

NOETHERIAN DOWN-UP ALGEBRAS

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ABSTRACT. Down-up algebras $A = A(\alpha, \beta, \gamma)$ were introduced by G. Benkart and T. Roby to better understand the structure of certain posets. In this paper, we prove that $\beta \neq 0$ is equivalent to A being right (or left) Noetherian, and also to A being a domain. Furthermore, when this occurs, we show that A is Auslander-regular and has global dimension 3.

§1. INTRODUCTION

Motivated by the study of posets, G. Benkart and T. Roby introduced certain *down-up algebras* in [BR], see also [B]. Specifically, let K be a field, fix parameters $\alpha, \beta, \gamma \in K$ and let $A = A(\alpha, \beta, \gamma)$ be the K -algebra with generators d and u , and relations

$$(1.1) \quad d^2u = \alpha dud + \beta ud^2 + \gamma d = (\alpha du + \beta ud + \gamma)d,$$

$$(1.2) \quad du^2 = \alpha udu + \beta u^2d + \gamma u = u(\alpha du + \beta ud + \gamma).$$

Note that

$$(ud)(du) = u(d^2u) = u(\alpha du + \beta ud + \gamma)d = (du^2)d = (du)(ud),$$

by (1.1) and (1.2), and therefore ud and du commute in A . Furthermore, A is clearly isomorphic to its opposite ring A^{op} via the map $d \mapsto u^{\text{op}}$ and $u \mapsto d^{\text{op}}$.

Our main result here is

Theorem. *If $A = A(\alpha, \beta, \gamma)$, then the following are equivalent.*

- (1) $\beta \neq 0$.
- (2) A is right (or left) Noetherian.
- (3) A is a domain.
- (4) $K[ud, du]$ is a polynomial ring in the two generators.

Furthermore, if these conditions hold, then A is Auslander-regular and has global dimension 3.

In particular, we answer some questions posed in a preliminary version of [B]. Note that condition (4) above is significant because $K[ud, du]$ plays the role of the enveloping algebra of a Cartan subalgebra in the highest weight theory of [BR].

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As is apparent, all our positive results for A occur when the parameter β is not zero. In this situation, we offer two distinct approaches to the study of A , namely via filtered rings and via generalized Weyl algebras.

§2. THE GENERALIZED WEYL ALGEBRA APPROACH

2.1. Suppose that $\beta \neq 0$. We first show that A embeds in a skew group ring. To this end, let $R = K[x, y]$ be a polynomial ring in two variables, and define $\sigma \in \text{Aut}_K(R)$ by $\sigma(x) = y$ and $\sigma(y) = \alpha y + \beta x + \gamma$. Note that σ is indeed an automorphism since $\beta \neq 0$, and we can form $S = R[z, z^{-1}; \sigma]$, the skew group ring of the infinite cyclic group $\langle z \rangle$ over R , with $rz = z\sigma(r)$ for all $r \in R$. Now consider the elements $D = z^{-1}$ and $U = xz$ in S . Then $UD = xzz^{-1} = x$ and $DU = z^{-1}xz = \sigma(x) = y$. In addition,

$$(2.1) \quad \begin{aligned} D^2U &= D \cdot DU = z^{-1}y = \sigma(y)z^{-1} \\ &= (\alpha y + \beta x + \gamma)z^{-1} = (\alpha DU + \beta UD + \gamma)D \end{aligned}$$

and

$$(2.2) \quad \begin{aligned} DU^2 &= DU \cdot U = yxz = xz\sigma(y) \\ &= U(\alpha y + \beta x + \gamma) = U(\alpha DU + \beta UD + \gamma). \end{aligned}$$

Hence, there is an algebra homomorphism $\theta: A(\alpha, \beta, \gamma) \rightarrow S$ given by $\theta(d) = D$ and $\theta(u) = U$. Since $\theta(ud) = UD = x$ and $\theta(du) = DU = y$ are algebraically independent, the same is true of the commuting elements ud and du . In particular, $K[ud, du]$ is isomorphic to a polynomial ring in the two variables and σ lifts to an automorphism of this algebra satisfying $\sigma(ud) = du$ and $\sigma(du) = \alpha du + \beta ud + \gamma$.

It remains to show that θ is a monomorphism. To this end, note that (1.1) implies

$$(2.3) \quad \begin{aligned} d(ud) &= (du)d = \sigma(ud)d, \\ d(du) &= (\alpha du + \beta ud + \gamma)d = \sigma(du)d, \end{aligned}$$

and similarly, (1.2) yields

$$(2.4) \quad \begin{aligned} (du)u &= u(\alpha du + \beta ud + \gamma) = u\sigma(du), \\ (ud)u &= u(du) = u\sigma(ud). \end{aligned}$$

Thus

$$(2.5) \quad dc = \sigma(c)d, \quad cu = u\sigma(c) \quad \text{for all } c \in K[ud, du],$$

and therefore $dK[ud, du] = K[ud, du]d$ and $K[ud, du]u = uK[ud, du]$. In particular, if we set

$$B = \sum_{k \geq 0} K[ud, du]d^k + \sum_{k \geq 0} K[ud, du]u^{k+1} \subseteq A,$$

then $dB \subseteq B$ and $uB \subseteq B$. Hence $AB \subseteq B$ and consequently $A = B$. In other words, A is spanned by the set $\mathcal{B} = \{(ud)^i(du)^j d^k, (ud)^i(du)^j u^{k+1} \mid i, j, k \geq 0\}$. But $\theta(\mathcal{B}) = \{x^i y^j z^{-k}, x^i y^j (xz)^{k+1} \mid i, j, k \geq 0\}$, and these elements are clearly linearly independent in S . It follows that \mathcal{B} is a basis for A and that θ is indeed a one-to-one map.

Corollary. *If $\beta \neq 0$, then $K[ud, du]$ is a polynomial ring in the two generators and $A(\alpha, \beta, \gamma)$ is a domain.*

2.2. If R is any K -algebra, σ any K -automorphism of R and x any central element of R , then the generalized Weyl algebra $R(\sigma, x)$ is defined to be the algebra generated by R and the two variables X^+ and X^- subject to the relations

$$(2.6) \quad \begin{aligned} X^-X^+ &= x, & X^+X^- &= \sigma(x), \\ X^+r &= \sigma(r)X^+, & X^-\sigma(r) &= rX^- \quad \text{for all } r \in R. \end{aligned}$$

Here, we take $R = K[x, y]$ and we let σ be described as in the preceding section. Then $\sigma(x) = y$, $\sigma(y) = \alpha y + \beta x + \gamma$, and it follows from the above that

$$\begin{aligned} X^+(X^+X^-) &= X^+y = \sigma(y)X^+ = (\alpha y + \beta x + \gamma)X^+ \\ &= (\alpha X^+X^- + \beta X^-X^+ + \gamma)X^+ \end{aligned}$$

and

$$\begin{aligned} (X^+X^-)X^- &= yX^- = X^-\sigma(y) = X^-(\alpha y + \beta x + \gamma) \\ &= X^-(\alpha X^+X^- + \beta X^-X^+ + \gamma). \end{aligned}$$

Thus there exists an algebra homomorphism $\varphi: A \rightarrow R(\sigma, x)$ given by $d \mapsto X^+$ and $u \mapsto X^-$. On the other hand, (2.5) implies that the map $\varphi': R(\sigma, x) \rightarrow A$ given by $X^+ \mapsto d$ and $X^- \mapsto u$ is also an algebra homomorphism. Therefore $\varphi' = \varphi^{-1}$ and φ is an isomorphism. In other words, we have shown

Theorem. *If $\beta \neq 0$, then the algebra $A = A(\alpha, \beta, \gamma)$ is isomorphic to a generalized Weyl algebra $R(\sigma, x)$ with $R = K[x, y]$.*

Consequently, [Bv1, Proposition 7] yields

Corollary. *If $\beta \neq 0$, then $A(\alpha, \beta, \gamma)$ is right and left Noetherian.*

§3. THE FILTERED RING APPROACH

3.1. To start with, let α, β, γ be arbitrary parameters and let $A = A(\alpha, \beta, c)$. We define a filtration on A for which the associated graded ring is isomorphic to $A(\alpha, \beta, 0)$. To this end, let $V = K + Ku + Kd$ and let $V_n = V^n$. Then $V_0 = K$, $V_1 = V$, and $\{V_n \mid n = 0, 1, 2, \dots\}$ is obviously a filtration of A . Certainly, $\bar{u} = u + K$ and $\bar{d} = d + K$ generate the associated graded ring $\text{Gr } A$, and it is clear, from (1.1) and (1.2), that \bar{u} and \bar{d} satisfy the generating relations of $A(\alpha, \beta, 0)$. Thus, there exists an epimorphism $\rho: A(\alpha, \beta, 0) \rightarrow \text{Gr } A$ given by $U \mapsto \bar{u}$ and $D \mapsto \bar{d}$. Here, of course, we use U and D to denote the obvious generators of $A(\alpha, \beta, 0)$. To see that ρ is an isomorphism, we use the PBW Theorem for down-up algebras as given in [B, Theorem 4.1]. Specifically, that result asserts that $\mathcal{C} = \{u^i(du)^j d^k \mid i, j, k \geq 0\}$ is a basis for A and that $V_n = V^n$ is spanned by those monomials with $i + 2j + k \leq n$. With this observation, it is clear that V_n/V_{n-1} has basis $\mathcal{C}_n = \{\bar{u}^i(\bar{d}\bar{u})^j \bar{d}^k \mid i, j, k \geq 0, i + 2j + k = n\}$, and hence $\bar{\mathcal{C}} = \bigcup_{n=0}^\infty \mathcal{C}_n$ is a basis of $\text{Gr } A$. But $\{U^i(DU)^j D^k \mid i, j, k \geq 0\}$ is a basis of $A(\alpha, \beta, 0)$, by [B, Theorem 4.1] again, and ρ maps this basis to $\bar{\mathcal{C}}$. Consequently, ρ is one-to-one, and we have shown

Lemma. *$A = A(\alpha, \beta, \gamma)$ has a filtration whose associated graded ring is isomorphic to $A(\alpha, \beta, 0)$.*

3.2. We now show that if $\beta \neq 0$ and if K is sufficiently big, then the algebra $A(\alpha, \beta, 0)$ is an iterated Ore extension. To start with, fix nonzero elements $\lambda, \mu \in K$, and let B be the algebra with generators a and b , and with relation

$$(3.1) \quad ba = \mu ab.$$

Then B is clearly an Ore extension of its polynomial subalgebra $K[a]$, and hence $\{a^i b^j \mid i, j \geq 0\}$ is a basis of B . Now let τ be the automorphism of B defined by $\tau(a) = \lambda a$, $\tau(b) = \mu b$, and let $\delta: B \rightarrow B$ be the K -linear map determined by

$$\delta(a^m b^n) = p_m(\lambda, \mu) a^{m-1} b^{n+1} \quad \text{for all } m, n \geq 0,$$

where

$$p_m = p_m(\lambda, \mu) = \sum_{i=0}^{m-1} \lambda^i \mu^{m-1-i}.$$

Since $p_{m+t} = \mu^t p_m + \lambda^m p_t$, it is easy to see that δ is a τ -derivation of B , that is,

$$\delta(rs) = \delta(r)s + \tau(r)\delta(s) \quad \text{for all } r, s \in B.$$

Hence, we can form the Ore extension $C = B[c; \tau, \delta] = C(\lambda, \mu)$. Basic properties of such extensions can be found in [GW, Chapter 1]. In particular, C is a free left and right B -module with basis $\{c^i \mid i \geq 0\}$, and with multiplication determined by

$$cr = \tau(r)c + \delta(r) \quad \text{for all } r \in B.$$

Indeed, since $\delta(a) = b$ and $\delta(b) = 0$, we have

$$(3.2) \quad ca = \lambda ac + b, \quad cb = \mu bc.$$

Furthermore, C has a basis over K consisting of all monomials $a^i b^j c^k$ with $i, j, k \geq 0$, and it is clear that $C = C(\lambda, \mu)$ is the K -algebra generated by a, b and c subject to the relations (3.1) and (3.2).

3.3. If $\eta \in K$ and if r and s are elements of any K -algebra, we introduce the notation $[r, s]_\eta = rs - \eta sr$. Now suppose that $\beta \neq 0$. If $\alpha^2 + 4\beta$ is a square in K , we say that $A = A(\alpha, \beta, \gamma)$ is *split*, and we can let λ and μ be the roots of the quadratic equation

$$\zeta^2 - \alpha\zeta - \beta = 0.$$

Thus $\lambda + \mu = \alpha$, $\lambda\mu = -\beta$, and $\lambda, \mu \neq 0$ since $\beta \neq 0$.

Now it is easily seen that the defining relations (1.1) and (1.2) for $A(\alpha, \beta, 0)$ can be expressed in the form

$$(3.3) \quad [[D, U]_\lambda, U]_\mu = 0 = [D, [D, U]_\lambda]_\mu.$$

Indeed,

$$\begin{aligned} [D, [D, U]_\lambda]_\mu &= [D, DU - \lambda UD]_\mu = D(DU - \lambda UD) - \mu(DU - \lambda UD)D \\ &= D^2U - (\lambda + \mu)DUD + \lambda\mu UD^2 = D^2U - \alpha DUD - \beta UD^2, \end{aligned}$$

so $0 = [D, [D, U]_\lambda]_\mu$ is equivalent to (1.1), and similarly, $[[D, U]_\lambda, U]_\mu = 0$ is equivalent to relation (1.2).

Finally, if we set $H = [D, U]_\lambda$, then (3.3) translates to

$$(3.4) \quad HU = \mu UH, \quad DH = \mu HD, \quad DU = \lambda UD + H.$$

In other words, $A(\alpha, \beta, 0)$ is generated by the elements U, D and H subject to the relations (3.4) and, in view of the comment at the end of §3.2, there is an algebra

isomorphism $\psi: C \rightarrow A(\alpha, \beta, 0)$ given by $\psi(c) = D$, $\psi(a) = U$ and $\psi(b) = H$. By combining all of this, we have therefore proved

Theorem. *Assume that $A = A(\alpha, \beta, \gamma)$ is split and that $\beta \neq 0$. Then A has a filtration whose associated graded ring $\text{Gr } A \cong A(\alpha, \beta, 0)$ is isomorphic to an iterated Ore extension of the form $K[a][b; \eta][c; \tau, \delta]$.*

We can also use the above result to prove that $A = A(\alpha, \beta, \gamma)$ is Noetherian when $\beta \neq 0$. Indeed, for this it suffices to extend the field and assume that A is split. Then $\text{Gr } A$ is an iterated Ore extension, so $\text{Gr } A$ is right and left Noetherian, and hence so is A .

§4. MAIN RESULTS

4.1. We start by considering the global dimension of $A = A(\alpha, \beta, \gamma)$. Since A and its associated graded ring $\text{Gr } A \cong A(\alpha, \beta, 0)$ are both isomorphic to their opposite rings, left and right global dimensions are equal here. Thus we can use gl dim to denote this common dimension.

Theorem. *If $\beta \neq 0$, then $\text{gl dim } A(\alpha, \beta, \gamma) = 3$.*

Proof. We first show that $\text{gl dim } A < \infty$ and for this, it suffices to assume that A is split. Indeed, if F is a field extension of K , then $A^F = F \otimes A$ is a free A -module and hence $\text{gl dim } A \leq \text{gl dim } A^F$ by [McR, Theorem 7.2.8]. Now if A is split, then Theorem 3.2 implies that A has a filtration with $\text{Gr } A$ isomorphic to an iterated Ore extension. Thus $\text{gl dim } \text{Gr } A < \infty$ by [McR, Theorem 7.5.3], and consequently $\text{gl dim } A < \infty$ by [McR, Corollary 7.6.18].

Now, by Theorem 2.2, A is isomorphic to a generalized Weyl algebra $R(\sigma, x)$ with $R = K[x, y]$. Here the automorphism σ of R is given by $\sigma(x) = y$ and $\sigma(y) = \alpha y + \beta x + \gamma$. Thus the maximal ideals $Q = (x, y)$ and $P = \sigma^{-1}(Q)$ of R both contain x , and it follows from [Bv2, Theorem 3.7] that $\text{gl dim } A = 3$. □

4.2. Recall that a Noetherian ring R is said to be *Auslander-regular* if R has finite global dimension and if, for every finitely generated R -module M and positive integer q , we have $j(N) \geq q$ for every submodule N of $\text{Ext}_R^q(M, R)$. Here $j(M) = \min\{j \mid \text{Ext}_R^j(M, R) \neq 0\}$. Furthermore, R is said to be *Cohen-Macaulay* if R has finite GK-dimension, and if the equality $\text{GKdim } M + j(M) = \text{GKdim } R$ holds for every finitely generated R -module M .

Lemma. *Let $\beta \neq 0$ and write $A = A(\alpha, \beta, \gamma)$.*

- (i) *A is Auslander-regular.*
- (ii) *If A is split, then it is also Cohen-Macaulay.*

Proof. (i) As is shown in [Bv2, pp. 88–89], a generalized Weyl algebra $R(\sigma, x)$ is always a factor ring of an iterated skew polynomial extension of R . The argument is as follows. First form the polynomial ring $R[z]$ and consider the generalized Weyl algebra $R[z](\sigma, x + z)$, where σ is extended to $R[z]$ by taking $\sigma(z) = z$. Thus, since z is in the center of $R[z](\sigma, x + z)$ and since $R[z](\sigma, x + z)/(z) \cong R(\sigma, x)$, it suffices to show that $R[z](\sigma, x + z)$ is an iterated skew polynomial ring extension of R . For this, note that $R[z](\sigma, x + z) \cong R[X^-; \sigma^{-1}][X^+; \sigma, \delta]$, where the automorphism σ is extended to $R[X^-; \sigma^{-1}]$ by $\sigma(X^-) = X^-$, and the σ -derivation δ is defined by $\delta(r) = 0$ for all $r \in R$ and $\delta(X^-) = \sigma(x) - x$.

We now proceed to show that A is Auslander-regular. To start with, Lemma 3.1 and [Bj, Theorem 4.1] allow us to assume that $\gamma = 0$. Consequently, σ is a graded automorphism of $R = K[x, y]$ and hence the Ore extensions above are all constructed via graded automorphisms. In other words, $R[z](\sigma, x + z)$ is an iterated Ore extension of a connected graded K -algebra, with each automorphism graded. Thus, [GZ, Lemma 3.8(2)] implies that $R[z](\sigma, x + z)$ is Auslander-regular (and also Cohen-Macaulay). But z is a central regular element, so we can conclude from [L, §3.4 Remark (3)] that the Auslander condition carries over to the factor ring $R[z](\sigma, x + z)/(z) \cong R(\sigma, x) \cong A$.

(ii) Now suppose that A is split. Then, by Lemma 3.1, A has a filtration with $A_0 = K$ and with $\text{Gr } A \cong A(\alpha, \beta, 0)$. Hence, by [GZ, Lemma 3.8(1)], it suffices to show that $A(\alpha, \beta, 0)$ is Auslander-regular and Cohen-Macaulay. To this end, observe that $A(\alpha, \beta, 0) \cong K[a][b; \eta][c; \tau, \delta]$, by Theorem 3.3, where η and τ are graded algebra automorphisms. Furthermore, in each Ore extension, the set of elements of total degree 0 is precisely equal to K . Thus, [GZ, Lemma 3.8(2)] implies that $A(\alpha, \beta, 0)$ is Auslander-regular and Cohen-Macaulay, as required. \square

4.3. Lemma. *If $\beta = 0$, then A is not right or left Noetherian.*

Proof. For convenience, set $x = du$ so that $\{u^i x^j d^k \mid i, j, k \geq 0\}$ is a K -basis for A . Since $\beta = 0$, (1.1) yields $dx = (\alpha x + \gamma)d$ and hence $dx^j = (\alpha x + \gamma)^j d$ for all $j \geq 0$. Furthermore, by (1.2), we have $(\alpha ud + \gamma - x)u = 0$.

For each $n \geq 0$, set

$$I_n = \sum_{i=0}^n u^i (\alpha ud + \gamma - x)A.$$

Then, since $(\alpha ud + \gamma - x)u = 0$, we have

$$I_n = \sum_{i=0}^n \sum_{j,k=0}^{\infty} K u^i (\alpha ud + \gamma - x) x^j d^k.$$

In particular, since $udx^j d^k = u(\alpha x + \gamma)^j d^{k+1}$, it follows that no element of I_n can contain the monomial $u^{n+1}x$ in its support. Thus $u^{n+1}(\alpha ud + \gamma - x) \notin I_n$ and hence I_{n+1} is properly larger than I_n . In other words, we have shown that $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ is a properly increasing sequence of right ideals of A , and therefore A is not right Noetherian. Since $A \cong A^{\text{op}}$, the result follows. \square

4.4. Proof of the main theorem. Suppose first that $\beta \neq 0$. Then, by Corollary 2.1, A is a domain and $K[ud, du]$ is a polynomial ring in the two generators. Furthermore, Corollary 2.2 implies that A is right and left Noetherian, Theorem 4.1 yields the appropriate information on the global dimension of A , and Lemma 4.2 asserts that A is Auslander-regular.

Conversely, suppose $\beta = 0$. Then A is not a domain since, as observed in [B], $d(du - \alpha ud - \gamma) = 0$. Furthermore, multiplying this relation on the left by u shows that du and ud are algebraically dependent. Finally, A is not left or right Noetherian by Lemma 4.3.

We remark in closing that the recent manuscript [Z] completely determines the center of $A(\alpha, \beta, \gamma)$, while [K] proves the equivalence of (1) and (3) by showing that A is a hyperbolic ring if $\beta \neq 0$.

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