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GROWTH AND COVERING THEOREMS ASSOCIATED WITH THE ROPER-SUFFRIDGE EXTENSION OPERATOR

IAN GRAHAM

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Abstract. The Roper-Suffridge extension operator, originally introduced in the context of convex functions, provides a way of extending a (locally) univalent function $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ to a (locally) univalent map $F \in \mathrm{Hol}(B_n, \mathbb{C}^n)$. If f belongs to a class of univalent functions which satisfy a growth theorem and a distortion theorem, we show that F satisfies a growth theorem and consequently a covering theorem. We also obtain covering theorems of Bloch type: If f is convex, then the image of F (which, as shown by Roper and Suffridge, is convex) contains a ball of radius $\pi/4$. If $f \in S$, the image of F contains a ball of radius $1/2$.

1. Notation and properties of the Roper-Suffridge extension operator

Let z be a point in $\mathbb{C}^n$ with coordinates $(z_1, \dots, z_n)$. Let

$$|z| = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{1/2}$$

be the Euclidean norm of z. Let $z' = (z_2, \dots, z_n)$ so that $z = (z_1, z')$. Let B_n be the unit ball in $\mathbb{C}^n$ with respect to the Euclidean norm. If $a \in \mathbb{C}^n$ and $\rho > 0$, we denote by $B_n(a, \rho)$ the open Euclidean ball with centre a and radius ρ. Let $\mathrm{Hol}(B_n, \mathbb{C}^n)$ denote the set of holomorphic maps from B_n to $\mathbb{C}^n$. A map $F \in \mathrm{Hol}(B_n, \mathbb{C}^n)$ will be said to be normalized if $F(0) = 0$ and $dF(0) = I$. We denote the class of normalized univalent maps in $\mathrm{Hol}(B_n, \mathbb{C}^n)$ by $S(B_n)$. A map $F \in S(B_n)$ is said to be convex if its image is a convex set in $\mathbb{C}^n$.

We shall use z_1 as a coordinate in $\mathbb{C}$. We shall denote the unit disc in $\mathbb{C}$ by $\mathbb{D}$, and the class of normalized univalent functions on $\mathbb{D}$ by S. Let K be the subclass of S consisting of convex functions. Let $\mathcal{B}_0$ denote the subclass of S consisting of functions with Bloch seminorm 1, i.e. such that

$$\sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)|f'(z)| \;=\; f'(0) \;=\; 1\,. \tag{1.1}$$

The Roper-Suffridge extension operator is defined for normalized locally univalent functions on $\mathbb{D}$ by

$$\Phi_n(f)(z) \;=\; F(z) \;=\; \left(f(z_1), \sqrt{f'(z_1)}\, z'\right). \tag{1.2}$$

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We choose the branch of the square root such that $\sqrt{f'(0)} = 1$. Actually we shall extend our use of this notation slightly. If f is not normalized, so that there isn't a canonical choice of the branch of the square root, there would be two possible ways to define $\Phi_n(f)$. Both of them have the same range, however, and their growth properties are identical. Without specifying a rule for choosing between them, we shall sometimes write $\Phi_n(f)$ in discussing results which are valid for both choices of the branch of the square root. In this paper we shall work almost entirely with functions in S. However, problems in choosing the branch of the square root arise after composing with disc automorphisms. We note that $\Phi_n(S) \subseteq S(B_n)$.

In [RS], Roper and Suffridge showed that Φ_n has the remarkable property that if f is convex, then so is F. Prior to this there was no general way of constructing convex maps of B_n out of convex functions on $\mathbb{D}$. Subsequently this operator was used in [GV] to show that certain constants in covering theorems for convex maps are sharp, and by Pfaltzgraff [P] in studying certain families of non-convex functions.

Other properties of Φ_n which are easily verified include the following:

(1.3). If $f(z_1) = z_1/(1-z_1)$, then F is the normalized Cayley transform

$$F(z) \;=\; \big(z_1/(1-z_1),\, z'/(1-z_1)\big)\,.$$

(1.4). If f is an automorphism of $\mathbb{D}$, then F (i.e., either choice for F) is an automorphism of B_n.

(1.5). If f is odd or more generally k-fold symmetric, then so is F, i.e. $F\big(e^{2\pi i/k}z\big) = e^{2\pi i/k}F(z)$.

A further property which we shall verify is

1.6 Lemma. *If f is a locally univalent holomorphic function on $\mathbb{D}$ and T is an automorphism of $\mathbb{D}$, then $\Phi_n(f)$ and $\Phi_n(f\circ T)$ have the same range.*

1.7 Remark. For specific f and T, the branches of the square roots can be chosen so that we obtain the formula

$$\Phi_n(f\circ T) \;=\; \Phi_n(f)\circ\Phi_n(T)\,.$$

But it is not possible to choose the branch of the square root in a systematic way for all f so that this formula is always valid.

Proof. We have

$$\begin{aligned}\Phi_n(f\circ T)(z) \;&=\; \big(f\circ T(z_1),\, \sqrt{(f\circ T)'(z_1)}\,z'\big)\\ &=\; \left(f\big(T(z_1)\big),\, \pm\sqrt{f'\big(T(z_1)\big)}\,\sqrt{T'(z_1)}\,z'\right).\end{aligned} \tag{1.8}$$

Whether the correct choice of sign (for a particular f and T) is plus or minus does not matter, for in either case $\big(T(z_1),\, \pm\sqrt{T'(z_1)}\,z'\big)$ gives an automorphism of B_n. Thus the right-hand side of (1.8) has the form $\big(f(\zeta_1),\, \sqrt{f'(\zeta_1)}\,\zeta'\big)$ for some $\zeta \in B_n$, and ζ ranges over B_n as z does. $\square$

This result is needed in proving covering theorems of Bloch type for families of mappings of the form $\Phi_n(\mathcal{F})$, where $\mathcal{F} \subset S$.

2. Growth theorem for families of the form $\Phi_n(\mathcal{F})$

In this section we show that, under conditions which are quite commonly encountered in practice, if $\mathcal{F}$ is a subfamily of S whose members satisfy a growth theorem and a distortion theorem, then the maps in $\Phi_n(\mathcal{F})$ satisfy a similar growth theorem. This is in contrast to the behaviour of the full class $S(B_n)$ [DR], though positive results have been obtained with additional geometric assumptions, such as starlikeness or convexity [BFG], [FT].

2.1 Theorem. *Suppose that $\mathcal{F}$ is a subfamily of S such that all $f \in \mathcal{F}$ satisfy*

(2.1.1) $\varphi(r) \;\le\; |f(z_1)| \;\le\; \psi(r), \quad r = |z_1|,$

(2.1.2) $\varphi'(r) \;\le\; |f'(z_1)| \;\le\; \psi'(r)$

where

(2.1.3) φ, ψ *are twice differentiable on* $[0,1)$,

(2.1.4) $\varphi(r) \le r, \quad \varphi'(r) \ge 0, \quad \varphi''(r) \le 0 \quad$ *on* $[0,1)$,

(2.1.5) $\psi(r) \ge r, \quad \psi'(r) \ge 0, \quad \psi''(r) \ge 0 \quad$ *on* $[0,1)$.

Then all maps $F \in \Phi_n(\mathcal{F})$ satisfy the growth theorem

(2.1.6) $$\varphi(r) \;\le\; |F(z)| \;\le\; \psi(r), \quad r = |z|.$$

Furthermore if for some $f \in \mathcal{F}$ the lower (respectively upper) estimate in (2.1.1) is sharp at $z_1 \in \mathbb{D}$, then the lower (respectively upper) estimate in (2.1.6) is sharp for $\Phi_n(f)$ at $(z_1, 0, \dots, 0)$.

To prove this we need

2.2 Lemma. *Suppose φ and ψ are functions which satisfy the conditions (2.1.3)–(2.1.5) of Theorem 2.1. Then for fixed $r \in [0,1)$*

(2.2.1) *the minimum of* $\big(\varphi(t)\big)^2 + (r^2 - t^2)\varphi'(t)$ *for* $t \in [0,r]$ *occurs when* $t = r$;

(2.2.2) *the maximum of* $\big(\psi(t)\big)^2 + (r^2 - t^2)\psi'(t)$ *for* $t \in [0,r]$ *occurs when* $t = r$.

Proof. The sign of the first derivative on $[0,r]$ is easily checked. □

Proof of Theorem 2.1. For $|z| = r$ we have to give upper and lower estimates for $|f(z_1)^2| + |z'|^2|f'(z_1)| = |f(z_1)|^2 + (r^2 - |z_1|^2)|f'(z_1)|$. This easily done using Lemma 2.2, and the sharpness is also clear. □

As special cases of Theorem 2.1 we obtain the following

2.3 Corollary.

(2.3.1) *If $f \in S$, then* $\frac{r}{(1+r)^2} \le |\Phi_n(f)(z)| \le \frac{r}{(1-r)^2}$.

(2.3.2) *If $f \in S$ and f is k-fold symmetric, then*

$$\frac{r}{(1+r^k)^{2/k}} \;\le\; |\Phi_n(f)(z)| \;\le\; \frac{r}{(1-r^k)^{2/k}}\,.$$

(2.3.3) *If $f \in K$, then* $\frac{r}{1+r} \le |\Phi_n(f)(z)| \le \frac{r}{1-r}$.

(Note: the growth theorem $\frac{r}{1+r} \le |F(z)| \le \frac{r}{1-r}$ is known for the full class of convex maps of B_n [FT]*.)*

(2.3.4) *If $f \in K$ and $f''(0) = 0$, then*

$$\arctan r \;\le\; |\Phi_n(f)(z)| \;\le\; \frac{1}{2}\log\frac{1+r}{1-r}\,.$$

(2.3.5) *If $f \in K$ and $f''(0) = 0, \dots, f^{(k)}(0) = 0$, then*

$$\int_0^r \frac{dt}{(1+t^k)^{2/k}} \;\le\; |\Phi_n(f)(z)| \;\le\; \int_0^r \frac{dt}{(1-t^k)^{2/k}}\,.$$

(2.3.6) *If $f \in \mathcal{B}_0$, then*

$$\frac{1}{2}\left(1 - \exp\left(-\frac{2r}{1-r}\right)\right) \;\le\; |\Phi_n(f)(z)| \;\le\; \frac{1}{2}\log\frac{1+r}{1-r}\,.$$

2.4 Remark. All of these results are sharp except for the lower estimate in (2.3.6). For the one-variable estimates corresponding to (2.3.4) and (2.3.5) see [GV] and the references there. The lower estimate in (2.3.6) is not sharp in one variable for functions in $\mathcal{B}_0$. The corresponding distortion estimate $|f'(z)| \ge \varphi'(r)$ is true for all normalized *locally* univalent Bloch functions with Bloch seminorm 1 [LM]. The distortion estimate is sharp for a function which is only locally univalent. From the distortion estimate and the local univalence of f one can deduce that $f(\{|z| < r\})$ contains a schlicht disc of radius $\varphi(r)$ centered at 0 (cf. the proof of the corollary on page 329 of [LM]). However univalence is needed to conclude that $|f(z)| \ge \varphi(r)$ whenever $|z| = r$.

3. Covering theorems for families of mappings of the form $\Phi_n(\mathcal{F})$

A covering theorem of Koebe type follows from Theorem 2.1 by a standard argument:

3.1 Theorem. *Suppose that the family $\mathcal{F} \subset S$ and the functions φ and ψ satisfy the hypotheses of Theorem 2.1. Then for all $f \in \mathcal{F}$, the image of $\Phi_n(f)$ contains the ball $B_n(0,\rho)$ where $\rho = \lim_{r \nearrow 1} \varphi(r)$.*

Proof. The existence of ρ follows from the fact that φ is a bounded increasing function. That $\Phi_n(f)(B_n) \supseteq B_n(0,\rho)$ follows from the fact that $\Phi_n(f)$ is an open mapping. □

3.2 Corollary.

(3.2.1) *If $f \in S$, then $\Phi_n(f)(B_n) \supseteq B_n(0, 1/4)$.*

(3.2.2) *If $f \in S$ and f is k-fold symmetric, then*

$$\Phi_n(f)(B_n) \;\supseteq\; B_n(0, 4^{-1/k})\,.$$

(3.2.3) *If $f \in K$, then $\Phi_n(f)(B_n) \supseteq B_n(0, 1/2)$.*

(Note: The 1/2-covering theorem is known for all convex maps of B_n [FT], [G]*.)*

(3.2.4) *If $f \in K$ and $f''(0) = 0$, then $\Phi_n(f)(B_n) \supseteq B_n(0, \pi/4)$.*

(3.2.5) *If $f \in K$ and $f''(0) = 0, \dots, f^{(k)}(0) = 0$, then*

$$\Phi_n(f)(B_n) \;\supseteq\; B_n(0, r_k) \quad \textit{where } r_k = \int_0^1 \frac{dt}{(1+t^k)^{1/k}}\,.$$

(3.2.6) *If $f \in \mathcal{B}_0$, then $\Phi_n(f)(B_n) \supseteq B_n(0, 1/2)$.*

All results are sharp except for (3.2.6).

Finally we shall obtain covering theorems of Bloch type for the families $\Phi_n(S)$ and $\Phi_n(K)$.

3.3 Theorem. *(a) The image of every $F \in \Phi_n(K)$ contains a ball of radius $\pi/4$. This result is sharp.*

(b) The image of every $F \in \Phi_n(S)$ contains a ball of radius $1/2$.

Proof. These results may be deduced from (3.2.4) and (3.2.6) respectively if we establish that, in addition to the stated assumptions, it is possible to assume that $f \in \mathcal{B}_0$ (which implies $f''(0) = 0$). To do so we apply Landau's reduction in the z_1 variable and use Lemma 1.6. To this end, suppose $f\colon \mathbb{D} \to \mathbb{C}$ is univalent, normalized, and f is not in $\mathcal{B}_0$. By dilating, we may assume that f is holomorphic on the closure of $\mathbb{D}$, so that $(1-|z|^2)|f'(z)|$ has an interior maximum at a. (a cannot be zero since f is not in $\mathcal{B}_0$.) Let T be a disc automorphism such that $T(0) = a$ and let $(f \circ T)'(0) = \alpha$. We must have $|\alpha| > 1$ since $a \neq 0$. Define $\Lambda_T(f)(z_1) = \alpha^{-1}\big(f \circ T(z_1) - f \circ T(0)\big)$ and consider the map $\Phi_n\big(\Lambda_T(f)\big)$ given by

$$\begin{aligned}\Phi_n\big(\Lambda_T(f)\big)(z) &= \left(\big(\Lambda_T(f)\big)(z_1),\ \sqrt{\big(\Lambda_T(f)\big)'(z_1)}\, z'\right)\\ &= A\big(\Phi_n(f \circ T)(z) - b\big)\end{aligned}$$

where $A = \operatorname{diag}(\alpha^{-1}, \alpha^{-1/2}, \dots, \alpha^{-1/2})$, the choice of $\alpha^{-1/2}$ is determined by the choice of branch of $\sqrt{(f \circ T)'}$ so that we have $\sqrt{\big(\Lambda_T(f)\big)'(z_1)} = \alpha^{-1/2}\sqrt{(f \circ T)'(z_1)}$, b has entries $\big(f \circ T(0), 0, \dots, 0\big)$, and the action of A on $\Phi_n(f \circ T)(z) - b$ is determined by writing the latter as a column vector. It can now be seen that if the image of $\Phi_n\big(\Lambda_T(f)\big)$ contains the ball $|w_1 - c_1|^2 + |w' - c'|^2 < \rho^2$, then the image of $\Phi_n(f \circ T)$ or of $\Phi_n(f)$ contains the ellipsoid $|\alpha|^{-2}|w_1 - c_1 - f \circ T(0)|^2 + |\alpha|^{-1}|w' - c'|^2 < \rho^2$. This ellipsoid contains the ball $|w_1 - c_1 - f \circ T(0)|^2 + |w' - c'|^2 < |\alpha|\rho^2$, which is larger than the original ball. We further note that $\Lambda_T(f)$ is convex whenever f is, so the same is true of $\Phi_n\big(\Lambda_T(f)\big)$. Thus given any f in S (resp. K) we can produce a g in $\mathcal{B}_0$ such that the largest ball in $\Phi_n(g)(\mathbb{B}_n)$ has smaller radius than that of the largest ball in $\Phi_n(f)(\mathbb{B}_n)$. So we may indeed reduce to the case $f \in \mathcal{B}_0$ and apply (3.2.4) and (3.2.6) as claimed.

The sharpness of part (a) follows by considering the map $\Phi_n(f)$ where $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$. □

3.4 Remark. In one variable it has been known since 1923 (a result of G. Szegö [S]) that the image of any convex map of $\mathbb{D}$ contains a disc of radius $\pi/4$. This result was rediscovered by M. Zhang [Z] and D. Minda [M] using ultrahyperbolic metrics, and can also be deduced from the growth theorem for convex functions with $f''(0) = 0$, cf. [GV]. In [GV] the question of whether the image of any convex map of B_n contains a ball of radius $\pi/4$ was considered. An affirmative answer was obtained for the case of odd mappings. (The extremal map in one variable is odd.) To this we have now added the case of convex maps of the family $\Phi_n(K)$. But the general case of the problem remains open.

References

[BFG] R.W. Barnard, C.H. FitzGerald and S. Gong, The growth and 1/4-theorems for starlike mappings in $\mathbb{C}^n$. *Pacific J. Math.* **150** (1991), 13–22. MR **92g:**32046

[DR] P. Duren and W. Rudin, Distortion in Several Variables. *Complex Variables* **5** (1986), 323–326. MR **88c:**32003

[FT] C. FitzGerald and C. Thomas, Some bounds on convex mappings in several complex variables. *Pacific J. Math.* **165** (1994), 295–320. MR **95k:**32021

[G] I. Graham, Sharp constants for the Koebe theorem and for estimates of intrinsic metrics on convex domains. *Proc. Symp. Pure Math.* **52** (1991), Part 2, 233–238. MR **92j:**32088

[GV] I. Graham and D. Varolin, Bloch constants in one and several variables. *Pacific J. Math.* **174** (1996), 347–357. MR **97f:**30006

[LM] X. Liu and D. Minda, Distortion theorems for Bloch functions. *Trans. Amer. Math. Soc.* **333** (1992), 325–338. MR **92k:**30041

[M] D. Minda, Lower bounds for the hyperbolic metric in convex regions. *Rocky Mtn. J. Math.* **13** (1983), 61–69. MR **84j:**30039

[P] J. Pfaltzgraff, Distortion of locally biholomorphic maps of the n-ball, *Complex Variables Theory Appl.* **33** (1997), 239–253. CMP 98:13

[RS] K. Roper and T. Suffridge, Convex mappings on the unit ball in $\mathbb{C}^n$. *J. d'Anal. Math.* **65** (1995), 333–347. MR **96m:**32023

[S] G. Szegö, Über eine Extremalaufgabe aus der Theorie der schlichten Abbildungen. *Sitzungsberichte der Berliner Mathematische Gesellschaft* **22** (1923), 38–47. [Gabor Szegö: Collected Papers, ed. by Richard Askey, Birkhäuser Verlag, Boston-Basel-Stuttgart 1982, Vol. I, pp. 607–618.]

[Z] M. Zhang, Ein überdeckungssatz für konvexe Gebiete. *Acad. Sinica Science Record* **5** (1952), 17–21. MR **15:**413d

Department of Mathematics, University of Toronto, Toronto, Canada M5S 1A1
E-mail address: graham@math.toronto.edu