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SPECTRUM PRESERVING LINEAR MAPPINGS FOR SCATTERED JORDAN-BANACH ALGEBRAS

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ABSTRACT. Given two semisimple complex Jordan-Banach algebras with identity A and B, we say that T is a spectrum preserving linear mapping from A to B if T is surjective and we have $\mathrm{Sp}(Tx) = \mathrm{Sp}(x)$, for all $x \in A$. We prove that if B is a scattered Jordan-Banach algebra, then T is a Jordan isomorphism.

The aim of this note is to provide a detailed proof of a question raised in [2], saying that surjective linear and spectrum preserving mappings on separable scattered Jordan-Banach algebras are Jordan isomorphisms. This is in fact an extension of Theorem 3.7 of [4] from the associative case to the more general situation of Jordan-Banach algebras. In the proof of our result we will use the extension of Harte's theorem obtained by the author in [6, 7] and the structure theorem for scattered Jordan-Banach algebras of Aupetit-Baribeau [3]. In [2] Aupetit generalized some results previously obtained in [4], from the associative setting to the Jordan case.

The next theorem contains Theorem 2.1 and Corollary 2.4 of [2].

Theorem 1.1. *Let T be a spectrum preserving linear mapping from A onto B. Then $Tx^2 - (Tx)^2 \in \mathrm{Ann}(\mathrm{Soc}\, B)$, for every $x \in A$. Moreover T is a Jordan isomorphism from $kh(\mathrm{Soc}\, A)$ onto $kh(\mathrm{Soc}\, B)$.*

We recall that a complex Jordan algebra A is non-associative and the product satisfies the identities $ab = ba$ and $(ab)a^2 = a(ba^2)$, for all a, b in A. A unital Jordan-Banach algebra is a Jordan algebra with a complete norm satisfying $\|xy\| \le \|x\|\,\|y\|$, for $x, y \in A$, and $\|1\| = 1$. An element $a \in A$ is said to be invertible if there exists $b \in A$ such that $ab = 1$ and $a^2 b = a$. For an element x of a Jordan-Banach algebra A, the spectrum of x is by definition the set of $\lambda \in \mathbb{C}$ for which $\lambda - x$ is not invertible. It is a non-empty subset of $\mathbb{C}$. Moreover $x \mapsto \mathrm{Sp}(x)$ is upper semicontinuous on A. As in the associative case, the set $\Omega(A)$ of invertible elements is open, but unfortunately it is not anymore a multiplicative group (see [6, 7]). We also denote by $\Omega_1(A)$ the connected component of $\Omega(A)$, which contains the identity. At this stage it is appropriate to notice that if A is a Jordan-Banach algebra and we define $\exp(A) = \{e^x : x \in A\}$, then $\exp(A) \subset \Omega_1(A)$. It is clear that e^{-x} is the inverse of e^x for $x \in A$. It is also possible to give a concise notion of exponential spectrum of an element x, that we denote $\varepsilon(x)$, as the compact set defined by $\lambda \notin \varepsilon(x)$ if and

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only if $\lambda - x \in \Omega_1(A)$. As in the associative case we have $\mathrm{Sp}(x) \subset \varepsilon(x) \subset \sigma(x)$, where $\sigma(x)$ denotes the full spectrum of x. To see the first inclusion, suppose $\lambda \notin \varepsilon(x)$; then $\lambda - x \in \Omega_1(A)$, which means $\lambda - x$ is invertible, so $\lambda \notin \mathrm{Sp}(x)$. For the second one, if $\lambda \notin \sigma(x)$, then by the Holomorphic Functional Calculus $\lambda - x = e^y$ for y in the closed associative subalgebra $C(1, x)$ generated by 1 and x. Then $x(t) = e^{ty}$ defines a continuous path of invertible elements joining $\lambda - x$ and 1. Hence $\lambda - x \in \Omega_1(A)$ and $\lambda \notin \varepsilon(x)$.

In [6] we proved the following result for which another application is given in this note.

Theorem 1.2 (Extension of Harte's Theorem [6, 7])**.** *Let T be a continuous morphism from a Jordan-Banach algebra A onto a Jordan-Banach algebra B. Then $T(\Omega_1(A)) = \Omega_1(B)$. Moreover,*

$$\varepsilon(Tx) = \bigcap_{y \in \ker T} \varepsilon(x + y)$$

and

$$\mathrm{Sp}(Tx) \subset \bigcap_{y \in \ker T} \mathrm{Sp}(x + y) \subset \sigma(Tx).$$

We recall that for a semisimple Jordan-Banach algebra A, the socle $\mathrm{Soc}\, A$ is the sum of all the quadratic minimal ideals of A. It is known that $\mathrm{Soc}\, A$ is an ideal which is the sum of simple ideals generated by minimal projections. The theory of the socle has been extensively studied in recent years; for more details and references see [3, 7]. The notion of annihilator of a set of a Jordan algebra has been introduced by E. Zelmanov [8]; in particular the annihilator of an ideal is also an ideal. It is shown in [5] that $\mathrm{Ann}(\mathrm{Soc}\, A) = \{a \in A : a \cdot \mathrm{Soc}\, A = \{0\}\}$.

In the proof of the following important lemma, we use implicitly that a projection which is in $kh(\mathrm{Soc}\, B)$ is actually in $\mathrm{Soc}\, B$ and if u is in $\mathrm{Soc}\, B$, then its non-zero spectrum consists of isolated points. These two assertions follow from Corollaries 2.5, 2.6 and 2.7 of [6].

Lemma 1.3. *The ideal $I = kh(\mathrm{Soc}\, B) \cap \mathrm{Ann}(\mathrm{Soc}\, B) = \{0\}$.*

Proof. First we prove that $u \in I$ implies $\rho(u) = 0$ where ρ denotes the spectral radius. Suppose $\rho(u) \neq 0$; then there exists $\alpha \neq 0$, $\alpha \in \mathrm{Sp}\, u$. Now, the non-zero Riesz projection p associated to α and u is in the socle of B and we have $p = \frac{u}{2\pi i} \int_\Gamma \frac{1}{\lambda} (\lambda - u)^{-1}\, d\lambda$, where Γ is a small circle centred at α. Since $u \in \mathrm{Ann}(\mathrm{Soc}\, B)$, which is an ideal, we deduce that $p \in \mathrm{Ann}(\mathrm{Soc}\, B)$, so $p \in \mathrm{Soc}(B) \cap \mathrm{Ann}(\mathrm{Soc}\, B) = \{0\}$, and this is absurd. Consequently, the ideal I contains only quasi-nilpotent elements, so $I \subset \mathrm{Rad}\, B = \{0\}$. □

We now prove the main result of this note for scattered Jordan-Banach algebras, that is, Jordan-Banach algebras for which the spectrum of every element is finite or countable. By Aupetit-Baribeau's theorem their socle is non-empty and they have a very particular algebraic structure [3].

Theorem 1.4 (Aupetit-Baribeau)**.** *Let J be a complex Jordan-Banach algebra with identity such that the spectrum of every element is at most countable. Moreover, suppose that J is separable. Then there exists an ordinal α_0 of the first or second class and a sequence $(I_\alpha)_{\alpha \le \alpha_0}$ of closed ideals of J such that $I_0 = \mathrm{Rad}\, J$, $I_{\alpha_0} = J$ and $I_{\alpha+1}/I_\alpha$ is a modular annihilator for $\alpha \le \alpha_0$.*

We are ready now to state and prove the main result of this note, which is an adaptation to the Jordan case of the proof of Theorem 3.7 of [4].

Theorem 1.5. *Let T be a spectrum preserving linear mapping from a Jordan-Banach algebra A onto a separable scattered Jordan-Banach algebra B. Then T is a Jordan isomorphism (i.e., $Tx^2 = (Tx)^2$ for all $x \in A$).*

Proof. Because $\mathrm{Sp}(Tx) = \mathrm{Sp}(x)$ for every $x \in A$ and B is a scattered Jordan-Banach algebra, it is clear that A is also a scattered Jordan-Banach algebra. Now let $I_1 = kh(\mathrm{Soc}\, A)$ and $J_1 = kh(\mathrm{Soc}\, B)$. By Theorem 1.1 we have $J_1 = T(I_1)$. Define semisimple Jordan-Banach algebras A_1 and B_1 by $A_1 = A/I_1, B_1 = B/J_1$. Let $\pi_1 : A \to A_1$ and $\gamma_1 : B \to B_1$ be the corresponding canonical maps. Define a linear map $T_1 : A_1 \to B_1$ by $T_1(\overline{a}) = \overline{Ta}$. By Theorem 1.2 and since the spectrum of every element of A and B is at most countable, we have $\mathrm{Sp}(x) = \sigma(x)$ and $\mathrm{Sp}(Tx) = \sigma(Tx)$. Then

$$\mathrm{Sp}(\overline{a}) = \bigcap_{x \in I_1} \mathrm{Sp}(a + x) = \bigcap_{y \in J_1} \mathrm{Sp}(Ta + y) = \mathrm{Sp}(T_1\overline{a}),$$

and hence T_1 is spectrum preserving. Furthermore if $a \in I_1$, then $Ta^2 - (Ta)^2 \in J_1 = kh(\mathrm{Soc}\, B)$ and also $Ta^2 - (Ta)^2 \in \mathrm{Ann}(\mathrm{Soc}\, B)$. Since $\mathrm{Soc}\, B \subset kh(\mathrm{Soc}\, B)$ implies $\mathrm{Ann}(kh(\mathrm{Soc}\, B)) \subset \mathrm{Ann}(\mathrm{Soc}\, B)$ we conclude that $Ta^2 - (Ta)^2 \in kh(\mathrm{Soc}\, B) \cap \mathrm{Ann}(\mathrm{Soc}\, B)$ which is zero by Lemma 1.3. Then $Ta^2 = (Ta)^2$ for every $a \in I_1$. Continuing inductively, we define

$$A_n = A_{n-1}/kh(\mathrm{Soc}\, A_{n-1}), \qquad B_n = B_{n-1}/kh(\mathrm{Soc}\, B_{n-1})$$

and we denote respectively by π_n, γ_n the canonical maps from A_{n-1} onto A_n and from B_{n-1} onto B_n. Let $I_n = \ker(\pi_1 \circ \cdots \circ \pi_n)$, $J_n = \ker(\gamma_n \circ \cdots \circ \gamma_1)$ and note that $J_n = T(I_n)$. Define a linear map $T_n : A_n \to B_n$ by $T_n(\overline{a}) = \overline{T_{n-1}(a)}$. Then T_n is spectrum preserving and by using Lemma 1.3 it is easy to see that for every $a \in I_n$, we have $Ta^2 = (Ta)^2$. If ω is the first ordinal number, we define

$$I_\omega = kh\left(\bigcup_{n \geq 1} I_n\right), \qquad J_\omega = kh\left(\bigcup_{n \geq 1} J_n\right)$$

and also notice that $T((\bigcup_{n \geq 1} I_n)) = (\bigcup_{n \geq 1} J_n)$. Define the linear mapping

$$T'_\omega : A/\overline{\left(\bigcup_{n \geq 1} I_n\right)} \to B/\overline{\left(\bigcup_{n \geq 1} J_n\right)} \quad \text{by } T'_\omega \overline{a} = \overline{Ta}.$$

By using Theorem 1.2 it follows that T'_ω is spectrum preserving and we have $J_\omega = T(I_\omega)$. Let $A_\omega = A/I_\omega$, $B_\omega = B/J_\omega$ and define a linear operator $T_\omega : A_\omega \to B_\omega$ by $T_\omega(\overline{a}) = \overline{Ta}$. Notice that T_ω is spectrum preserving and that A_ω and B_ω are semisimple. We claim that $Ta^2 = (Ta)^2$ for every $a \in I_\omega$. Let $a \in I_\omega$ and suppose that $u = Ta^2 - (Ta)^2 \neq 0$. By Lemma 1.3 we conclude that $\gamma_n \circ \cdots \circ \gamma_1(u) \neq 0$ for $n = 1, 2, \ldots$, and by Theorem 1.1 $(\gamma_n \circ \cdots \circ \gamma_1(u))y = 0$ for every $y \in \mathrm{Soc}\, B_n$. Since $u \neq 0$ and B is semisimple, there exists $b \in B$ such that $\mathrm{Sp}(ub) \neq \{0\}$ and since $ub \in J_\omega$, once more by Theorem 1.2 we have

$$(**) \qquad \bigcap_{y \in \overline{\bigcup_{n \geq 1} I_n}} \mathrm{Sp}(ub + y) = 0.$$

We now prove that there exists an integer n such that $\mathrm{Sp}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \neq \mathrm{Sp}(\gamma_{n+1} \circ \cdots \circ \gamma_0(ub))$, where $B_0 = B$ and γ_0 is the identity map on B. Suppose the contrary: then $\mathrm{Sp}(ub) = \mathrm{Sp}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \subset \mathrm{Sp}(ub + y)$ for every $n \geq 1$ and $y \in J_n$, consequently, by continuity of the spectrum, $\mathrm{Sp}(ub) \subset \mathrm{Sp}(ub + y)$ for every $y \in \overline{\bigcup_{n \geq 1} J_n}$. Now by $(**)$ we have $\mathrm{Sp}(ub) = \{0\}$, which is a contradiction. Hence suppose $\mathrm{Sp}_{B_n}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \neq \mathrm{Sp}_{B_{n+1}}(\gamma_{n+1} \circ \cdots \circ \gamma_0(ub))$ for some $n \geq 0$ and let $x = (\gamma_n \circ \cdots \circ \gamma_0)(ub)$. Then there exists an isolated point $\lambda \neq 0$ of $\mathrm{Sp}_{B_{n+1}}(x)$ such that $\lambda \notin \mathrm{Sp}_{B_{n+1}}(\gamma_{n+1}(x))$. If we denote by p the spectral projection associated to p and λ we have $p \in \mathrm{Soc}\, B_0$ and $0 \neq xp = (\gamma_n \circ \cdots \circ \gamma_0(u))(\gamma_n \circ \cdots \circ \gamma_0(b))p$, which contradicts the fact that $(\gamma_n \circ \cdots \circ \gamma_0(u))y = 0$ for every $y \in \mathrm{Soc}\, B_n$. So finally we have proved that for every $a \in I_\omega : Ta^2 - (Ta)^2 = 0$. Now, continuing by transfinite induction, there exists an ordinal β in the first class of ordinals such that $A = I_\beta$ ([3]). By the previous arguments it is easy to see that $B = I_\beta$ and $Ta^2 = (Ta)^2$ for every $a \in A$. □

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