied it to poles and values of L-functions of $Sp(n)$.

Given an irreducible cuspidal representation π of $U(2)$, we would like to ask if its global theta lifting $\Theta(\pi)$ in $U(2,1)$, to be defined below, is non-trivial. Let $\bigotimes \pi_\upsilon$ be the decomposition of π as a restricted tensor product where υ runs through all the places of F. We say that π_υ has a local theta lift (or Howe lift) to $U(2,1)(F_\upsilon)$ if π_υ occurs in the (local) Howe correspondence for the dual reductive pair. (See for example [MVW].)

We will show that, under a certain condition on π, if π_υ has a local theta lifting everywhere, then π has a global theta lifting $\Theta(\pi)$. More precisely,

Theorem 1.1. *Let $\pi = \bigotimes \pi_\upsilon$ be a cuspidal representation of $U(2)$ such that π_υ belongs to the discrete series for all ramified places υ and the standard Langlands L-function $L(s,\pi,\gamma^3)$ has no pole at $s = 1$. Then the global theta lifting $\Theta(\pi)$ of π to $U(2,1)$ is non-trivial if and only if the local theta lifting of π_υ to $U(2,1)_\upsilon$ is nontrivial for all υ.*

All the notations and terminologies will be introduced in section 2 and the proof of the theorem will be given in section 4.

We remark here that, for a general dual pair $U(m)$ and $U(n)$ such that $m > 2n$ (the convergent range), it has been proved in [Li2] that this is the case using the classical Siegel-Weil formula on $U(m)$ and $U(n,n)$. Note that this pair lies in the convergent range for the classical formula to work. Since our dual pair falls outside

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this range, we have to resort to the regularized version of the Siegel-Weil formula which we will recall in section 3.

This paper is based on part of the author's dissertation [Tan1] under the guidance of Jonathan Rogawski. This result was first announced in [GRS1] based on the approach in [Tan1]. The result was later improved in [GRS2] using a sophisticated arguments involving endoscopic L-packets. It should be mentioned that, in that paper, the hypothesis that π_v belongs to the discrete series at all ramified places has been removed. However, this condition is needed in our approach in order to conclude the non-vanishing of certain local integrals defined in section 4. We believe that the local theta lifting of π_v should be enough to imply these local integrals are nonzero provided they converge. Nevertheless, our approach is more elementary and our emphasis is on the application of the Siegel-Weil formula.

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2. Set up

Fix a quadratic extension E of F. Let $G = U(2)$ act on the two-dimensional Hermitian space U over E with Hermitian form $(,)$ and $H = U(2,1)$ act on the three dimensional skew-Hermitian space V over E with skew-Hermitian form $(,)'$. We form the tensor product $W = U \otimes_E V$ together with the symplectic form $\langle , \rangle = (,) \otimes \overline{(,)'}$. Let $X \oplus Y$ be a complete polarization of W.

Let G' be $U(2,2)$ so that $G' \times H \subset Sp(W')$, where $W' = W \oplus W$ with Hermitian form $\langle\langle , \rangle\rangle := \langle , \rangle \oplus -\langle , \rangle$. Then

$$(X \oplus X) \bigoplus (Y \oplus Y)$$

is a complete polarization of W'. We can choose a basis for W' so that, with respect to this decomposition, the Hermitian form $\langle\langle , \rangle\rangle$ is given by the matrix

$$\begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}.$$

We also have another complete polarization of W'. Let

$$U^d := \{(u,u) | u \in U\},\ U_d := \{(u,-u) | u \in U\}$$

and

$$W^d := U^d \otimes V,\ W_d := U_d \otimes V.$$

Then $W' = W_d \oplus W^d$.

Let P' be the parabolic subgroup of G' that stabilizes U^d (and hence W^d).

We can embed the product $G \times G$ into G' by an embedding ι such that

$$\iota(g_1, g_2)(u_1, u_2) = (g_1 u_1, g_2 u_2)$$

for all $g_i \in G$ and $u_i \in U$. We identify $G \times G$ with its image under ι.

Given a non-trivial additive character ψ of $\mathbb{A}/F$, there is a metaplectic representation, depending on ψ, on the metaplectic cover of $Sp(W')(\mathbb{A})$. Here $\mathbb{A}$ denotes the adele ring of F. By [GR], Proposition 3.1.1, we have an explicit splitting of the metaplectic cover over $G'(\mathbb{A}) \times H(\mathbb{A})$. This splitting is not unique and depends on the choice of ψ and a Hecke character γ whose restriction to F is $\omega_{E/F}$, the quadratic character of the extension E/F defined by class field theory. Then the metaplectic representation induces, via the splitting, an oscillator representation $\omega = \omega(\psi,\gamma)$ of $G'(\mathbb{A}) \times H(\mathbb{A})$. We have

$$\omega \circ \iota = \omega_+ \otimes \omega_-$$

where ω_+ (resp. ω_-) is the oscillator representation of $G \times H$ determined by ψ (resp. $\bar{\psi}$) and γ. We can also define similarly the local oscillator representation ω_v (resp. $\omega_{\pm,v}$) of $G'_v \times H_v$ (resp. $G_v \times H_v$) but we often abuse notation by omitting the subscript v from ω_v and $\omega_{\pm,v}$.

Now, for φ_1, φ_2 any two Schwartz functions in $S(X)$, the tensor product $\varphi = \varphi_1 \otimes \bar{\varphi}_2$ belongs to $S(X \oplus X)$. We pass from $S(X \oplus X)$ to $S(W_d)$ via the partial Fourier transform $\varphi \longmapsto \varphi^*$ where

$$\varphi^*(w) = \int_X \psi(2\langle v, y\rangle)\varphi(v+x, v-x)dv.$$

Here we have identified $w \in W_d$ with $(x,y) \in (X \oplus Y) = W$ by mapping $(w,-w)$ to w. We then have

$$\varphi^*(0) = \int_X \varphi_1(v)\bar{\varphi}_2(v)dv = \langle \varphi_1, \varphi_2\rangle \tag{2.1}$$

where $\langle , \rangle$ denotes the inner product in $L^2(X)$. Also, the two theta series

$$\sum_{w \in W_d(F)} \varphi^*(w) \text{ and } \sum_{x,y \in X(F)} \varphi(x,y)$$

agree by the Poisson summation formula.

Now we define the theta lifting $\Theta(\pi)$ for a cuspidal representation π of G. First of all, we have our usual theta function

$$\theta^{\pm}(g,h,\varphi) = \sum_{x \in X(F)} \omega_{\pm}(g,h)\varphi(x)$$

where $\varphi \in S(X(\mathbb{A}))$. For a cusp form f of G in π, we define

$$\theta_{\varphi}^{\pm,f}(h) := \int_{G(F)\backslash G(\mathbb{A})} f(g)\theta^{\pm}(g,h,\varphi)dg. \tag{2.2}$$

This function is well defined and slowly increasing on $H(F)\backslash H(\mathbb{A})$. Then $\Theta(\pi)$ is the representation whose space is spanned by the functions $\theta_{\varphi}^{+,f}$ where φ runs through $S(X(\mathbb{A}))$ and f runs through the cusp forms in π.

Let us also recall the standard Langlands L-function $L(s,\pi,\gamma^3)$ attached to π and twisted by the character γ^3. (For a general exposition of this type of L-functions, please refer to [Bor].) It is defined by an Euler product of local L-factors over those places v such that $G_v = G(F_v)$ is unramified over F_v and π_v is unramified. These local factors are explicitly given as follows:

When v is inert and $\pi_v \subseteq \mathrm{Ind}_{B_v}^{G_v}(\tau)$, the normalized induced representation of G_v from the character τ of $B_v \cong GL(1,E_v)$,

$$L(s,\pi_v,\gamma_v^3) = L_{E_v}(s,\tau\gamma_v^3)L_{E_v}(s,\bar{\tau}^{-1}\gamma_v^3) \tag{2.3}$$

(we use a bar to denote the non-trivial Galois automorphism of E/F).

When v splits in E and $\pi_v = \mathrm{Ind}_{B_v}^{G_v}(\chi_1\chi_2)$ where χ_i are characters of $GL(1,F_v)$,

(2.4)

$$L(s,\pi_v,\gamma_v^3) = L_{F_v}(s,\chi_1\gamma_v^3)L_{F_v}(s,\chi_2\gamma_v^3)L_{F_v}(s,\chi_1^{-1}\gamma_v^{-3})L_{F_v}(s,\chi_2^{-1}\gamma_v^{-3}).$$

Here $L_{F_v}(s,\chi) = \frac{1}{1-\chi q_v^{-s}}$ and is the local Tate L-factors associated to the Hecke character χ of F where q_v is the order of the residue field at F_v.

Via the base change lift from G to $GL(2,E)$ (see [Rog], section 4.2), we can regard $L(s,\pi,\gamma^3)$ as a standard Langlands L-function of $GL(2,E)$. More precisely, if Π_E is the automorphic representation of $GL(2,E)$ corresponding to π by base change lift, then

$$\Pi_{E,v} = \mathrm{Ind}_{B_{2,v}}^{GL(2,E_v)}(\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \mapsto \tau(\alpha/\bar{\beta}))$$

when v is inert and

$$\Pi_{E,\varpi_1} = \mathrm{Ind}_{B_{2,\varpi_1}}^{GL(2,E_{\varpi_1})}(\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \mapsto \chi_1(\alpha)\chi_2(\beta)),$$

$$\Pi_{E,\varpi_2} = \mathrm{Ind}_{B_{2,\varpi_2}}^{GL(2,E_{\varpi_2})}(\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \mapsto \chi_1^{-1}(\alpha)\chi_2^{-1}(\beta))$$

when v splits as $\varpi_1\varpi_2$ in E. Then

$$L(s,\pi,\gamma^3) = L(s,\Pi_E \otimes \gamma^3)$$

where the right-hand side above is the standard Langlands L-function attached to $\Pi_E \otimes \gamma^3$ (without twist). Therefore, the study of such L-function of $U(2)$ reduces to that of $GL(2)$ (over E) whose analytic properties are more well known. In particular, we know that the possible pole of the standard L-function of $GL(2)$ can only occur at $s = 1$. Therefore the L-function condition of G in Theorem 1.1 can be replaced by that of $GL(2)$.

3. Regularized Siegel-Weil formula

In this section, we recall the regularized Siegel-Weil formula for the dual pair G' and H which is the key to the proof of our theta lifting problem. Details of this formula can be found in [Tan2]. As is mentioned in the introduction, the regularized Siegel-Weil formula was first formulated (the so-called *first term identity*) by S. Kudla and S. Rallis in [KR] for the dual pair $Sp(n) \times O(m)$. In the other paper [KRS], a further result (*second term identity*) was obtained for the pair $Sp(2)\times O(4)$. These two papers have provided a prototype for the regularized Siegel-Weil formula in the unitary case.

There are two objects involved in the formula, namely the Siegel-Eisenstein series and the regularized theta integral. Let $I(s) = \mathrm{Ind}_{P'}^{G'}(\gamma^3\|\cdot\|^s)$ be the normalized induced representation of $G'(\mathbb{A})$ inducing from the character $\gamma^3\|\cdot\|^s$ of the Levi factor of $P'(\mathbb{A})$. We denote an element in (the space of) $I(s)$ by $\Phi(s)$ or simply Φ.

Now we define the Siegel-Eisenstein series

$$E(g,s,\Phi) = \sum_{\varepsilon \in P'(F)\backslash G'(F)} \Phi(\varepsilon g, s)$$

for $\Phi(s)$ an element in $I(s)$. This series converges absolutely for $Re(s) > 1$ and has a meromorphic continuation to the whole complex plane. It has a simple pole at $s = \frac{1}{2}$. Let

$$\frac{A_{-1}(g,\Phi)}{s-\frac{1}{2}} + A_0(g,\Phi) + \cdots$$

be the Laurent expansion of $E(g,s,\Phi)$ at $s = \frac{1}{2}$. Then A_{-1} defines an intertwining map from $I(\frac{1}{2})$ to $\mathcal{A}(G')$, the space of automorphic forms on G'.

We now turn over to the regularized theta integral. We first define the theta function

$$\theta(g,h,\varphi)=\sum_{w\in W_d(F)}\omega(g,h)\varphi(w)$$

where $\varphi \in S(W_d(\mathbb{A}))$. We also need to introduce a Hecke operator z of H_v for some unramified inert place v. This is a locally constant, compactly supported bi K-invariant function on H_v. Given a representation σ of H_v, a Hecke operator z acts on the space of σ via

$$\sigma(z)=\int_{H_v} z(h)\sigma(h)dh.$$

If σ is unramified, then z acts on the K-fixed vectors of σ by scalars. More precisely, suppose χ is some unramified character of the Borel of H_v such that

$$\chi\begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & \bar{a}^{-1}\end{pmatrix}=\rho(a^2b)\|a\|^s$$

for some $s\in\mathbb{C}$ and some unitary character ρ. If v^0 is a K-fixed vector in σ, then $\sigma(z)v^0=P(q^{\pm s})$ where $P(q^{\pm s})$ is a symmetric polynomial in $q^{\pm s}$ over $\mathbb{C}$ and q is the residual characteristic of F_v. (In fact, $z\mapsto P(q^{\pm s})$ defines a one-to-one correspondence between the set of all Hecke operators of H_v and $\mathbb{C}[q^s]^{W_{H_v}}$ where W_{H_v} is the Weyl group of H_v. This is called the Satake isomorphism.) For our choice of z, the corresponding $P(q^{\pm s})$ is

$$P(q^{\pm s})=q^s+q^{-s}-q-q^{-1}.$$

By Corollary 2.3.2 in [Tan2], we have

Lemma 3.1. *$\theta(g,h,\omega(z)\varphi)$ is rapidly decreasing on $H(F)\backslash H(\mathbb{A})$ for all g and φ.*

In order to compare our theta integral with the Siegel-Eisenstein series, we need to incorporate an auxiliary Eisenstein series $E(h,s)$ (see [Tan1, section 3.3]) of H into the definition. Then the regularized theta integral is defined by

$$\mathcal{E}(g,s,\varphi)=\frac{1}{P(q^{\pm s})}\int_{H(F)\backslash H(\mathbb{A})}\theta(g,h,\omega(z)\varphi)E(h,s)\gamma^{-2}(\det h)dh.$$

$E(h,s)$ has a simple pole at $s=1$ with constant residue. So we may write

$$E(h,s)=\frac{\kappa}{s-1}+\kappa_0(h)+\cdots. \tag{3.2}$$

Also $P(q^{\pm s})$ has a zero at $s=1$. So

$$\frac{1}{P(q^{\pm s})}=\frac{a}{s-1}+b+\cdots. \tag{3.3}$$

Hence $\mathcal{E}(g,s,\varphi)$ has a double pole at $s=1$. Let us write its Laurent expansion as

$$\frac{B_{-2}(g,\varphi)}{(s-1)^2}+\frac{B_{-1}(g,\varphi)}{(s-1)}+B_0(g,\varphi)+\cdots.$$

In view of (3.2) and (3.3), we have the expressions

$$B_{-2}(g,\varphi)=a\kappa\int_{H(F)\backslash H(\mathbb{A})}\theta(g,h,\omega(z)\varphi)\gamma^{-2}(\det h)dh$$

and

(3.4)

$$B_{-1}(g,\varphi) = \frac{b}{a}B_{-2}(g,\varphi) + a\int_{H(F)\backslash H(\mathbb{A})} \theta(g,h,\omega(z)\varphi)\kappa_0(h)\gamma^{-2}(\det h)dh.$$

Note that the integrals in the above expressions are convergent in view of Lemma 3.1. Again, B_{-1} and B_{-2} define $G'(\mathbb{A})$-intertwining maps from $S(W_d(\mathbb{A}))$ to $\mathcal{A}(G')$.

In order to link the two objects we have just defined, we define a map

$$\begin{array}{ccc} S(W_d(\mathbb{A})) & \longrightarrow & I(\frac{1}{2}), \\ \varphi & \mapsto & \Phi(\frac{1}{2}) \end{array}$$

such that

$$\Phi(\frac{1}{2})(g) = \omega(g)\varphi(0).$$

This gives an intertwining map between the two representations. We denote the image of this map by $\Pi(V)$.

If we decompose V as $V_0 \oplus V_{1,1}$ where $V_{1,1}$ is a hyperbolic plane in V, then V_0 is the one dimensional anisotropic subspace associated to the skew-Hermitian form $(,)'|_{V_0}$. We can define similarly $\Pi(V_0)$ as the space of all $\Phi(-\frac{1}{2})$ where

$$\Phi(-\frac{1}{2})(g) = \omega'(g)\varphi(0)$$

and ω' is the oscillator representation of $G' \times U(1)$ as φ runs over $S(U_d \otimes V_0(\mathbb{A}))$.

We may now summarize some of the main results in [Tan2]:

(i) $\mathrm{Im}A_{-1} \cong \bigoplus \Pi(V_{00})$ where V_{00} runs over all one dimensional skew Hermitian spaces;

(ii) $\mathrm{Im}A_{-1}|_{\Pi(V)} \cong \Pi(V_0)$;

(iii) $\mathrm{Im}B_{-2} \cong \Pi(V_0)$;

(iv) $\Pi(V_0)$ can be embedded in $\mathcal{A}(G')$ in exactly one way. In particular, we have

(v) (First term identity) There is a non-zero constant c such that, for all $\varphi \in S(W_d)$,

$$A_{-1}(g,\Phi) = cB_{-2}(g,\varphi)$$

where Φ is associated to φ.

We also have

(vi) (Second term identity) There is a non-zero constant c such that, for all $\varphi \in S(W_d)$,

$$A_0(g,\Phi) = cB_{-1}(g,\varphi) + \Psi(g)$$

for some Ψ in $\mathrm{Im}A_{-1}$.

4. The non-vanishing result

Let us consider the integral

$$\mathcal{Z}(s,f_1,f_2,\Phi) := \int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} f_1(g_1)\bar{f}_2(g_2)E((g_1,g_2),s,\Phi)dg_1dg_2$$

where $f_1, f_2 \in \pi$, $\Phi = \Phi(s) \in I(s)$ and $E(g',s,\Phi)$ is our Siegel-Eisenstein series.

Using the fundamental identity, which was first introduced in [GPS], we have

$$\mathcal{Z}(s,f_1,f_2,\Phi) = \int_{G(\mathbb{A})} \Phi((g,1))\langle\pi(g)f_1,f_2\rangle dg \tag{4.1}$$

where $\langle f_1,f_2\rangle = \int_{G(F)\backslash G(\mathbb{A})} \gamma^3(\det g)f_1(g)\bar{f}_2(g)dg$.

Now suppose Φ, f_1, f_2 are decomposable as local factors. Then the global zeta integral on the right-hand side of (4.1) admits an Euler product:

$$\prod_v \int_{G_v} \Phi_v((g,1))\langle \pi_v(g) f_{1,v}, f_{2,v}\rangle dg. \tag{4.2}$$

Our next lemma says that, for almost all v, the local zeta integrals in (4.2) are essentially the local factors of the (twisted) Langlands L-function associated with π_v when $\Phi_v, f_{1,v}, f_{2,v}$ are chosen properly.

Let $\mathcal{S}$ be a finite set of places in F including all archimedean places such that everything is unramified outside $\mathcal{S}$ (i.e. $G_v, \pi_v, \gamma_v, \psi_v$ etc. are unramified).

Then by a tedious but similar computation as in [Li2] (see [Tan1, chapter 6] for details), we obtain:

Lemma 4.3. *Let $v \notin \mathcal{S}$. For Φ_v^0 the normalized K_v'-fixed vector in $I_v(s)$ and $f_{0,v}$ a K_v-fixed vector in π_v such that $\langle f_{0,v}, f_{0,v}\rangle = 1$,*

$$\int_{G_v} \Phi_v^0((g,1))\langle \pi_v(g) f_{0,v}, f_{0,v}\rangle dg = \frac{1}{\xi_v(s)} L_v(s+\frac{1}{2}, \pi_v, \gamma_v^3)$$

where $\xi_v(s) = L_{F_v}(2s+1)L_{F_v}(2s+2, \omega_{E/F})$ and $L_v(s,\pi_v,\gamma_v^3)$ is given by (2.3) and (2.4).

Let us choose two cusp forms f_1, f_2 in π. For almost all v,

$$f_{1,v} = f_{2,v} = f_{0,v} \tag{4.4}$$

where $f_{0,v}$ is as in Lemma 4.3. We may assume (4.4) is satisfied for every v outside $\mathcal{S}$. We also choose $\Phi(\frac{1}{2})$ from $\Pi(V)$, i.e. $\Phi(\frac{1}{2})(g') = \omega(g')\varphi^*(0)$ where $\varphi^* \in S(W_d)$ is the partial Fourier transform of $\varphi = \varphi_1 \otimes \bar{\varphi}_2 \in S(X \oplus X)$. In view of (2.1),

$$\Phi(\frac{1}{2})((g,1)) = \langle \omega_+(g)\varphi_1, \varphi_2\rangle.$$

At almost all places, Φ_v is the normalized K'-fixed vector in $I_v(s)$. So we might as well assume that, outside the finite set $\mathcal{S}$, $\Phi_v = \Phi_v^0$. In particular, for $v \notin \mathcal{S}$, $\varphi_{1,v}$ and $\varphi_{2,v}$ are the characteristic functions of the lattice $X(\mathcal{O}_{E_v})$.

We shall examine the analytic property of $\mathcal{Z}(s, f_1, f_2, \Phi)$ at the point $s = \frac{1}{2}$. From Lemma 4.3,

$$\begin{aligned} &\mathcal{Z}(\frac{1}{2}, f_1, f_2, \Phi) \\ &\quad = L^{\mathcal{S}}(1,\pi,\gamma^3)\prod_{v\in\mathcal{S}}\int_{G_v}\langle \omega_+(g)\varphi_{1,v}, \varphi_{2,v}\rangle\langle \pi_v(g) f_{1,v}, f_{2,v}\rangle dg. \end{aligned} \tag{4.5}$$

where $L^{\mathcal{S}}(s,\pi,\gamma^3) = \prod_{v\notin\mathcal{S}} L_v(s,\pi_v,\gamma_v^3)$ is the partial direct product of the local L-factors. By the discussion in section 2, the possible pole of this object can only occur at $s=1$. Whenever π_v is a discrete series, the local integrals

$$\int_{G_v}\langle \omega_+(g)\varphi_{1,v}, \varphi_{2,v}\rangle\langle \pi_v(g) f_{1,v}, f_{2,v}\rangle dg \tag{4.6}$$

converge absolutely. In fact, this is true more generally for $G = U(n)$ and $H = U(m)$ with $n \le m$ and π in the discrete series of G (see [Li1]). Hence

Lemma 4.7. *If $L(s,\pi,\gamma^3)$ does not have a pole at $s=1$ and π_v are discrete for all $v \in \mathcal{S}$, then $\mathcal{Z}(s, f_1, f_2, \Phi)$ has no pole at $s = \frac{1}{2}$.*

Now let us recall that $\mathcal{Z}(s, f_1, f_2, \Phi)$ is defined by integrating the Eisenstein series $E((g_1, g_2), s, \Phi)$ against $f_1(g_1)\bar{f}_2(g_2)$ over $G(F) \times G(F)\backslash G(\mathbb{A}) \times G(\mathbb{A})$. But $E(g', s, \Phi)$ has a pole at $s = \frac{1}{2}$ with residue $A_{-1}(g', \Phi)$ (see section 3). Hence the discussion above implies that

$$\int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} f_1(g_1)\bar{f}_2(g_2)A_{-1}((g_1, g_2), \Phi)dg_1dg_2 = 0. \tag{4.8}$$

We also have $\mathrm{Im}A_{-1}|_{\Pi(V)} = \Pi(V_0)$. Thus we have shown that $f_1\bar{f}_2$ is orthogonal to $\Pi(V_0)$. If we replace V by another three-dimensional skew Hermitian space V^*, we get a new group H^*. Since f_1, f_2 are cusp forms of G, by repeating the arguments for the dual pair $G' \times H^*$, we obtain $f_1\bar{f}_2$ being orthogonal to $\Pi(V_0^*)$ where V_0^* is the complementary one-dimensional subspace of V^*. In fact, by allowing V to run through all (global) three-dimensional skew-Hermitian spaces, we obtain

Lemma 4.9. *Under the same condition as in Lemma 4.7, $f_1\bar{f}_2$ is orthogonal to $\bigoplus \Pi(V_0)$ where V_0 ranges over all one-dimensional skew-Hermitian spaces.*

Now we turn to the non-vanishing of $\mathcal{Z}(\frac{1}{2}, f_1, f_2, \Phi)$. We have to check that the factors in the right-hand side of (4.5) are non-zero. By a well known result of [Jac], the L-function $L(s, \Pi_E \otimes \gamma^3)$ of $GL(2, E)$ is non-zero at $s = 1$. (In fact, this is true for all $GL(n)$ and all s with $Re(s) = 1$.) Therefore, $L(s, \pi, \gamma^3)$ is also non-vanishing at $s = 1$.

So it remains to check that the local zeta integrals for $\upsilon \in \mathcal{S}$ are also non-zero. Whenever π_υ is discrete and has a theta lifting ρ_υ in H_υ, it follows from [Li1], section 2 that (4.6) is not identically zero for all $\varphi_{i,\upsilon}$ and $f_{i,\upsilon}$. Hence we have

Lemma 4.10. *Under the same condition as in Lemma 4.7, if π_υ has non-trivial theta-lifting in H_υ for all υ, then $\mathcal{Z}(s, f_1, f_2, \Phi)$ is holomorphic at $s = \frac{1}{2}$ and non-zero for some choice of Φ such that $\Phi(\frac{1}{2}) \in \Pi(V)$ and $f_1, f_2 \in \pi$.*

Finally, we shall see how the results we have obtained so far together with the regularized Siegel-Weil formula imply that the theta lift of π to H is non-trivial.

The point is to express $\mathcal{Z}(\frac{1}{2}, f_1, f_2, \Phi)$ in terms of theta integral associated to G' and H.

In view of (4.8), we can write $\mathcal{Z}(\frac{1}{2}, f_1, f_2, \Phi)$ as

$$\int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} f_1(g_1)\bar{f}_2(g_2)A_0((g_1, g_2), \Phi)dg_1dg_2$$

where $A_0(g, \Phi)$ is the second term in the Laurent expansion of $E(g, s, \Phi)$ at $s = \frac{1}{2}$. Now we invoke the second term identity that we have stated in section 3 to pass from the Eisenstein series to the regularized theta integral. We recall that

$$A_0(g', \Phi) = cB_{-1}(g', \varphi^*) + \Psi(g')$$

where B_{-1} is the second term of the regularized theta integral, c is a constant and $\Psi \in \bigoplus \Pi(V_{00})$. By Lemma 4.9, Ψ is orthogonal to $f_1\bar{f}_2$. So we get

$$\mathcal{Z}(\frac{1}{2}, f_1, f_2, \Phi) = c\int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} f_1(g_1)\bar{f}_2(g_2)B_{-1}((g_1, g_2), \varphi^*)dg_1dg_2.$$

In view of (3.4) and the fact that $B_{-2}(g', \varphi^*)$ belongs to $\Pi(V_0)$ and hence is orthogonal to $f_1\bar{f}_2$, $\mathcal{Z}(\frac{1}{2}, f_1, f_2, \Phi)$ becomes a double integral

$$\int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} f_1(g_1)\bar{f}_2(g_2) \times \int_{H(F)\backslash H(\mathbb{A})} a\kappa_0(h)\gamma^{-2}(\det h)\theta(g', h, \omega(z)\varphi^*)dh\, dg_1 dg_2.$$

$\theta((g_1, g_2), h, \omega(z)\varphi^*)$ is rapidly decreasing on $H(F)\backslash H(\mathbb{A})$. Furthermore, f_1, f_2 are cusp forms on G and hence also rapidly decreasing on $G(F)\backslash G(\mathbb{A})$. We can then apply Fubini's Theorem to interchange the two integrals. So we get

$$\mathcal{Z}(\frac{1}{2}, f_1, f_2, \Phi) = \int_{H(F)\backslash H(\mathbb{A})} a\kappa_0(h)\gamma^{-2}(\det h) \times \int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} f_1(g_1)\bar{f}_2(g_2)\theta((g_1, g_2), h, \omega(z)\varphi^*)dg_1 dg_2\, dh. \tag{4.11}$$

Now we are only one step from proving the non-vanishing of $\Theta(\pi)$. What we really have to show is that the function $\theta_\varphi^{+,f}$ we defined in section 2 is non-zero for some f and φ. Using our choice of $f_1, f_2, \varphi_1, \varphi_2$, we compute that

$$\begin{aligned}
&\theta_{\varphi_1}^{+,f_1}(h)\theta_{\bar{\varphi}_2}^{-,\bar{f}_2}(h) \\
&= \int_{G(F)\backslash G(\mathbb{A})} \sum_{x\in X(F)} \omega_+(g, h)\varphi_1(x)f_1(g)dg \\
&\quad \times \int_{G(F)\backslash G(\mathbb{A})} \sum_{x\in X(F)} \omega_-(g, h)\bar{\varphi}_2(x)\bar{f}_2(g)dg \\
&= \int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} f_1(g_1)\bar{f}_2(g_2) \sum_{x,y\in X(F)} \omega((g_1, g_2), h)\varphi(x, y)dg_1 dg_2.
\end{aligned}$$

If we convolve $\theta_{\varphi_1}^{+,f_1}(h)\theta_{\bar{\varphi}_2}^{-,\bar{f}_2}(h)$ with z', the Hecke operator in H_v corresponding to z under Howe correspondence (see [MVW]), we get precisely the inner integral in (4.11). Therefore, under the conditions of Lemma 4.10, the non-vanishing of $\mathcal{Z}(\frac{1}{2}, f_1, f_2, \Phi)$ implies $\theta_{\varphi_1}^{+,f_1}$ is non-zero and hence $\Theta(\pi)$ is non-trivial.

We hence proved Theorem 1.1.

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