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INFINITE HOMOGENEOUS ALGEBRAS ARE ANTICOMMUTATIVE

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ABSTRACT. A (non-associative) algebra A, over a field k, is called homogeneous if its automorphism group permutes transitively the one dimensional subspaces of A. Suppose A is a nontrivial finite dimensional homogeneous algebra over an infinite field. Then we prove that $x^2 = 0$ for all x in A, and so $xy = -yx$ for all $x, y \in A$.

INTRODUCTION

The algebras to be discussed are assumed to be finite dimensional over a field k and not necessarily associative. Thus the multiplication $A \times A \to A$ can be an arbitrary bilinear map. Such an algebra A is *nontrivial* if $\dim A > 1$ and $A^2 \neq 0$. Also, $\mathrm{Aut}(A)$ denotes the group of algebra automorphisms of A.

An algebra A is said to be *extremely homogeneous* if $\mathrm{Aut}(A)$ acts transitively on $A \setminus \{0\}$. The concept of an extremely homogeneous algebra arose from a particular problem in the structure of certain finite p-groups as studied by Boen, Rothaus, and Thompson [2]. Extremely homogeneous algebras were also investigated by Kostrikin [6].

An algebra A is *homogeneous* if $\mathrm{Aut}(A)$ permutes transitively the one dimensional subspaces of A. Homogeneous algebras over finite fields were studied by Shult [11] and Gross [4]. The classification of homogeneous algebras over finite fields was completed by Ivanov [5]. He showed that if A is a nontrivial homogeneous algebra over a finite field k, then $k = \mathrm{GF}(2)$ and A is of a type previously described by Kostrikin.

The condition of homogeneity is very strong indeed. For instance, if k is algebraically closed, then the second author [10] has shown that there are no nontrivial homogeneous algebras. When k is the real field, the first author [3] has classified all homogeneous algebras. In that case there are up to isomorphism exactly three nontrivial homogeneous algebras. Also, MacDougall and Sweet [7], [8], [9] have found all homogeneous algebras of dimension $2, 3$, or 4 with no restrictions on the ground field.

If $a \in A$, we denote by $\langle a \rangle$ the subalgebra of A generated by a. If A is homogeneous, then $\langle a \rangle$ is also homogeneous and it is easy to show that $\langle a \rangle = \langle x \rangle$ for any

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$x \in \langle a \rangle \setminus \{0\}$. Indeed if $x \in \langle a \rangle \setminus \{0\}$, there is some $\varphi \in \mathrm{Aut}(A)$ taking $k \cdot a$ to $k \cdot x$, hence carrying $\langle a \rangle$ to $\langle x \rangle$, so φ yields an automorphism of A. It follows that if A is homogeneous and $a \in A$, then $\langle a \rangle$ is an algebra with no proper subalgebras. Algebras with no proper subalgebras were investigated by Artamonov [1]. In particular he showed that if A is a nontrivial algebra with no proper subalgebras and $\dim A$ is a prime, then $\mathrm{Aut}(A)$ is a finite group.

The purpose of this paper is to show that if A is a homogeneous algebra over an infinite field, then $x^2 = 0$ for all $x \in A$. This of course implies that A is anticommutative. It also implies that if A is a nontrivial homogeneous algebra with no proper subalgebras, then the underlying field k must be finite (and so these algebras are known, they must be of Kostrikin type).

In the remainder of the paper we use the following notation: A is a homogeneous algebra over an infinite field k, K is an algebraic closure of k, $A_K = K \otimes_k A$ is the K-algebra obtained from A by extension of scalars, and G_k (resp. G_K) is the automorphism group of the algebra A (resp. A_K). For $x \in A_K$, $L_x : A_K \to A_K$ denotes left multiplication by x (and similarly R_x denotes right multiplication by x). If A is an algebra, then A^+ is the commutative algebra obtained from A by introducing the new multiplication $x \cdot y = xy + yx$ (note that $\mathrm{Aut}(A)$ is a subgroup of $\mathrm{Aut}(A^+)$ and so A homogeneous implies A^+ homogeneous). By A^{opp} we denote the opposite algebra of A in which $x \cdot y = yx$. As usual we set $k^* = k \setminus \{0\}$ and $K^* = K \setminus \{0\}$.

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Results and proofs

The following three lemmas are needed for the proof of the main theorem.

Lemma 1. *Let A be a nontrivial homogeneous algebra over an infinite field k. If $a^2 \in k \cdot a$, $a \in A$, then $a^2 = 0$.*

Proof. This result follows directly from Theorem 7 of [8]. □

Lemma 2. *Let $A = \langle a_0 \rangle$ be a nontrivial homogeneous algebra over an infinite field k. If $a \in A \setminus \{0\}$, then the stabilizer $H \subset G_K$ of the 1-dimensional subspace $K \cdot a$ is trivial.*

Proof. Assume that H contains an element $h \neq 1$. Then $h(a) = \lambda a$ for some $\lambda \in K^*$, $\lambda \neq 1$. We define a^{2^r}, $r \geq 0$, inductively by $a^1 = a$, $a^{2^{r+1}} = (a^{2^r})^2$. Since $A = \langle a_0 \rangle$ has dimension > 1, we must have $a_0^2 \neq 0$, hence by homogeneity all nonzero elements of A have nonzero square, hence $a^{2^r} \neq 0$ for all $r \geq 0$ and $h(a^{2^r}) = \lambda^{2^r} a^{2^r}$. As h has only finitely many eigenvalues, λ is a root of unity, say, $\lambda^d = 1$, $d > 1$.

As $A = \langle a \rangle$, A has a basis $\{b_1, b_2, \ldots, b_n\}$ where each b_i is a product of n_i a's under some bracketing. Let m be a positive integer such that $t = dm > n_i$ for all i. Then $h(a^t) = \lambda^t a^t = a^t$, where a^t is any product of t a's. If $a^t \neq 0$ for some bracketing, then $A = \langle a^t \rangle$ which implies that $h = 1$, a contradiction. Hence we have $a^t = 0$ for all bracketing. But then $L_a^{t-n_i}(b_i) = 0$, and so $L_a^t(b_i) = 0$ for all i. Thus $L_a^t = 0$, and similarly $R_a^t = 0$. Hence A is a special nil algebra as defined in [10]. Since k is infinite, it follows from Theorem 2 of the above paper that $A^2 = 0$, which is a contradiction. □

Lemma 3. *Let $A = \langle a_0 \rangle$ be a nontrivial commutative homogeneous algebra over an infinite field k where* $\operatorname{char} k \neq 2$. *Then A has no zero divisors.*

Proof. Suppose $ab = 0$ for some $a, b \in A \setminus \{0\}$. Since $A = \langle a \rangle$ and $\dim A > 1$, we have $a^2 \neq 0$ and so a and b are linearly independent. Hence there exist $g_1, g_2 \in G_k$ such that

$$g_1(a) = \lambda(a+b), \quad g_2(a) = \mu(a-b), \quad \lambda, \mu \in k^*. \tag{1}$$

But then, using $ab = ba = 0$,

$$g_1(a^2) = \lambda^2(a^2+b^2), \quad g_2(a^2) = \mu^2(a^2+b^2)$$

and so $g_2^{-1}g_1$ is in the stabilizer of $k \cdot a^2$. Lemma 2 implies that $g_1 = g_2$ and (1) gives a contradiction. □

We can now prove our main result.

Theorem. *Let A be a nontrivial homogeneous algebra over an infinite field k. Then $x^2 = 0$ for all $x \in A$.*

Proof. Assume that $x^2 \neq 0$ for some $x \in A$. Since A is homogeneous, this implies that $x^2 \neq 0$ for all nonzero $x \in A$. Our goal is to obtain a contradiction. If $\operatorname{char} k \neq 2$, by replacing A with A^+, we may assume that A is commutative.

Let us fix a nonzero element $b \in A$. By Lemma 1, $b^2 \notin k \cdot b$. By replacing A with the subalgebra generated by b, we may assume that $A = \langle b \rangle$. Suppose L_b is nilpotent. If $\operatorname{char} k \neq 2$, then $R_b = L_b$ is also nilpotent and so A is a special nilalgebra. As noted in the proof of Lemma 2, this is impossible. If $\operatorname{char} k = 2$, we obtain the same contradiction if L_b and R_b are both nilpotent. So, if necessary, by replacing A by A^{opp}, we can assume that L_b is not nilpotent.

Let us fix a nonzero eigenvalue $\mu \in K$ of L_b and set $a := \mu^{-1}b \in A_K$. Let $P : A_K \to K$ be the polynomial function defined by

$$P(x) := \det(1 - L_x)$$

and let $P = P_1P_2 \cdots P_m$ be its prime factorization. Since 1 is an eigenvalue of L_a, we have $P(a) = 0$. We may assume that the factors are numbered so that $P_1(a) = 0$ and normalized so that $P_i(0) = 1$ for all i. Let H be the hypersurface in A_K defined by $P(x) = 0$ and let H_1 be its irreducible component defined by $P_1(x) = 0$.

For $g \in G_K$ and $x \in A_K$ we have $g \circ L_x \circ g^{-1} = L_{g(x)}$ and consequently P is G_K invariant. Hence the orbit $\mathcal{O} := G_K \cdot a$ is contained in H.

Let

$$\pi : A_K \setminus \{0\} \to \mathbf{P}(A_K)$$

be the canonical projection onto the projective space $\mathbf{P}(A_K)$. The image of $A \setminus \{0\}$ under π will be identified with the projective space $\mathbf{P}(A)$ whose points are the 1-dimensional k-subspaces of A. We have

$$\mathbf{P}(A) = \pi(G_k \cdot b) \subset \pi(\mathcal{O}) \subset \mathbf{P}(A_K).$$

Since k is infinite, $\mathbf{P}(A)$ is Zariski dense in $\mathbf{P}(A_K)$. Hence $\dim \mathcal{O} \geq n-1$, and since $\mathcal{O} \subset H$, we infer that $\dim \mathcal{O} = n - 1$.

We claim that the affine algebraic group G_K is connected (in the Zariski topology). Indeed let G_K^0 be the identity component of G_K and let $g \in G_K$ be arbitrary.

Both $\pi(G_K^0 \cdot a)$ and $\pi(gG_K^0 \cdot a)$ are nonempty open subsets of $\mathbf{P}(A_K)$. Hence they have nontrivial intersection, i.e., there exist $g_1, g_2 \in G_K^0$ such that $gg_1 \cdot Ka = g_2 \cdot Ka$. By Lemma 2 we conclude that $gg_1 = g_2$ and so $g = g_2 g_1^{-1} \in G_K^0$ and our claim is verified.

It follows that each factor P_i of the G_K-invariant polynomial P is G_K-invariant. Hence H_1 is G_K-invariant and so $\mathcal{O} \subset H_1$ and $\mathcal{O}$ is open and dense in H_1. Let d be the degree of P_1 and write $P_1 = P_{10} + P_{11} + \cdots + P_{1d}$ where P_{1i} is a homogeneous polynomial of degree i. We set $f = -P_{1d}$.

We claim that $d > 1$. Assume that $d = 1$. Then H_1 is the hyperplane $f(x) = 1$; hence $f(a) = 1$ and $f(b) = \mu$. If $g \in G_k$ and $g \neq 1$, then also $f(g \cdot b) = \mu$. Thus $c = b - g \cdot b$ is a nonzero vector of A such that $f(c) = 0$ contradicting the homogeneity. Hence our claim is proved.

Since k is infinite and $f \neq 0$, f cannot vanish identically on $\mathbf{P}(A)$. As $\pi(G_k \cdot b) = \mathbf{P}(A)$, it follows that $f(b) \neq 0$, and so $f(a) \neq 0$. We conclude that the polynomial $P_1(ta)$ in the variable t must also have degree d. But $P_1(a) = 0$ and so $t = 1$ is a root of the polynomial $P_1(ta)$. Let $\lambda \in K$ be an arbitrary root of that polynomial. Then $\lambda \neq 0$ and $\lambda\mathcal{O}$ and $\mathcal{O}$ are both open dense subsets of H_1. Hence $\lambda\mathcal{O} \cap \mathcal{O} \neq \emptyset$, and consequently $\lambda\mathcal{O} = \mathcal{O}$. Now Lemma 2 implies that $\lambda = 1$. We conclude that $t = 1$ is the only root of the polynomial $P_1(ta)$ and, by using $P_1(0) = 1$, we infer that

$$P_1(ta) = (1 - t)^d.$$

Since P_1 is G_K-invariant and $\mathcal{O}$ is dense in H_1, we have

$$P_1(tx) = (1 - t)^d, \quad \forall x \in H_1.$$

This polynomial identity (in the variable t) implies that each of the polynomials

$$P_{1i}(x) - (-1)^i \binom{d}{i}, \quad 0 \leq i \leq d,$$

vanishes on H_1. Since P_1 is irreducible, P_1 must divide each of these polynomials. It follows that $P_{1i} = 0$ for $0 < i < d$, $\operatorname{char} k = p > 0$, and

$$\binom{d}{i} \equiv 0 \pmod{p}, \quad 0 < i < d.$$

The last condition implies that $d = p^s$ for some integer $s \geq 1$. Since $P_{10} = P_1(0) = 1$, we have $P_1 = 1 - f$ where $f = -P_{1d}$ is homogeneous of degree d.

For $x \in A_K$ and $1 \leq i \leq p$, we define the powers x^i recursively by $x^1 = x$, and $x^i = x \cdot x^{i-1}$ for $1 < i \leq p$. Next we define the powers x^{p^i} for $i > 1$ by

$$x^{p^i} = (x^{p^{i-1}})^p.$$

It follows from these definitions that for all $x \in A_K$ and $g \in G_K$ we have

$$g(x^{p^i}) = g(x)^{p^i}.$$

We claim that $x^{p^i} \neq 0$ for nonzero $x \in A$ and $i \geq 0$. If $p > 2$, this follows from Lemma 3, and for $p = 2$ from our assumption that $A = \langle b \rangle$.

Let N be the smallest positive integer such that

$$a, a^p, a^{p^2}, \ldots, a^{p^N}$$

are K-linearly dependent. Then

$$\sum_{i=0}^{N} \alpha_i a^{p^i} = 0 \tag{2}$$

with $\alpha_i \in K$ and $\alpha_N = 1$.

Assume that $\alpha_0 = 0$. As $b = \mu a \in A \setminus \{0\}$, there exists $g \in G_k$ such that $g(a^p) = \lambda a$ for some $\lambda \in K^*$. Equation (2) implies that

$$\sum_{i=1}^{N} \alpha_i \, g(a^p)^{p^{i-1}} = 0.$$

But then

$$\sum_{i=0}^{N-1} \alpha_{i+1} \, \lambda^{p^i} a^{p^i} = 0$$

and this contradicts our choice of N. So we have shown that $\alpha_0 \neq 0$.

As $\mathcal{O}$ is dense in H_1, it follows from (2) that

$$\sum_{i=0}^{N} \alpha_i \, x^{p^i} = 0, \quad \forall x \in H_1.$$

Hence each coordinate function of the polynomial map

$$x \to \sum_{i=0}^{N} \alpha_i \, x^{p^i}$$

must be divisible by $P_1 = 1 - f$, and so

$$\sum_{i=0}^{N} \alpha_i \, x^{p^i} = (1 - f(x)) \sum_{i=1}^{p^N - d} Q_i(x)$$

where Q_i is a homogeneous polynomial map of degree i. By extracting the homogeneous components of degrees congruent to $1 \bmod p$, we obtain that

$$\alpha_0 \, x = (1 - f(x)) \sum_{i \equiv 1 \pmod{p}} Q_i(x).$$

It follows that $Q_1(x) = \alpha_0 \, x \neq 0$. Hence the right hand side has degree $\geq d + 1$, and we have a contradiction. $\square$

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