

## ON RANDOM ALGEBRAIC POLYNOMIALS

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ABSTRACT. This paper provides asymptotic estimates for the expected number of real zeros and  $K$ -level crossings of a random algebraic polynomial of the form  $a_0 \binom{n-1}{0}^{1/2} + a_1 \binom{n-1}{1}^{1/2} x + a_2 \binom{n-1}{2}^{1/2} x^2 + \cdots + a_{n-1} \binom{n-1}{n-1}^{1/2} x^{n-1}$ , where  $a_j (j = 0, 1, \dots, n-1)$  are independent standard normal random variables and  $K$  is a constant independent of  $x$ . It is shown that these asymptotic estimates are much greater than those for algebraic polynomials of the form  $a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$ .

### 1. INTRODUCTION

Let  $(\Omega, \mathcal{A}, \Pr)$  be a fixed probability space and let  $\{a_j(\omega)\}_{j=0}^{n-1}$  be a sequence of independent random variables defined on  $\Omega$ . The random algebraic polynomial was introduced in the pioneer work of Littlewood and Offord [9] and [10] as

$$Q(x) \equiv Q_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega) x^j,$$

and since then has been greatly studied. Denote by  $N_K(\alpha, \beta)$  the number of real roots of the equation  $P(x) = K$  in the interval  $(\alpha, \beta)$  and by  $EN_K(\alpha, \beta)$  its expected value. In particular it is shown (for example see Kac [8] or Wilkins [12]) that if the coefficients are assumed to have a standard normal distribution and  $n$  is sufficiently large,  $EN_0(-\infty, \infty) \sim (2/\pi) \log n$ . Recently (see Farahmand [4]), it was shown that this asymptotic value remains valid for  $EN_K(-\infty, \infty)$  as long as  $K$  is bounded. For  $K$  large such that  $K^2/n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $EN_K(-\infty, \infty)$  is asymptotically reduced to  $(1/\pi) \log(n/K^2)$  in  $(-1, 1)$  while it remains the same as for  $K = 0$  in  $(-\infty, -1) \cup (1, \infty)$  ([4]). In contrast, a random trigonometric polynomial

$$T(x) \equiv T_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega) \cos j\theta$$

has more roots in  $(0, 2\pi)$ . In fact  $EN_K(0, 2\pi) \sim 2n/\sqrt{3}$  for  $K = o(\sqrt{n})$ ; see [5] and [6].

Motivated by the interesting results obtained in Littlewood and Offord [10] we considered the case when the coefficients  $a_j(\omega)$  have variance  $1/j!$ . It is presumably

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the case, possibly under some mild conditions for  $K$  and for  $n$  sufficiently large, that  $EN_K(-\infty, \infty)$  is  $o(\sqrt{n})$ . This author, however, was unable to make any substantial progress towards this conjecture. Instead in this paper we study the polynomials

$$P(x) \equiv P_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega) \binom{n-1}{j}^{1/2} x^j.$$

This is, indeed, the same as saying that the  $j$ th coefficient of  $Q(x)$  has variance  $\binom{n-1}{j}$ . Besides the mathematical interest, as reported in Edelman and Kostlan [3, page 11], these polynomials have some relationship with physics [1]. We prove the following theorems. Theorem 1 was known to Edelman and Kostlan, however to be complete we give a proof here. Theorem 3, and its comparison with Theorem 1, shows that for  $P(x)$  there are as many extrema as the number of zero crossings. Therefore, unlike  $Q(x)$ , all the oscillations of  $P(x)$  are between two zero crossings, asymptotically in expectation.

**Theorem 1.** *If the coefficients  $a_j$  of  $P(x)$  are independent standard normal random variables, then*

$$EN_0(-\infty, \infty) = \sqrt{n-1}.$$

**Theorem 2.** *Denote  $\Upsilon(n) = (2n-1)!!/(2n)!!$  and assume  $K^2\Upsilon(n) \rightarrow 0$ . With the same assumptions as in Theorem 1 for the coefficients of  $P(x)$ , we have*

$$EN_K(-\infty, \infty) \sim \sqrt{n-1}.$$

**Theorem 3.** *Denote  $M(-\infty, \infty)$  the number of extrema of  $P(x)$  in  $(-\infty, \infty)$ . Then the expected value of  $M(-\infty, \infty)$  satisfies*

$$EM(-\infty, \infty) \sim \sqrt{n-1}.$$

## 2. A FORMULA FOR THE EXPECTED NUMBER OF REAL ROOTS

Let

$$(2.1) \quad A^2 = \text{var}\{P(x) - K\},$$

$$(2.2) \quad B^2 = \text{var}\{P'(x)\},$$

$$(2.3) \quad C = \text{cov}\{P(x) - K, P'(x)\}$$

and

$$\eta = -CK/A\sqrt{A^2B^2 - C^2}.$$

Then by using the expected number of level crossings given by Cramer and Leadbetter [2, page 285] for our equation  $P(x) - K = 0$ , we can obtain

$$EN_K(\alpha, \beta) = \int_{\alpha}^{\beta} \frac{B\sqrt{1 - C^2/A^2B^2}}{A} \phi\left(-\frac{K}{A}\right) [2\phi(\eta) + \eta\{2\Phi(\eta) - 1\}] dx,$$

where as usual

$$\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^t \exp(-y^2/2) dy$$

and

$$\phi(t) = (2\pi)^{-1/2} \exp(-t^2/2).$$

Let  $\Delta^2 = A^2B^2 - C^2$  and  $\operatorname{erf}(x) = \int_0^x \exp(-t^2) dt$ ; then we can write the extension of a formula obtained by Rice [11] for the case of  $K = 0$  as

$$(2.4) \quad EN_K(\alpha, \beta) = I_1(\alpha, \beta) + I_2(\alpha, \beta),$$

where

$$(2.5) \quad I_1(\alpha, \beta) = \int_{\alpha}^{\beta} \frac{\Delta}{\pi A^2} \exp\left(-\frac{B^2 K^2}{2\Delta^2}\right) dx$$

and

$$(2.6) \quad I_2(\alpha, \beta) = \int_{\alpha}^{\beta} \frac{\sqrt{2}KC}{\pi A^3} \exp\left(-\frac{K^2}{2A^2}\right) \operatorname{erf}\left(\frac{KC}{\sqrt{2}A\Delta}\right) dx.$$

### 3. PROOF OF THEOREM 1

We need the following, where (3.2) and (3.3) are obtained by differentiation of (3.1),

$$(3.1) \quad A^2 = \sum_{j=0}^{n-1} \binom{n-1}{j} x^{2j} = (x^2 + 1)^{n-1},$$

$$(3.2) \quad B^2 = \sum_{j=0}^{n-1} j^2 \binom{n-1}{j} x^{2j-2} = (n-1)(x^2 + 1)^{n-3}(nx^2 - x^2 + 1),$$

and

$$(3.3) \quad C = \sum_{j=0}^{n-1} j \binom{n-1}{j} x^{2j-1} = (n-1)x(x^2 + 1)^{n-2}.$$

Therefore, since from (3.1)-(3.3)

$$(3.4) \quad \Delta^2 = A^2B^2 - C^2 = (n-1)(x^2 + 1)^{2n-4},$$

from (2.4)-(2.6) we obtain

$$(3.5) \quad \begin{aligned} EN_0(0, \infty) &= \pi^{-1} \int_0^{\infty} \frac{\Delta}{A^2} dx \\ &= \frac{\sqrt{n-1}}{\pi} \int_0^{\infty} \frac{dx}{x^2 + 1} = \frac{\sqrt{n-1}}{2}. \end{aligned}$$

This gives the proof of Theorem 1. It is, indeed, interesting to note that from (3.5)

$$\frac{\sqrt{n-1}}{\pi(x^2 + 1)}$$

is the density function of the number of real zeros of  $P(x)$ ; see also [3, page 12]. Obtaining a closed form of the above density function is uncommon. An asymptotic result, for most cases, is the best that can be achieved.

### 4. PROOF OF THEOREM 2

Because  $|\operatorname{erf} u| < \sqrt{\pi}/2$ , it follows that from (2.6), (3.1), and (3.3) that

$$\begin{aligned} 0 \leq I_2(-\infty, \infty) &\leq \frac{|K|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{C}{A^3} \exp\left(-\frac{K^2}{2A^2}\right) dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^{|K|/\sqrt{2}} e^{-\nu^2} d\nu, \end{aligned}$$

if  $\nu = |K|/(A\sqrt{2})$ . Therefore, because  $\operatorname{erf} u \leq u$  when  $u \geq 0$ ,

$$(4.1) \quad 0 \leq I_2(-\infty, \infty) \leq \frac{2}{\sqrt{\pi}} \operatorname{erf} \left( \frac{K}{\sqrt{2}} \right) \leq \min \left\{ 1, \frac{\sqrt{2}|K|}{\sqrt{\pi}} \right\}.$$

Moreover, it follows from (2.5), (3.1), (3.2) and (3.4) that

$$I_1(-\infty, \infty) = \frac{\sqrt{n-1}}{\pi} \int_{-\infty}^{\infty} \frac{e^{-s}}{x^2+1} dx,$$

in which  $s = K^2(nx^2 - x^2 + 1)/\{2(x^2 + 1)^{n-1}\}$ . If  $x = \tan \theta$ , we find that

$$I_1(-\infty, \infty) = \frac{2\sqrt{n-1}}{\pi} \int_0^{\pi/2} e^{-s} d\theta,$$

in which  $s = (K^2/2) \{(n-1)\sin^2 \theta + \cos^2 \theta\} \cos^{2n-4} \theta$ . Therefore,

$$(4.2) \quad I_1(-\infty, \infty) = \frac{2\sqrt{n-1}}{\pi} \int_0^{\pi/2} \{1 - (1 - e^{-s})\} d\theta = \sqrt{n-1} - R,$$

in which from [7, page 369],

$$\begin{aligned} 0 \leq R &= \frac{2\sqrt{n-1}}{\pi} \int_0^{\pi/2} (1 - e^{-s}) d\theta \leq \frac{2\sqrt{n-1}}{\pi} \int_0^{\pi/2} s d\theta = K^2 \beta_n, \\ \beta_n &= \frac{\sqrt{n-1}(3n-4)(2n-5)!!}{(2n-2)(2n-4)!!} = \sqrt{n-1} \Upsilon(n-2) \frac{3n-4}{2n-2}. \end{aligned}$$

A straight-forward algebraic calculation shows that  $\beta_{n+1} < \beta_n$  when  $n \geq 2$ . We conclude that  $0 \leq R \leq K^2 \beta_2 = K^2/2$  and then that  $R = o(\sqrt{n})$  because  $K^2 = o\{\Upsilon^{-1}(n)\}$  and  $\Upsilon(n) \sim \sqrt{n\pi}$ . When this last result is combined with (4.1), (4.2) and (2.4), it is clear that Theorem 2 is true. In fact, we have actually proved the better result that

$$-\frac{K^2}{2} \leq EN_k(-\infty, \infty) - \sqrt{n-1} \leq \min \left\{ 1, \frac{\sqrt{2}|K|}{\sqrt{\pi}} \right\},$$

from which we can also infer not only Theorem 2 but also that

$$EN_K(-\infty, \infty) = \sqrt{n-1} + O(1)$$

when  $K$  is bounded.

## 5. EXTREMA

The expected number of extrema of  $P(x)$ , denoted by  $EM(-\infty, \infty)$ , is simply the expected number of real zeros of  $P'(x) = \sum_{j=1}^{n-1} a_j j \binom{n-1}{j}^{1/2} x^{j-1}$ . Therefore we apply  $EN_0(-\infty, \infty)$  for  $P'(x)$ . To this end, by successive differentiation of (2.2)

we obtain

$$\begin{aligned} A^2 &= \sum_{j=0}^{n-1} j^2 \binom{n-1}{j} x^{2j-2} \\ (5.1) \quad &= (n-1)(x^2+1)^{n-3}(nx^2-x^2+1), \end{aligned}$$

$$\begin{aligned} C &= \sum_{j=1}^{n-1} j^2(j-1) \binom{n-1}{j} x^{2j-3} \\ (5.2) \quad &= (n-1)(n-2)x(x^2+1)^{n-4} \{(n-1)x^2+2\} \end{aligned}$$

and

$$\begin{aligned} B^2 &= \sum_{j=1}^{n-1} j^2(j-1)^2 \binom{n-1}{j} x^{2j-4} \\ (5.3) \quad &= (n-1)(n-2)(x^2+1)^{n-5} \{(n-1)(n-2)x^4+4(n-2)x^2+2\}. \end{aligned}$$

Therefore, from (5.1)-(5.3) we obtain

$$(5.4) \quad \frac{\Delta}{A^2} = \frac{\sqrt{(n-2)\{(n-1)(n-2)x^4+2(n-1)x^2+2\}}}{(x^2+1)\{(n-1)x^2+1\}}.$$

From (5.4) it then follows that, for all non-zero  $x$

$$\lim_{n \rightarrow \infty} \frac{\Delta}{A^2 \sqrt{n-2}} = \frac{1}{x^2+1}.$$

Then since  $\Delta/A^2 \sqrt{n-2} \leq \sqrt{2}/(x^2+1)$  for all real  $x$ , the dominated convergent theorem for Lebesgue integrals shows that

$$\int_{-\infty}^{\infty} \frac{\Delta}{A^2 \sqrt{n-2}} dx = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi.$$

Theorem 3 is then an immediate consequence of this result.

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