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MATRIX PRESENTATIONS OF BRAIDS AND APPLICATIONS

SANG YOUL LEE

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ABSTRACT. We show that there exists a one-to-one correspondence between the class of certain block tridiagonal matrices with the entries $-1, 0$, or 1 and the free monoid generated by $2n$ generators $\sigma_1, \cdots, \sigma_n, \sigma_1^{-1}, \cdots, \sigma_n^{-1}$ and relation $\sigma_i^{\pm 1}\sigma_j^{\pm 1} = \sigma_j^{\pm 1}\sigma_i^{\pm 1}$ $(|i-j| \geq 2)$ and give some applications for braids. In particular, we give new formulation of the reduced Alexander matrices for closed braids.

1. INTRODUCTION

The Artin's braid group on $n+1$ strings $B_{n+1}(n > 0)$ has a standard presentation as a group with generators $\sigma_1, \sigma_2, \cdots, \sigma_n$ and defining relations:

$$(I)\ \sigma_i\sigma_j = \sigma_j\sigma_i\ (|i-j| \geq 2),$$
$$(II)\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\ (1 \leq i \leq n-1).$$

By $(\gamma, n+1)$ we mean a braid γ in B_{n+1}. The *closure* of a braid $(\gamma, n+1)$ is denoted by $(\gamma, n+1)^\wedge$ or simply by $\gamma^\wedge$. In [Al], Alexander showed that every oriented link in S^3 is ambient isotopic to the closure of some braid and Markov's theorem [Bir] said that two braids have the ambient isotopic closures if and only if they are Markov equivalent, i.e., there exists a finite sequence of the following two moves taking one to the other: $(M1)$ Replace $(\gamma, n+1)$ by $(\omega, n+1)$, where ω is a conjugate of γ; $(M2)$ Replace $(\gamma, n+1)$ by $(\gamma\sigma_{n+1}^{\pm 1}, n+2)$ (or vice versa).

In section 2 of this paper, we show that the class Ω_n of certain block tridiagonal matrices with entries $-1, 0$, or 1 and the free monoid $\mathcal{F}_n$, under juxtaposition, of all words generated by $2n$ generators $\sigma_1, \cdots, \sigma_n, \sigma_1^{-1}, \cdots, \sigma_n^{-1}$ are in one-to-one correspondence up to the relation $\sigma_i^{\pm 1}\sigma_j^{\pm 1} = \sigma_j^{\pm 1}\sigma_i^{\pm 1}$ $(|i-j| \geq 2)$ in $\mathcal{F}_n$. Since a braid word γ in B_{n+1} can be regarded as an element of $\mathcal{F}_n$, this correspondence gives a *matrix presentation* $M(\gamma) \in \Omega_n$ of the braid γ and, conversely, the matrix $M(\gamma)$ reproduces the braid γ up to the defining relation (I).

In [Mur], Murasugi defined an integral matrix M, called Murasugi's matrix, with respect to an oriented link diagram L in S^2 and showed that the S-equivalence class of the symmetrized matrix $M^* + M^{*T}$ of its principal minor M^* is an oriented link type invariant. He thereby attached a class of quadratic forms to an oriented link L and defined the *signature* of L, denoted by $\sigma(L)$, to be the signature of $M^* + M^{*T}$

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and the *nullity* of L, denoted by $\mathcal{N}(L)$, to be the nullity of $M^* + M^{*T}$ plus 1. Some authors([GL], [Tr]) developed more simple algorithms to compute the signature and nullity of an oriented link via its link diagram.

In section 3, for a braid $\gamma \in B_{n+1}$, we characterize a Murasugi's matrix with respect to the closure $\gamma^\wedge$ of γ from the matrix presentation $M(\gamma) \in \Omega_n$ of γ by using simple matrix calculations. This gives a new formulation of a reduced Alexander matrix $\Lambda_{M(\gamma)}(t)$ for the closure of a braid γ and simultaneously gives an algorithm to compute the signature and nullity of the closure $\gamma^\wedge$ of γ by using the reduced Alexander matrix $\Lambda_{M(\gamma)}(t)$ for $\gamma^\wedge$ without using the braid version of Reidemeister move III.

2. Certain block matrices and words

Let A be a block tridiagonal matrix of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & & & & \\ A_{21} & A_{22} & A_{23} & & & \\ & A_{32} & A_{33} & A_{34} & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{n-1n-2} & A_{n-1n-1} & A_{n-1n} \\ & & & & A_{nn-1} & A_{nn} \end{pmatrix} \tag{2.1}$$

satisfying the following conditions: for each $i \in \{1, 2, \cdots, n\}$,

(2.1.1). Each diagonal block $A_{ii} = (0)$, the 1×1 zero matrix, or the $s_i \times s_i$ matrix of the form: for $\tau_k = \pm 1$ $(k = 1, 2, \cdots, s_i)$, $A_{ii} = (-\tau_1)$ $(s_i = 1)$, and

$$A_{ii} = \begin{pmatrix} -\tau_1 & 0 & 0 & \cdots & 0 & \tau_1 \\ \tau_2 & -\tau_2 & 0 & \cdots & 0 & 0 \\ 0 & \tau_3 & -\tau_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\tau_{s_i-1} & 0 \\ 0 & 0 & 0 & \cdots & \tau_{s_i} & -\tau_{s_i} \end{pmatrix} (s_i > 1).$$

(2.1.2). If $A_{ii} = (0)$, then $A_{ii-1}, A_{ii+1}, A_{i-1i}$, and A_{i+1i} are all zero matrices.

If $A_{ii}, A_{i+1i+1} \neq (0)$, let c_j $(j = 1, 2, \cdots, s_i)$ denote the j^{th}-column vector of the matrix A_{i+1i}, i.e., $A_{i+1i} = [c_1\ c_2\ \cdots\ c_{s_i}]$. Then A_{i+1i} and A_{ii+1} satisfies the following:

(1) One and only one of the following occurs.
  (i) All c_j $(j = 1, 2, \cdots, s_i)$ are nonzero and $A_{ii+1} = O_{s_i \times s_{i+1}}$.
  (ii) For some k $(1 \leq k \leq s_i)$, $c_1, \cdots, c_k$ are all zero and $c_{k+1}, \cdots, c_{s_i}$ are all nonzero and, in this case, $A_{ii+1} = [r_1\ \cdots\ r_k\ O\ \cdots\ O]^T$, where $r_j = (0\ \cdots\ 0\ 1)$ or $(0\ \cdots\ 0\ -1)$ for all $j = 1, 2, \cdots, k$.

(2) Each nonzero column (resp., row) of A_{i+1i}(resp., A_{ii+1}) contains one and only one nonzero entry which is equal to -1 times the diagonal entry of A_{ii} corresponding to the column(resp., row).

(3) Let c_k be the first nonzero column of A_{i+1i}. Then for each p $(k \leq p \leq s_i - 1)$, the nonzero entry of the $(p+1)^{th}$-column c_{p+1} of A_{i+1i} cannot occur in the higher row than the row in which the nonzero entry of the p^{th}-column c_p of A_{i+1i} occurs.

Let Ω_n denote the set of all block tridiagonal matrices described in (2.1) and let $\mathcal{F}_n$ be the free monoid, under juxtaposition, of all words generated by $2n$ generators $\sigma_1, \sigma_2, \cdots, \sigma_n, \sigma_1^{-1}, \sigma_2^{-1}, \cdots, \sigma_n^{-1} (n \geq 1)$.

For a matrix $A = (A_{ij})_{1\leq i,j\leq n} \in \Omega_n$, we define a word $W(A) \in \mathcal{F}_n$ by the following procedure:

First, let

$$W_n = \sigma_n^{\tau_1}\sigma_n^{\tau_2}\cdots\sigma_n^{\tau_{s_n}},$$

where $-\tau_j$ $(j = 1, \cdots, s_n)$ is the (j, j)-diagonal entry of the block A_{nn} of A. If $A_{nn} = (0)$, then W_n is the empty word.

Next, for $i \in \{1, 2, \cdots, n-1\}$, we define the word W_i from the word W_{i+1} as follows: If $A_{ii} = (0)$, then $W_i = W_{i+1}$.

Now we assume that $A_{ii} \neq (0)$. If $A_{i+1i+1} = (0)$, then

$$W_i = \sigma_i^{\tau_1}\sigma_i^{\tau_2}\cdots\sigma_i^{\tau_{s_i}}W_{i+1}.$$

If $A_{i+1i+1} \neq (0)$, then we define W_i to be the word obtained from W_{i+1} by inserting the letters $\sigma_i^{\tau_1}, \sigma_i^{\tau_2}, \cdots, \sigma_i^{\tau_{s_i}}$ into the word W_{i+1}, where $-\tau_j$ $(j = 1, \cdots, s_i)$ is the (j, j)-diagonal entry of the block A_{ii} of A, as follows:

(1) If $A_{i+1i} = O_{s_{i+1}\times s_i}$, then

$$W_i = \sigma_i^{\tau_1}\sigma_i^{\tau_2}\cdots\sigma_i^{\tau_{s_i}}W_{i+1}.$$

(2) If $A_{i+1i} \neq O_{s_{i+1}\times s_i}$ and the k^{th}-column is the first nonzero column of A_{i+1i}, then for each $q = k, \cdots, s_i$,

(i) If the nonzero entry of the q^{th}-column occurs in the p^{th}-row $(1 \leq p \leq s_{i+1})$ and the nonzero entry of the $(q-1)^{th}$-column does not occur in the same p^{th}-row, then insert $\sigma_i^{\tau_1}\sigma_i^{\tau_2}\cdots\sigma_i^{\tau_{k-1}}$ before $\sigma_{i+1}^{\tau_1}$ and insert $\sigma_i^{\tau_q}$ after $\sigma_{i+1}^{\tau_p}$, i.e.,

$$W_i = \cdots\sigma_i^{\tau_1}\sigma_i^{\tau_2}\cdots\sigma_i^{\tau_{k-1}}\sigma_{i+1}^{\tau_1}\cdots\sigma_{i+1}^{\tau_p}\sigma_i^{\tau_q}\cdots\sigma_{i+1}^{\tau_{p+1}}\cdots.$$

(ii) If the nonzero entry of the q^{th}-column $(q \geq k+1)$ occurs in the p^{th}-row $(1 \leq p \leq s_{i+1})$ and the nonzero entry of the $(q-1)^{th}$-column also occurs in the same p^{th}-row, then insert $\sigma_i^{\tau_1}\sigma_i^{\tau_2}\cdots\sigma_i^{\tau_{k-1}}$ before $\sigma_{i+1}^{\tau_1}$ and insert $\sigma_i^{\tau_q}$ after $\sigma_i^{\tau_{q-1}}$, i.e.,

$$W_i = \cdots\sigma_i^{\tau_1}\sigma_i^{\tau_2}\cdots\sigma_i^{\tau_{k-1}}\sigma_{i+1}^{\tau_1}\cdots\sigma_{i+1}^{\tau_p}\cdots\sigma_i^{\tau_{q-1}}\sigma_i^{\tau_q}\cdots\sigma_{i+1}^{\tau_{p+1}}\cdots.$$

Repeating this process for $i = n-1, n-2, \cdots, 2, 1$, we obtain the word $W_1 \in \mathcal{F}_n$. Finally, we define the word $W(A)$ to be the word W_1.

Now we are going to consider the reverse procedure. For a word $\gamma = \sigma_{i_1}^{\tau_{i_1}}\sigma_{i_2}^{\tau_{i_2}}\cdots\sigma_{i_m}^{\tau_{i_m}}$ $(\tau_{i_k} = \pm 1, k = 1, 2, \cdots, m) \in \mathcal{F}_n$, we define a block matrix $M(\gamma) = (A_{ij})_{1\leq i,j\leq n} \in \Omega_n$ as follows:

For each $i \in \{1, 2, \cdots, n\}$, let s_i $(s_i \geq 0)$ denote the number of the generators σ_i, σ_i^{-1} which occur in the word γ and rewrite the s_i generators as $\sigma_{(i,1)}^{\tau(i,1)}, \sigma_{(i,2)}^{\tau(i,2)}, \cdots, \sigma_{(i,s_i)}^{\tau(i,s_i)}$ keeping the order from left to right, where $\tau(i, k)$ denotes the exponent of the generator σ_i in γ which constitutes $\sigma_{(i,k)}$ and $s_1 + s_2 + \cdots + s_n = m$. The resultant is denoted by $\bar{\gamma}$ and called the *rewriting word* of γ (see Example 2.1).

Each diagonal block A_{ii} $(1 \le i \le n)$ of $M(\gamma)$ is defined to be the square matrix given by $A_{ii} = (0)$ $(s_i = 0), A_{ii} = (-\tau(i,1))$ $(s_i = 1)$, and

$$A_{ii} = \begin{pmatrix} -\tau(i,1) & 0 & 0 & \cdots & 0 & \tau(i,1) \\ \tau(i,2) & -\tau(i,2) & 0 & \cdots & 0 & 0 \\ 0 & \tau(i,3) & -\tau(i,3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\tau(i,s_i-1) & 0 \\ 0 & 0 & 0 & \cdots & \tau(i,s_i) & -\tau(i,s_i) \end{pmatrix} (s_i > 1).$$

For $i \neq j$, define A_{ij} as follows:

(1) For $|i-j| > 1, A_{ij} = O_{s_i \times s_j}$.

(2) For $|i-j| = 1$, if $s_i = 0$ and so $A_{ii} = (0)$, then $A_{ii-1} = O_{1\times s_{i-1}}, A_{ii+1} = O_{1\times s_{i+1}}, A_{i-1i} = O_{s_{i-1}\times 1}$, and $A_{i+1i} = O_{s_{i+1}\times 1}$. If $s_i \ge 1$, then

$$A_{i+1i} = (a_{pq}^{i+1i})_{1\le p\le s_{i+1}, 1\le q\le s_i},$$

where $a_{pq}^{i+1i} = \tau(i,q)$ if $1 \le p \le s_{i+1}-1$ and $\sigma_{(i,q)}^{\tau(i,q)}$ occurs between $\sigma_{(i+1,p)}^{\tau(i+1,p)}$ and $\sigma_{(i+1,p+1)}^{\tau(i+1,p+1)}$ in $\bar{\gamma}$, $a_{s_{i+1}q}^{i+1i} = \tau(i,q)$ if $\sigma_{(i,q)}^{\tau(i,q)}$ occurs farther to the right than $\sigma_{(i+1,s_{i+1})}^{\tau(i+1,s_{i+1})}$ in $\bar{\gamma}$; otherwise, $a_{pq}^{i+1i} = 0$, and

$$A_{ii+1} = (a_{pq}^{ii+1})_{1\le p\le s_i, 1\le q\le s_{i+1}},$$

where $a_{pq}^{ii+1} = \tau(i,p)$ if $q = s_{i+1}$ and $\sigma_{(i,p)}^{\tau(i,p)}$ occurs farther to the left than $\sigma_{(i+1,1)}^{\tau(i+1,1)}$ in $\bar{\gamma}$; otherwise, $a_{pq}^{ii+1} = 0$.

It is obvious that the matrix $M(\gamma)$ is contained in Ω_n and $M(W(A)) = A$ for any matrix $A \in \Omega_n$.

Example 2.1. (1) Let $A = (A_{ij})_{1\le i,j\le 3}$ be the 11×11 matrix in Ω_3 given by

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & & & & \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & & & & \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & & & & \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ & & & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ & & & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1, \end{pmatrix}.$$

Then $W_3 = \sigma_3\sigma_3\sigma_3^{-1}\sigma_3^{-1}, W_2 = \sigma_2\sigma_2\sigma_3\sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2$, and

$$W_1 = \sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2\sigma_1 = W(A) \in \mathcal{F}_3.$$

Conversely, let $\gamma = W(A)$. Then the rewritten word $\bar{\gamma}$ of γ is

$$\bar{\gamma} = \sigma_{(2,1)}\sigma_{(1,1)}\sigma_{(2,2)}\sigma_{(1,2)}^{-1}\sigma_{(3,1)}\sigma_{(3,2)}\sigma_{(2,3)}^{-1}\sigma_{(3,3)}^{-1}\sigma_{(3,4)}^{-1}\sigma_{(2,4)}\sigma_{(1,3)}.$$

This gives that $M(\gamma) = M(W(A)) = A$.

Now let $\omega = \sigma_2\sigma_1\sigma_2\sigma_3\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2\sigma_1 \in \mathcal{F}_3$. Then the rewriting word $\bar{\omega}$ of ω is $\bar{\omega} = \sigma_{(2,1)}\sigma_{(1,1)}\sigma_{(2,2)}\sigma_{(3,1)}\sigma_{(3,2)}\sigma_{(1,2)}^{-1}\sigma_{(2,3)}^{-1}\sigma_{(3,3)}^{-1}\sigma_{(3,4)}^{-1}\sigma_{(2,4)}\sigma_{(1,3)}$ and $M(\omega) = A$. So $W(M(\omega)) = W(A) = \gamma$.

(2) Let A and B be the $n \times n$ matrices in Ω_1 and Ω_n, respectively, given by

$$A = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 1[0] & 0 & \cdots & 0 & 0 \\ 0[1] & -1 & 1[0] & \cdots & 0 & 0 \\ 0 & 0[1] & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1[0] \\ 0 & 0 & 0 & \cdots & 0[1] & -1 \end{pmatrix}.$$

Then $W(A) = \sigma_1^n \in \mathcal{F}_1$ and $W(B) = \sigma_1\sigma_2\cdots\sigma_n[\sigma_n\sigma_{n-1}\cdots\sigma_1]$ in $\mathcal{F}_n$. Conversely, $M(W(A)) = M(\sigma_1^n) = A$ and $M(W(B)) = M(\sigma_1\sigma_2\cdots\sigma_n[\sigma_n\sigma_{n-1}\cdots\sigma_1]) = B$.

Let γ and ω be two words in $\mathcal{F}_n$. By $\gamma \approx \omega$ we mean that γ can be transformed to ω by applying a finite number of relations: $\sigma_i^{\pm 1}\sigma_j^{\pm 1} = \sigma_j^{\pm 1}\sigma_i^{\pm 1}(|i-j| \geq 2)$ and vice versa. For $\gamma \in \mathcal{F}_n$, the equivalence class of γ is denoted by $[\gamma]$, i.e., $[\gamma] = \{\omega \in \mathcal{F}_n \mid \omega \approx \gamma\}$.

Lemma 2.2. *Let γ and ω be two words in $\mathcal{F}_n$. Then $\gamma \approx \omega$ if and only if $M(\gamma) = M(\omega)$.*

Proof. Let $M(\gamma) = (A_{ij})_{1\leq i,j\leq n}$ and $M(\omega) = (B_{ij})_{1\leq i,j\leq n}$ be the matrices for γ and ω in Ω_n defined as above. For each $i \in \{1, 2, \cdots, n\}$, let $\gamma(i)$ and $\omega(i)$ denote the words obtained from γ and ω by replacing all generators σ_j and $\sigma_j^{-1}(j \neq i)$ with the empty word and similarly, for $i \in \{1, 2, \cdots, n-1\}$, let $\gamma(i, i+1)$ and $\omega(i, i+1)$ denote the words obtained from γ and ω by replacing all generators σ_j and $\sigma_j^{-1}(j \neq i, i+1)$ with the empty word, respectively. Then it is clear from the definition that the words $\gamma(i), \gamma(i+1)$ and $\gamma(i, i+1)$ uniquely define the blocks $A_{ii}, A_{i+1i+1}, A_{ii+1}$, and A_{i+1i} of $M(\gamma)$ and the words $\omega(i), \omega(i+1)$ and $\omega(i, i+1)$ uniquely define the blocks $B_{ii}, B_{i+1i+1}, B_{ii+1}$, and B_{i+1i} of $M(\omega)$, and vice versa. Now $\gamma \approx \omega$ if and only if $\gamma(i) = \omega(i)$ $(1 \leq i \leq n)$ and $\gamma(i, i+1) = \omega(i, i+1)$ $(1 \leq i \leq n-1)$, equivalently $A_{ii} = B_{ii}$ $(1 \leq i \leq n)$ and $A_{i+1i} = B_{i+1i}, A_{ii+1} = B_{ii+1}$ $(1 \leq i \leq n-1)$. Hence $\gamma \approx \omega$ if and only if $M(\gamma) = M(\omega)$. □

Lemma 2.3. *For any word γ in $\mathcal{F}_n$, $W(M(\gamma)) \approx \gamma$.*

Proof. Let γ be any word in $\mathcal{F}_n$ and let $M(\gamma)$ be the matrix for γ in Ω_n. Now let $\omega = W(M(\gamma))$. Then $M(\omega) = M(W(M(\gamma))) = M(\gamma)$. By Lemma 2.2, $\gamma \approx \omega = W(M(\gamma))$. □

Theorem 2.4. *Let Ω_n be the set of all block tridiagonal matrices described in* (2.1) *and let $[\mathcal{F}_n] = \{[\gamma] \mid \gamma \in \mathcal{F}_n\}$, where $[\gamma] = \{\omega \in \mathcal{F}_n \mid \omega \approx \gamma\}$. Then Ω_n and $[\mathcal{F}_n]$ are in one-to-one correspondence.*

Proof. Let $\Phi : [\mathcal{F}_n] \to \Omega_n$ be the mapping defined by $\Phi([\gamma]) = M(\gamma)$ for $[\gamma] \in [\mathcal{F}_n]$ and let $\Psi : \Omega_n \to [\mathcal{F}_n]$ be the mapping defined by $\Psi(A) = [W(A)]$ for $A \in \Omega_n$. By Lemma 2.2, Φ is well defined and, by Lemma 2.3, $\Psi(\Phi([\gamma])) = [W(M(\gamma))] = [\gamma]$ for any $[\gamma] \in [\mathcal{F}_n]$. Since $\Phi(\Psi(A)) = M(W(A)) = A$ for any matrix $A \in \Omega_n$, Φ is a bijection and $\Phi^{-1} = \Psi$. This completes the proof. □

Let B_{n+1} be the Artin's braid group on $n+1$ strings and let γ be any braid word of B_{n+1}. Then we may regard γ as an element of $\mathcal{F}_n$. In this regard, the trivial braid 1_{n+1} of B_{n+1} corresponds to the empty word in $\mathcal{F}_n$. This gives that each braid word $\gamma \in B_{n+1}$ can be presented by a matrix $M(\gamma) \in \Omega_n$ and, conversely, any matrix $A \in \Omega_n$ presents a braid word $W(A) \in B_{n+1}$. In what follows the matrix $M(\gamma) \in \Omega_n$ corresponding to the braid word $\gamma \in B_{n+1}$ is called the *matrix presentation* of the braid word γ. Note that each braid of B_{n+1} has many matrix presentations.

3. Alexander matrices for closed braids

Let $A = (A_{ij})_{1\le i,j\le n}$ be a matrix in Ω_n, where A_{ii} is a $s_i \times s_i$ matrix ($s_i \ge 1$), and let

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & & & & & \\ \bar{A}_{21} & \bar{A}_{22} & & & & \\ & \bar{A}_{32} & \bar{A}_{33} & & & \\ & & \ddots & \ddots & & \\ & & & \bar{A}_{n-1n-2} & \bar{A}_{n-1n-1} & \\ & & & & \bar{A}_{nn-1} & \bar{A}_{nn} \end{pmatrix},$$

where $\bar{A}_{ii} = A_{ii}$ $(1 \le i \le n)$ and $\bar{A}_{i+1i} = A_{i+1i} + A^T_{ii+1}$ $(1 \le i \le n-1)$.

Define $D_A = \bar{A}_{11} \oplus \cdots \oplus \bar{A}_{nn}$ and $L_A = \bar{A} - D_A$.

Throughout this section we denote by C and E the block diagonal matrices given by $C = C_{s_1} \oplus \cdots \oplus C_{s_n}$ and $E = E_{s_1} \oplus \cdots \oplus E_{s_n}$, where C_{s_i} is a $s_i \times s_i$ circulant matrix and E_{s_i} is a $s_i \times s_i$ skew-symmetric matrix: $C_{s_i} = E_{s_i} = (0)$ if $s_i = 1, E_{s_i} = O_{2\times 2}$ if $s_i = 2$, and

$$C_{s_i} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, E_{s_i} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

Let Z be an infinite cyclic group generated by t and let JZ be the integral group ring of Z. For a matrix $A \in \Omega_n$, let $\Lambda_A(t)$ be the matrix with entries in the ring JZ defined by

$$\Lambda_A(t) = L_A D_A - t(L_A D_A)^T + \frac{1}{2}(1-t)CD_A + (1+t)E.$$

Then we have the following

Theorem 3.1. *Let γ be a braid in B_{n+1} and let $M(\gamma) \in \Omega_n$ be a matrix presentation of γ. Then a reduced Alexander matrix for the closure $\gamma^\wedge$ of γ is given by*

$$\Lambda_{M(\gamma)}(t) = L_{M(\gamma)} D_{M(\gamma)} - t(L_{M(\gamma)} D_{M(\gamma)})^T + \frac{1}{2}(1-t)CD_{M(\gamma)} + (1+t)E.$$

Proof. The theorem follows from Theorem 4.1([Mur]) and the identity $M'(\gamma) - tM'(\gamma)^T = \Lambda_{M(\gamma)}(t)$, where $M'(\gamma)$ is the Murasugi's matrix with respect to $\gamma^\wedge$ given by the following:

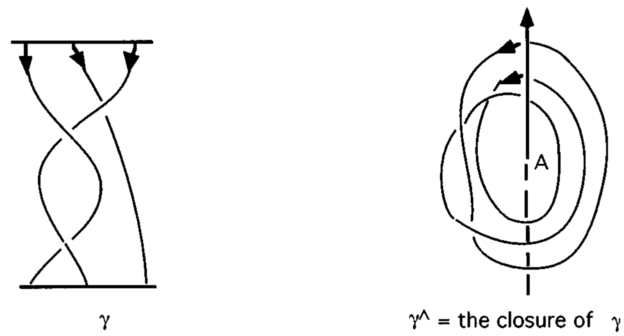

FIGURE 3.1

Lemma 3.2. *Let γ be a braid word in B_{n+1} and let $A = M(\gamma) \in \Omega_n$ be the matrix presentation of γ. Then $M'(\gamma) = (L_A + \frac{1}{2}C)D_A + E$ is a Murasugi's matrix with respect to the closure $\gamma^\wedge$ of γ. Consequently, if two braids γ and ω are Markov equivalent, then $M'(\gamma)$ and $M'(\omega)$ are S-equivalent.*

Proof. Let γ be a braid word in B_{n+1} and let $M(\gamma) = (A_{ij})_{1 \leq i,j \leq n}$ be the matrix presentation in Ω_n of γ. Assume that each A_{ii} in $M(\gamma)$ is an $s_i \times s_i$ matrix ($s_i \geq 1$). The closure $\gamma^\wedge$ of γ is an oriented link diagram in $\mathbb{R}^2$ obtained from γ by joining the $n+1$ points at the top of the braid γ to the corresponding $n+1$ points at the bottom without further crossings as in Figure 3.1.

In what follows we refer to [Mur, pp. 389–391] for the definitions of the Seifert circuit, Seifert domain, α-region, β-region, and the indices $\eta(c), d_r(c)$ and $\epsilon_r(c)$.

Case I. γ involves all the generators $\sigma_1, \sigma_2, \cdots, \sigma_n$ of B_{n+1}. It is obvious that the number of Seifert circuits in $\gamma^\wedge$ is equal to $n+1$. Among them, $n-1$ Seifert circuits are of the second type and so there are n Seifert domains, say, $D_1, D_2, \cdots, D_n$. Note that the unbounded region in D_1 and the region in D_n which intersects the braid axis A of $\gamma^\wedge$ are β-regions. Otherwise, they are all α-regions.

Now let $\bar{\gamma}$ be the rewriting word of $\gamma \in B_{n+1}$ and, for each $i \in \{1, 2, \cdots, n\}$, let W_p^i ($p = 1, 2, \cdots, s_i$) denote the word obtained from $\bar{\gamma}$ such that the initial letter of W_p^i is $\sigma_{(i,p)}^{\tau(i,p)}$ and the terminal letter of W_p^i is $\sigma_{(i,p+1)}^{\tau(i,p+1)}(s_i + 1 = 1)$ cyclically. Define $\bar{W}_p^i$ ($p = 1, 2, \cdots, s_i$) to be the word obtained from W_p^i by replacing all $\sigma_{(k,q)}^{\tau(k,q)}(k \neq i-1, i, i+1)$ with the empty word. Then each D_i ($i = 1, 2, \cdots, n$) contains s_i α-regions, denoted by $X_1^i, X_2^i, \cdots, X_{s_i}^i$, which can be identified with the words $\bar{W}_1^i, \bar{W}_2^i, \cdots, \bar{W}_{s_i}^i$, respectively, in such a way that all vertices in the boundary of the region X_p^i are just the letters in $\bar{W}_p^i$ ($p = 1, 2, \cdots, s_i$).

From Definition 3.3 [Mur, p.391] and the identification of X_p^i with $\bar{W}_p^i$, Murasugi's matrix M with respect to $\gamma^\wedge$ is given by the block matrix:

$$M = (M_{ij})_{i,j=1,2,\cdots,n},$$

$$M_{ii} = (a_{pq}^{(i)})_{p,q=1,2,\cdots,s_i},$$

where for $p \neq q$ $a_{pq}^{(i)} = -\sum \eta(c) d_{\bar{W}_p^i}(c)$, where the summation extends over all common letters c that are in the two different α-regions $\bar{W}_p^i$ and $\bar{W}_q^i$ contained in

D_i, and $a^{(i)}_{pp} = -\sum_{q=1,p\neq q}^{s_i} a^{(i)}_{pq}$,

$$M_{ij} = (b^{(ij)}_{pq})_{p=1,2,\cdots,s_i;q=1,2,\cdots,s_j}(i \neq j),$$

where $b^{(ij)}_{pq} = -\sum \eta(c) d_{\bar{W}^i_p}(c)\epsilon_{\bar{W}^j_q}(c)$, where the summation extends over all common letters c that are in the two α-regions $\bar{W}^i_p$ and $\bar{W}^j_q$ contained in D_i and D_j respectively.

(3.2.1). $M_{ii} = \frac{1}{2}C_{s_i}\bar{A}_{ii} + E_{s_i}$ $(i = 1, 2, \cdots, n)$.

If $s_i = 1$, then D_i has one and only one α-region $\bar{W}^i_1$, and $M_{ii} = (0)$.

If $s_i = 2$, then

$$M_{ii} = (a^{(i)}_{pq})_{p,q=1,2} = \begin{pmatrix} -\frac{\tau(i,1)+\tau(i,2)}{2} & \frac{\tau(i,1)+\tau(i,2)}{2} \\ \frac{\tau(i,1)+\tau(i,2)}{2} & -\frac{\tau(i,1)+\tau(i,2)}{2} \end{pmatrix} = \frac{1}{2}C_{s_i}\bar{A}_{ii}.$$

If $s_i \geq 3$, then for $1 \leq p, q \leq s_i$ $(p \neq q)$,

$$a^{(i)}_{pq} = \begin{cases} \frac{\tau(i,p)-1}{2} & \text{if } p = q+1 \ (1 \leq q \leq s_i - 1) \text{ or } p = 1, q = s_i, \\ \frac{1+\tau(i,q)}{2} & \text{if } q = p+1 \ (1 \leq p \leq s_i - 1) \text{ or } p = s_i, q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So $M_{ii} = (a^{(i)}_{pq})_{p,q=1,2,\cdots,s_i}$ is the matrix given by $M_{ii} =$

$$\begin{pmatrix} -\frac{\tau(i,1)+\tau(i,2)}{2} & \frac{1+\tau(i,2)}{2} & 0 & \cdots & 0 & \frac{\tau(i,1)-1}{2} \\ \frac{\tau(i,2)-1}{2} & -\frac{\tau(i,2)+\tau(i,3)}{2} & \frac{1+\tau(i,3)}{2} & \cdots & 0 & 0 \\ 0 & \frac{\tau(i,3)-1}{2} & -\frac{\tau(i,3)+\tau(i,4)}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{\tau(i,s_i-1)+\tau(i,s_i)}{2} & \frac{1+\tau(i,s_i)}{2} \\ \frac{1+\tau(i,1)}{2} & 0 & 0 & \cdots & \frac{\tau(i,s_i)-1}{2} & -\frac{\tau(i,1)+\tau(i,s_i)}{2} \end{pmatrix}$$

$= \frac{1}{2}C_{s_i}\bar{A}_{ii} + E_{s_i}$.

(3.2.2). For $1 \leq i, j \leq n (i \neq j)$, $M_{ij} = \begin{cases} \bar{A}_{ij}\bar{A}_{jj} & \text{if } i = j+1, \\ O_{s_i \times s_j} & \text{otherwise.} \end{cases}$

(i) $|i - j| > 2$: there do not exist common letters that are in the two α-regions $\bar{W}^i_p$ and $\bar{W}^j_q$ for all $1 \leq p \leq s_i, 1 \leq q \leq s_j$. Hence $b^{(ij)}_{pq} = 0$ and so $M_{ij} = O_{s_i \times s_j}$.

(ii) $|i-j| = 1$: $b^{(i+1i)}_{pq} = -\sum \eta(c) d_{\bar{W}^{i+1}_p}(c)\epsilon_{\bar{W}^i_q}(c)$, where the summation extends over all letters that are in the two α-regions $\bar{W}^{i+1}_p$ and $\bar{W}^i_q$. Since $d_{\bar{W}^{i+1}_p}(c) = 0$ for all common letters c that are not in the two regions $\bar{W}^{i+1}_p$ and $\bar{W}^i_q$, we have

$$\begin{aligned} b^{(i+1i)}_{pq} = &-\sum_{k=1}^{s_i} \eta(\sigma^{\tau(i,k)}_{(i,k)}) d_{\bar{W}^{i+1}_p}(\sigma^{\tau(i,k)}_{(i,k)})\epsilon_{\bar{W}^i_q}(\sigma^{\tau(i,k)}_{(i,k)}) \\ &-\sum_{k=1}^{s_{i+1}} \eta(\sigma^{\tau(i+1,k)}_{(i+1,k)}) d_{\bar{W}^{i+1}_p}(\sigma^{\tau(i+1,k)}_{(i+1,k)})\epsilon_{\bar{W}^i_q}(\sigma^{\tau(i+1,k)}_{(i+1,k)}). \end{aligned}$$

Since each letter $\sigma^{\tau(i+1,k)}_{(i+1,k)}$ does not belong to D_i (for the definition, see [Mur, p.390]), $\epsilon_{\bar{W}^i_q}(\sigma^{\tau(i+1,k)}_{(i+1,k)}) = 0$ for all $k = 1, 2, \cdots, s_{i+1}$.

Now let $\bar{a}_{pk}^{i+1i} = -\eta(\sigma_{(i,k)}^{\tau(i,k)})d_{\bar{W}_p^{i+1}}(\sigma_{(i,k)}^{\tau(i,k)})$. Then

$$\bar{a}_{pk}^{i+1i} = \begin{cases} 0 & \text{if } \sigma_{(i,k)}^{\tau(i,k)} \text{ is not in } \bar{W}_p^{i+1}, \\ \tau(i,k) & \text{if } \sigma_{(i,k)}^{\tau(i,k)} \text{ is in } \bar{W}_p^{i+1}. \end{cases}$$

This gives that $(\bar{a}_{pk}^{i+1i})_{1\le p\le s_{i+1}, 1\le k\le s_i} = \bar{A}_{i+1i}$. Let $\bar{a}_{kq}^{ii} = \epsilon_{\bar{W}_q^i}(\sigma_{(i,k)}^{\tau(i,k)})$. Then we can check that $(\bar{a}_{kq}^{ii})_{1\le k,q\le s_i} = \bar{A}_{ii}$. Hence

$$\begin{aligned} M_{i+1i} &= (b_{pq}^{(i+1i)})_{p=1,2,\cdots,s_{i+1};q=1,2,\cdots,s_i} \\ &= (-\sum_{k=1}^{s_i} \eta(\sigma_{(i,k)}^{\tau(i,k)})d_{\bar{W}_p^{i+1}}(\sigma_{(i,k)}^{\tau(i,k)})\epsilon_{\bar{W}_q^i}(\sigma_{(i,k)}^{\tau(i,k)}))_{p=1,2,\cdots,s_{i+1};q=1,2,\cdots,s_i} \\ &= (\sum_{k=1}^{s_i} \bar{a}_{pk}^{i+1i}\bar{a}_{kq}^{ii})_{p=1,2,\cdots,s_{i+1};q=1,2,\cdots,s_i} \\ &= \bar{A}_{i+1i}\bar{A}_{ii}. \end{aligned}$$

It is obvious that $M_{ii+1} = O_{s_i\times s_{i+1}}$.

(iii) $|i-j| = 2$: Since each letter c that is in the two α-regions X_p^i and X_q^j for all p, q are not contained in D_i or D_j, $\epsilon_{\bar{W}_q^j}(c) = 0$. Hence

$$M_{ij} = (b_{pq}^{(ij)})_{p=1,2,\cdots,s_i;q=1,2,\cdots,s_j} = (-\sum \eta(c)d_{\bar{W}_p^i}(c)\epsilon_{\bar{W}_q^j}(c)) = O_{s_i\times s_j}.$$

Combining (3.2.1) and (3.2.2), we obtain that $M = L_A D_A + \frac{1}{2}CD_A + E = M'(\gamma)$.

Case II. γ does not involve the generator $\sigma_{i_1}, \cdots, \sigma_{i_k}$ ($i_j \in \{1, 2, \cdots, n\}$) with $i_1 < i_2 < \cdots < i_k$. Then $\gamma \approx \omega = \omega_1(1, i_1 - 1)\omega_2(i_1 + 1, i_2 - 1)\cdots\omega_k(i_k + 1, n)$, where for $1 \le j \le k$, $\omega_j(r, s)$ denotes a braid word which involves all generators $\sigma_r, \sigma_{r+1}, \cdots, \sigma_s$ if $r \le s$, and $\omega_j(r, s)$ denotes the empty word if $r > s$. By Lemma 2.2, $M(\gamma) = M(\omega)$ and so $M'(\gamma) = M'(\omega) = M'(\omega_1) \oplus (0) \oplus M'(\omega_2) \oplus (0) \oplus \cdots \oplus (0) \oplus M'(\omega_k)$, where $M'(\omega_j) = M'(\omega_j(r, s))$ is the empty matrix if $r > s$. Note that the closure $\omega^\wedge$ is a split diagram with k components $\omega_1^\wedge, \omega_2^\wedge, \cdots, \omega_k^\wedge$. Let M_j denote the Murasugi's matrix with respect to $\omega_j^\wedge$. Then by Case I, we see that $M_j = M'(\omega_j)$. Hence the Murasugi's matrix M with respect to $\omega^\wedge$ is the matrix $M = M_1\oplus\cdots\oplus M_k\oplus O_{(k-1)\times(k-1)}$ and there is a unimodular integral matrix U such that $UMU^T = M'(\omega)$. This completes the proof of Lemma 3.2 and so Theorem 3.1. □

Now let $\Lambda_{M(\gamma)}^*(t)$ denote the principal minor of the reduced Alexander matrix $\Lambda_{M(\gamma)}(t)$ for $\gamma^\wedge$ obtained by deleting the row and column containing $(1, 1)$-entry in each diagonal block matrix $\frac{1}{2}(1-t)C_{s_i}\bar{A}_{ii} + (1+t)E_{s_i}$ corresponding to the nonzero block $\bar{A}_{ii}$ of D_A. Then we have the following:

Theorem 3.3. *Let γ be a braid word in B_{n+1} and let $\gamma^\wedge$ be the closure of γ. Then $\sigma(\gamma^\wedge) = \sigma(\Lambda_{M(\gamma)}^*(-1))$ and $\mathcal{N}(\gamma^\wedge) = \mathcal{N}(\Lambda_{M(\gamma)}^*(-1)) + 1$.*

Proof. Let $M'(\gamma)$ be the Murasugi matrix with respect to $\gamma^\wedge$ given by Lemma 3.2 and let $M^*(\gamma)$ denote the principal minor of $M'(\gamma)$ obtained by deleting the row and column containing $(1, 1)$-entry in each diagonal block matrix $\frac{1}{2}C_{s_i}\bar{A}_{ii} + E_{s_i}$ corresponding to the nonzero block $\bar{A}_{ii}$ of D_A. Then $\sigma(\gamma^\wedge) = \sigma(M^*(\gamma) + M^*(\gamma)^T)$ and $\mathcal{N}(\gamma^\wedge) = \mathcal{N}(M^*(\gamma) + M^*(\gamma)^T) + 1$. But $M^*(\gamma) + M^*(\gamma)^T = \Lambda_{M(\gamma)}^*(-1)$. This completes the proof. □

Theorem 3.4. *Let A be any matrix in Ω_n and let $W(A)$ be the braid in B_{n+1} corresponding to A. Then $\sigma(W(A)^\wedge) = \sigma(\Lambda^*_A(-1))$ and $\mathcal{N}(W(A)^\wedge) = \mathcal{N}(\Lambda^*_A(-1))+1$.*

Proof. Since $M(W(A)) = A$, $\Lambda^*_{M(W(A))}(t) = \Lambda^*_A(t)$. By Theorem 3.3, $\sigma(W(A)^\wedge) = \sigma(\Lambda^*_{M(W(A))}(-1)) = \sigma(\Lambda^*_A(-1))$ and $\mathcal{N}(W(A)^\wedge) = \mathcal{N}(\Lambda^*_{M(W(A))}(-1)) + 1 = \mathcal{N}(\Lambda^*_A(-1)) + 1$. □

Remark 3.5. For $i \in \{1, 2, \cdots, n\}$, let B^i_{n+1} denote the subgroup of B_{n+1} generated by $\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_n$. A braid $\gamma \in B_{n+1}$ is called a *split braid* if γ is conjugate into B^i_{n+1} for some $i \in \{1, 2, \cdots, n\}$. A braid $\gamma \in B_{n+1}$ is called a *positive braid* if it can be presented by a braid word such that the exponents of the generators in the word are all positive.

(1) Let γ be a nonsplit braid word in B_{n+1}. Then γ involves all generators in B_{n+1} and so $\ell(\gamma) = \mathcal{R}(\Lambda^*_{M(\gamma)}(-1)) + \mathcal{N}(\gamma^\wedge) + n - 1$, where $\ell(\gamma)$ is the letter length of γ and $\mathcal{R}(\Lambda^*_{M(\gamma)}(-1))$ denotes the rank of the matrix $\Lambda^*_{M(\gamma)}(-1)$. Hence we can see that $|\sigma(\gamma^\wedge)| \leq \mathcal{L}(\gamma) - \mathcal{N}(\gamma^\wedge) - n + 1$, where $\mathcal{L}(\gamma) = min\{\ell(\omega)|\omega$ is equivalent to $\gamma\}$. This shows that if $\ell(\gamma) = |\sigma(\gamma^\wedge)| + \mathcal{N}(\gamma^\wedge) + n - 1$, then γ is a minimal length braid representation in B_{n+1}(cf. [Ke]).

(2) Let γ be a nonsplit positive braid in B_{n+1} and let $\Delta_{\gamma^\wedge}(t)$ denote the reduced Alexander polynomial of $\gamma^\wedge$. Since $deg(\Delta_{\gamma^\wedge}(t)) = \ell(\gamma) - n$, $\sigma(\gamma^\wedge) + \mathcal{N}(\gamma^\wedge) - 1 \leq deg(\Delta_{\gamma^\wedge}(t))$. Moreover, the equality holds if and only if $\Lambda^*_{M(\gamma)}(-1)$ is a positive semidefinite, that is, the eigenvalues of $\Lambda^*_{M(\gamma)}(-1)$ are all nonnegative.

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Department of Mathematics, College of Natural Science, Kyungpook National University, Taegu 702-701, Korea

E-mail address: syleek@chollian.dacom.co.kr

Current address: Department of Mathematics, Pusan National University, Pusan 609-735, Korea

E-mail address: syleek@chollian.net