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ON MODELS OF $U_q(sl(2))$ AND q–APPELL FUNCTIONS USING A q–INTEGRAL TRANSFORMATION

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Abstract. We discuss few models of the quantum universal enveloping algebra of $sl(2)$ from the special function point of view. Two sets of such models are given, one acting on the space of ${}_1\phi_0$ functions while the other on the space of q–Appell functions. These models are closely related through a q–integral transformation. Some interesting identities are obtained.

1. Introduction

Quantum groups have generated considerable interest among mathematicians and physicists alike. The theory of q–special functions has found a group theoretic interpretation using the techniques of quantum groups and quantum algebras. In this direction major contributions have come from the works of Floreanini and Vinet [6]–[12], Kalnins and Miller [18], to name a few.

In a recent paper [29], the author has constructed models of irreps of classical $sl(2)$ in terms of difference-differential operators using an integral transformation which is inspired by the well known Euler integral representation for the hypergeometric functions ${}_2F_1$ with extension to some other functions as well. The models in [29] have culminated in various recurrence relations and identities involving ordinary hypergeometric series. To have an extension of this to q–series, we introduce a q–integral transformation, which is a generalization of and is motivated by Thomae's integral representation of ${}_2\phi_1$, given in [13], to construct models of $U_q(sl(2))$, the quantum universal enveloping algebra of $sl(2)$, in terms of difference-dilation operators. These models act on a space of basis functions involving q–Appell functions and give rise to identities involving certain generalized q–hypergeometric series in two and three variables. A q–difference equation satisfied by q–Appell functions is also obtained. Section–wise treatment is as follows.

In Section 2, we list various definitions and results needed for our discussion. In Section 3, we discuss models of $U_q(sl(2))$ and give its two variable models in terms of dilation operators in which ${}_1\phi_0$ functions appear as the basis functions. In Section 4, a q–integral transformation is defined and transforms of certain expressions are obtained. Next, we reproduce two theorems which help us in upgrading the models of Section 3 into new models of $U_q(sl(2))$ in terms of difference-dilation operators

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in which the basis functions involve $\Phi^{(2)}$, the q–Appell functions. These models, in Section 5, are exploited for identities, which are believed to be new. Finally in Section 6, we outline the corresponding transformed models for a m–fold q–integral transformation.

2. Preliminaries

The generalized basic (or q–) hypergeometric series is defined by [13]

$$ {}_r\phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(b_1,\ldots,b_s,q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}x^n, \tag{1}$$

where

$$(a_1,\ldots,a_r;q)_n=\prod_{i=1}^{r}(a_i;q)_n \tag{2}$$

and q–shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n=\begin{cases}(1-a)(1-aq)\cdots\left(1-aq^{n-1}\right), & n=1,2,\ldots,\\ 1, & n=0,\\ \left[\left(1-aq^{-1}\right)\cdots(1-aq^{-n})\right]^{-1}, & n=-1,-2,\ldots,\end{cases} \tag{3}$$

$$(a;q)_\infty=\prod_{r=0}^{\infty}(1-aq^r). \tag{4}$$

The series ${}_r\phi_s$ terminates if one of the numerator parameter is of the form q^{-m}, $m=0,1,2,\ldots$ and $q\neq 0$. When $0<|q|<1$, the series ${}_r\phi_s$ converges absolutely for all z if $r\le s$, and for $|z|<1$ if $r=s+1$. If $|q|>1$ and $|z|<|b_1\cdots b_s|/|a_1\cdots a_r|$, then also ${}_r\phi_s$ converges absolutely. The series ${}_r\phi_s$ diverges for $z\neq 0$ when $0<|q|<1$ and $r>s+1$, and when $|q|>1$ and $|z|>|b_1\cdots b_s|/|a_1\cdots a_r|$, unless it terminates.

We now list q–analogues of some important functions which will be needed in the discussion. The q–analogues of the exponential function are

$$e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}=\frac{1}{(x;q)_\infty},\quad |x|<1, \tag{5}$$

and

$$E_q(x)=\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{x^n}{(q;q)_n}=(-x;q)_\infty. \tag{6}$$

$E_q(x)$ converges for all x. Note that $e_q(x)\,E_q(-x)=1$. The q–analogue of the binomial function is

$$ {}_1\phi_0\left(\begin{matrix}a\\ -\end{matrix};x\right)=\frac{(ax;q)_\infty}{(x;q)_\infty},\quad |x|<1,\ |q|<1. \tag{7}$$

The gamma function has the following q–analogue:

$$\Gamma_q(\alpha)=\frac{e_q(q^\alpha)}{e_q(q)}(1-q)^{1-\alpha},\quad \alpha\neq 0,-1,-2,\cdots, \tag{8}$$

and satisfies the equation

$$\Gamma_q(\alpha+1)=\frac{1-q^\alpha}{1-q}\Gamma_q(\alpha). \tag{9}$$

The q–derivative operator is defined as

$$\Delta_z f(z) = \frac{(1-T_z)\, f(z)}{(1-q)\, z} \tag{10}$$

where the q–dilation operator T_z is defined by $T_z f(z) = f(qz)$. Indeed, as $q \to 1$, the q–derivative operator becomes an ordinary differential operator, provided f is differentiable at z.

The q–integral is defined as

$$\int_0^1 f(t)\, d_q t = (1-q) \sum_{n=0}^{\infty} f(q^n)\, q^n. \tag{11}$$

If f is continuous on $[0,1]$, then the q–integral reduces to the ordinary integral.

Also we need the q–analogue of Appell function F_2, [33], as

$$\Phi^{(2)}\left(\begin{matrix} a; b, b' \\ c, c' \end{matrix}; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+n}\,(b;q)_m\,(b';q)_n}{(c,q;q)_m\,(c',q;q)_n} x^m y^n \tag{12}$$

and the q–analogue of generalized Kampé de Fériet function $F^{A:B;C;D}_{E:F;G;H}$, as

$$\begin{aligned} &\phi^{A:B;C;D}_{E:F;G;H}\left(\begin{matrix} (a_i):(b_j);(c_k);(d_l) \\ (e_m):(f_n);(g_u);(h_v) \end{matrix}; x, y, z\right) \\ &= \sum_{p,r,t=0}^{\infty} \frac{((a_i);q)_{p+r+t}\,((b_j);q)_p\,((c_k);q)_r\,((d_l);q)_t}{((e_m);q)_{p+r+t}\,((f_n),q;q)_p\,((g_u),q;q)_r\,((h_v),q;q)_t}\, x^p\, y^r\, z^t \end{aligned} \tag{13}$$

where (a_i) abbreviates parameters $a_1, \ldots, a_A$, and where $((a_i);q)_{p+r+t}$ denotes $(a_1, \ldots, a_A; q)_{p+r+t}$, etc. The arguments and the complex parameters in (12) and (13) are so constrained that the multiple series converges.

3. Models of $U_q(sl(2))$

The commutation relations among the generators e, f, h

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = \frac{q^{h/2} - q^{-h/2}}{q^{1/2} - q^{-1/2}} \tag{14}$$

define an associative algebra [14], which may be regarded as the q–deformation of the universal enveloping algebra of $sl(2)$, referred to as $U_q(sl(2))$ in this paper. The Casimir element which belongs to the center of $U_q(sl(2))$ is given by

$$\mathbb{C} = \frac{k^2 q^{1/2} + k^{-2} q^{-1/2} - 2}{q^{1/2} - q^{-1/2}} + fe. \tag{15}$$

As the parameter $q \to 1^-$, these reduce to familiar commutator rules for $sl(2)$.

Let V_q be a complex vector space consisting of q–special functions with a basis $\{\phi_\lambda : \lambda \in S\}$ such that the functions $\{f_\lambda = \lim_{q\to 1} \phi_\lambda : \lambda \in S\}$ form a basis for the vector space, say V. Let $A(V_q)$ be the associative algebra of all linear operators on V_q over a complex field. Then a representation ρ_q of $sl(2)$ on V_q is defined as a mapping $\rho_q \colon sl(2) \to A(V_q)$ satisfying

(i) $\rho_q(ax + by) = a\rho_q(x) + b\rho_q(y)$.

(ii) There exists a Lie algebra representation ρ of $sl(2)$ on V such that

$$\lim_{q\to 1} \rho_q(x)\phi_\lambda = \rho(x) f_\lambda,$$

for all $x, y \in sl(2)$ and $a, b \in C$.

This representation ρ_q of $sl(2)$ is said to be irreducible if there is no proper subspace W_q of V_q which is invariant under ρ_q. Define

$$J_q^+ = \rho_q(\mathcal{J}^+), \quad J_q^- = \rho_q(\mathcal{J}^-), \quad J_q^0 = \rho_q(\mathcal{J}^0) \tag{16}$$

where $J_q^+, J_q^-, J_q^0 \in A(V_q)$ and $\{\mathcal{J}^+, J^-, J^0\}$ form a basis of $sl(2)$. On his work on irreps of $sl(2)$, Manocha [25] has defined the following commutator rules satisfied by J_q–operators:

$$\begin{aligned} &J_q^0 J_q^+ - qJ_q^+ J_q^0 = J_q^+, \\ &qJ_q^0 J_q^- - J_q^- J_q^0 = -J_q^-, \\ &qJ_q^+ J_q^- - J_q^- J_q^+ = 2q^{2u} J_q^0 - (1-q)q^{2u} J_q^0 J_q^0, \quad u \in C. \end{aligned} \tag{17}$$

If we define an operator C_q on V_q by

$$C_q = qJ_q^+ J_q^- + q^{2u} J_q^0 J_q^0 - q^{2u} J_q^0, \tag{18}$$

then it can be verified that

$$\begin{aligned} &qJ_q^+ C_q = C_q J_q^+, \\ &J_q^- C_q = qC_q J_q^-, \\ &J_q^0 C_q = C_q J_q^0. \end{aligned} \tag{19}$$

As will be seen, the operator C_q will be used to derive a q–difference equation satisfied by q–Appell function. Further, as $q \to 1$, the operators J_q^+, J_q^-, J_q^0 reduce to J^+, J^-, J^0 which satisfy the commutation relations obeyed by $sl(2)$ and the operator C_q reduces to the Casimir operator C and the relations (19) exhibit that C commutes with J^+, J^- and J^0. Hence, as $q \to 1$, ρ_q reduces to a Lie algebra representation ρ of $sl(2)$.

Notice that the commutation relations given by eqs. (17) are equivalent to that given by (14) in the sense that if the operators e, f, h satisfy (14), then J_q−operators defined by

$$J_q^+ = e, \quad J_q^- = q^{2u+\frac{h-1}{2}} f, \quad J_q^0 = \frac{1-q^{h/2}}{1-q} \tag{20}$$

satisfy (17).

Following the analysis as in Manocha [25] and Miller [27], we have the following theorem:

Theorem 3.1. *Every representation ρ_q of $sl(2)$ is isomorphic to a representation in the following list:*

(i) *The representation $D_q(\alpha, u), \alpha \in C \setminus \{0\}, 0 \le \operatorname{Re}\alpha < 1$, such that $\alpha - 2u$ is not an integer. $S = \{\alpha + n : n = 0, \pm1, \pm2, \dots\}$.*

(ii) *The representation $\uparrow_q (u)$, $u \in C$ and $2u$ is not a non-negative integer. $S = \{0, 1, 2, \dots\}$.*

(iii) *The representation $D_q(2u)$ where $2u$ is a non-negative integer. $S = \{0, 1, 2, \dots, 2u\}$.*

For each of these representations there is a basis of V_q consisting of vectors $\{f_\lambda\}$, defined for each $\lambda \in S$ such that

$$\begin{aligned} J_q^0 f_\lambda &= \frac{1-q^{\lambda-u}}{1-q} f_\lambda, \\ J_q^+ f_\lambda &= \frac{q^{2u}-q^\lambda}{1-q} f_{\lambda+1}, \\ J_q^- f_\lambda &= -\frac{1-q^\lambda}{1-q} f_{\lambda-1}. \end{aligned} \tag{21}$$

Guided by the above theorem, we give below two-variable models of representation $D_q(\alpha)$ in the particular case when $u=0$.

Model I

$$\begin{aligned} J_q^+ &= t\left(zT_t\Delta_z + t\Delta_t\right), \\ J_q^- &= t^{-1}T_z^{-1}\left(z(1-zT_t)\Delta_z - t\Delta_t\right), \\ J_q^0 &= t\Delta_t, \end{aligned} \tag{22}$$

$$f_\lambda = \frac{(q^\lambda z;q)_\infty}{(z;q)_\infty} t^\lambda = {}_1\phi_0\left(\begin{matrix} q^\lambda \\ - \end{matrix}; z\right) t^\lambda, \qquad \lambda \in S. \tag{23}$$

Model II

$$\begin{aligned} J_q^+ &= t\left(z(1-z)T_t\Delta_z + t\Delta_t\right), \\ J_q^- &= t^{-1}T_z^{-1}\left(z\Delta_z - t\Delta_t\right), \\ J_q^0 &= t\Delta_t, \end{aligned} \tag{24}$$

$$f_\lambda = \frac{(z;q)_\infty}{(q^\lambda z;q)_\infty} t^\lambda = {}_1\phi_0\left(\begin{matrix} q^{-\lambda} \\ - \end{matrix}; q^\lambda z\right) t^\lambda, \qquad \lambda \in S, \tag{25}$$

where $S = \{\alpha + n : \alpha \in C \setminus \{0\}, 0 \le \text{Re } \alpha < 1, n = 0, \pm 1, \pm 2, \dots\}$. The dilation models given above satisfy (17) as well as (21) with $u=0$.

A model of representation $\uparrow_q$ is same as Model II above with $S = \{0, 1, 2, \dots\}$. However, we concentrate on the model of representation $D_q(\alpha)$ and will return to the representation $\uparrow_q$ later in Section 5, where we obtain identities based on a model of representation $\uparrow_q$.

The models of irreducible representation of $U_q(sl(2))$ can be found in the literature too. For example, in [12], Floreanini and Vinet have considered two such models. Also, models of $U_q(su_2)$ are discussed in detail by Kalnins *et al.* [17, 19, 20, 21] and by Rideau and Winternitz [28]. The metaplectic representation of $su_q(1,1)$ are presented in [8] and are connected with q–Gegenbauer polynomials. Models of q–harmonic oscillators given by Biedenharn [1] and Mcfarlane [26] are closely related to the representation of $U_q(sl(2))$. Work on these representations has been carried out by Drinfeld [3], Faddeev [5], Kulish and Reshetikhin [23], Sklyanin [31, 32] and Woronowich [34, 35] among others. Some models of q-representations of the Lie algebra $\mathcal{G}(a,b)$ are contained in Sahai [30]. For more details, see Chari and Pressley [2], Jantzen [15], Kassel [22] and Majid [24].

4. A q–integral transformation and transformed models

In this section, we introduce a two-fold q–integral transformation, based on Thomae's integral representation of ${}_2\phi_1$ [13, 4], *viz.*

(26)
$$ {}_2\phi_1\left(\begin{matrix} q^\alpha, q^\beta \\ q^\gamma \end{matrix}; q, z\right) = \Gamma_q\left(\begin{matrix} \gamma \\ \beta, \gamma-\beta \end{matrix}\right)\int_0^1 t^{\beta-1}\frac{(tq,;q)_\infty}{(tq^{\gamma-\beta};q)_\infty}\,{}_1\phi_0\left(\begin{matrix} q^\alpha \\ - \end{matrix}; q, tz\right)d_qt $$

and then compute transforms of certain operator expressions. These transforms are later used to obtain new models from the ones given in Section 3 in which the basis functions will be in terms of q–Appell function.

Define

$$ (27)\qquad \begin{aligned} h(\beta,\beta',\gamma,\gamma') = \mathcal{I}[f(u,v)] &= \Gamma_q\left(\begin{matrix} \gamma, \gamma' \\ \beta, \beta', \gamma-\beta, \gamma'-\beta' \end{matrix}\right) \\ &\times \int_0^1\int_0^1 u^{\beta-1}v^{\beta'-1}\frac{(qu,qv;q)_\infty}{(q^{\gamma-\beta}u, q^{\gamma'-\beta'}v;q)_\infty}f(u,v)d_qud_qv, \end{aligned} $$

$$ \operatorname{Re}\gamma > \operatorname{Re}\beta > 0, \qquad \operatorname{Re}\gamma' > \operatorname{Re}\beta' > 0. $$

Defining the operators

$$ (28)\qquad \begin{aligned} E_\beta(h(\beta,\beta',\gamma,\gamma')) &= h(\beta+1,\beta',\gamma,\gamma'), \\ L_\beta(h(\beta,\beta',\gamma,\gamma')) &= h(\beta-1,\beta',\gamma,\gamma'), \end{aligned} $$

and denoting $E_{\beta\gamma} = E_\beta E_\gamma$, *etc.* we have the following transforms of operator expressions needed for our discussion:

$$ (29)\qquad \begin{aligned} \mathcal{I}[u\Delta_u f] &= x\Delta_x h, \\ \mathcal{I}[v\Delta_v f] &= y\Delta_y h, \\ \mathcal{I}[u^2\Delta_u f] &= \frac{1-q^\beta}{1-q^\gamma}x\Delta_x E_{\beta\gamma}h, \\ \mathcal{I}[v^2\Delta_v f] &= \frac{1-q^{\beta'}}{1-q^{\gamma'}}y\Delta_y E_{\beta'\gamma'}h, \\ \mathcal{I}[\Delta_u f] &= \frac{1-q^{\gamma-1}}{1-q^{\beta-1}}x\Delta_x L_{\beta\gamma}h, \\ \mathcal{I}[\Delta_v f] &= \frac{1-q^{\gamma'-1}}{1-q^{\beta'-1}}y\Delta_y L_{\beta'\gamma'}h, \\ \mathcal{I}[uf] &= \frac{1-q^\beta}{1-q^\gamma}E_{\beta\gamma}h, \\ \mathcal{I}[vf] &= \frac{1-q^{\beta'}}{1-q^{\gamma'}}E_{\beta'\gamma'}h. \end{aligned} $$

To obtain models of $U_q(sl(2))$ in terms of difference-dilation operators (difference in β, β', γ, γ' and dilation in x and y) in which the basis functions are in terms of q–Appell functions, we need the following theorems.

Theorem 4.1 ([25])**.** *Let ρ_q be a representation of $sl(2)$ in terms of $\{J_q^+, J_q^-, J_q^0\}$ with basis functions $\{f_\lambda : \lambda \in S\}$. Then ρ_q is also a representation of $sl(2)$ in terms of $\{K_q^+, K_q^-, K_q^0\}$ with basis functions $\{h_\lambda : \lambda \in S\}$ where*

$$ (30)\qquad K_q^+ = \mathcal{I}J_q^+\mathcal{I}^{-1}, \quad K_q^- = \mathcal{I}J_q^-\mathcal{I}^{-1}, \quad K_q^0 = \mathcal{I}J_q^0\mathcal{I}^{-1}, \quad h_\lambda = \mathcal{I}f_\lambda. $$

Theorem 4.2 ([13]). *If a and b are indeterminates such that $ab = qba$, q commutes with a and b and the associative law holds, then*

$$(a+b)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k a^{n-k}, \tag{31}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q–binomial coefficient.

To upgrade the two-variable models to three-variable ones, we substitute $z = ux + vy$. This gives

$$z\Delta_z = u\Delta_u + v\Delta_v T_u, \tag{32}$$

where Δ_z is defined by (10) and

$$T_z f(z) = f(qz) = T_u T_v f(ux + vy). \tag{33}$$

Putting $z = ux + vy$, the Model I takes the following shape:

$$\begin{aligned} J_q^+ &= t\left[(u\Delta_u + v\Delta_v T_u)\, T_t + t\Delta_t\right], \\ J_q^- &= t^{-1}T_u^{-1}T_v^{-1}\left[(1 - T_t(ux+vy))\,(u\Delta_u + v\Delta_v T_u) - t\Delta_t\right], \\ J_q^0 &= t\Delta_t, \end{aligned} \tag{34}$$

with the basis functions as

$$f_\lambda = {}_1\phi_0 \left(\begin{matrix} q^\lambda \\ - \end{matrix}; ux + vy \right) t^\lambda, \qquad \lambda \in S. \tag{35}$$

The J–model (34) along with (35) is now ready to be transformed into a K–model involving the q–Appell function $\Phi^{(2)}$ as the basis functions with the help of Theorems 4.1 and 4.2. We have

Model IA.

(36)

$$\begin{aligned} K_q^+ &= \mathcal{I}J_q^+\mathcal{I}^{-1} = t\left[(x\Delta_x + y\Delta_y T_x)\, T_t + t\Delta_t\right], \\ K_q^- &= \mathcal{I}J_q^-\mathcal{I}^{-1} \\ &= t^{-1}T_x^{-1}T_y^{-1}\Bigg[x\Delta_x + y\Delta_y T_x - xT_t\left(\frac{1-q^\beta}{1-q^\gamma}x\Delta_x E_{\beta\gamma} + \frac{1-q^\beta}{1-q^\gamma}y\Delta_y T_x E_{\beta\gamma}\right) \\ &\qquad - yT_t\left(\frac{1-q^{\beta'}}{1-q^{\gamma'}}x\Delta_x E_{\beta'\gamma'} + \frac{1-q^{\beta'}}{1-q^{\gamma'}}y\Delta_y T_x E_{\beta'\gamma'}\right) - t\Delta_t\Bigg], \\ K_q^0 &= \mathcal{I}J_q^0\mathcal{I}^{-1} = t\Delta_t, \end{aligned}$$

with the basis functions as

$$h_\lambda(x,y,t) = \Phi^{(2)} \left(\begin{matrix} q^\lambda; b, b' \\ c, c' \end{matrix}; x, y \right) t^\lambda, \qquad \lambda \in S, \tag{37}$$

where $b = q^\beta, b' = q^{\beta'}, c = q^\gamma, c' = q^{\gamma'}$. Note that Theorem 4.2 is utilized in obtaining $\mathcal{I}({}_1\phi_0) = \Phi^{(2)}$.

Likewise Model II is transformed to the following

Model IIA.

(38)

$$\begin{aligned}K_q^+ &= t\left[(x\Delta_x + y\Delta_y T_x)\,T_t - xT_t\left(\frac{1-q^\beta}{1-q^\gamma}x\Delta_x E_{\beta\gamma} + \frac{1-q^\beta}{1-q^\gamma}y\Delta_y T_x E_{\beta\gamma}\right)\right.\\ &\qquad \left. - yT_t\left(\frac{1-q^{\beta'}}{1-q^{\gamma'}}x\Delta_x E_{\beta'\gamma'} + \frac{1-q^{\beta'}}{1-q^{\gamma'}}y\Delta_y T_x E_{\beta'\gamma'}\right) + t\Delta_t\right],\\ K_q^- &= t^{-1}T_x^{-1}T_y^{-1}\left(x\Delta_x + y\Delta_y T_x - t\Delta_t\right),\\ K_q^0 &= t\Delta_t,\end{aligned}$$

with the basis functions as

$$h_\lambda(x,y,t) = \Phi^{(2)}\left(\begin{matrix}q^{-\lambda}; b, b' \\ c, c'\end{matrix}; q^\lambda x, q^\lambda y\right)\, t^\lambda, \qquad \lambda \in S. \tag{39}$$

The K–operators given above satisfy the following relations:

$$\begin{aligned}K_q^0 h_\lambda &= \frac{1-q^\lambda}{1-q}h_\lambda,\\ K_q^+ h_\lambda &= \frac{1-q^\lambda}{1-q}h_{\lambda+1},\\ K_q^- h_\lambda &= -\frac{1-q^\lambda}{1-q}h_{\lambda-1},\\ C_q h_\lambda &= 0 \text{ where } C_q = qK_q^+K_q^- + K_q^0K_q^0 - K_q^0\end{aligned} \tag{40}$$

as well as the commutation relations

$$\begin{aligned}K_q^0K_q^+ - qK_q^+K_q^0 &= K_q^+,\\ qK_q^0K_q^- - K_q^-K_q^0 &= -K_q^-,\\ qK_q^+K_q^- - K_q^-K_q^+ &= 2K_q^0 - (1-q)K_q^0K_q^0\end{aligned} \tag{41}$$

and as such give difference-dilation models of $U_q(sl(2))$.

Note that the equation $C_q h_\lambda = 0$ in (40) can be used to obtain a q-difference equation satisfied by $\Phi^{(2)}$. For example, if h_λ is given by (37), then we get the q-difference equation

(42)

$$\begin{aligned}&\left[\left\{\left(1-aq^{-1}\right) + aq^{-1}\left(xD_x^+ + yD_y^+T_x\right)\right\}\right.\\ &\times\left\{a\left(xD_x^- + yD_y^-T_x\right)\left(\frac{1-b}{1-c}xT_{bc} + \frac{1-b'}{1-c'}yT_{b'c'} - 1\right) - (1-a)\right\}\\ &\left.+q(1-a)\left(1-aq^{-1}\right)\right]\Phi^{(2)}\left(\begin{matrix}a; b, b' \\ c, c'\end{matrix}; x, y\right) = 0\end{aligned}$$

where $D_x^+ = x^{-1}(1-T_x)$ and $D_x^- = x^{-1}(1-T_x^{-1})$ are q–difference operators [11]. Observe that $\frac{1}{1-q}D_x^+ \to \frac{\partial}{\partial x}$ and $\frac{1}{1-q^{-1}}D_x^- \to \frac{\partial}{\partial x}$ as $q \to 1$.

Further, we have

$$\begin{aligned}qK_q^+C_q &= C_qK_q^+,\\ K_q^-C_q &= qC_qK_q^-,\\ K_q^0C_q &= C_qK_q^0.\end{aligned} \tag{43}$$

The eqs. (43) will become instrumental in obtaining identities.

5. Identities

5.1. **Based on Model IA.** We have shown in eq. (40) that

$$h_\lambda(x,y,t) = \Phi^{(2)}\left(\begin{matrix} q^\lambda; b, b' \\ c, c' \end{matrix}; x, y\right) t^\lambda, \qquad \lambda \in S, \tag{44}$$

is a solution of $C_q h_\lambda(x,y,t) = 0$ where $C_q = qK_q^+K_q^- + K_q^0K_q^0 - K_q^0$. This suggests that

$$\begin{aligned} u(x,y,t) &= \phi^{1:1;1;2}_{-:1;1;1}\left(\begin{matrix} a : b; b'; b'', b''' \\ - : c; c'; c'' \end{matrix}; x, y, t\right) t^\alpha \\ &= \sum_{n=0}^{\infty} \frac{(a, b'', b'''; q)_n}{(c'', q; q)_n} \Phi^{(2)}\left(\begin{matrix} aq^n; b, b' \\ c, c' \end{matrix}; x, y\right) t^{\alpha+n} \end{aligned} \tag{45}$$

is a solution of $C_q u(x,y,t) = 0$. Using the fact that $qK_q^+C_q = C_qK_q^+$, we have

$$C_q[e_q(sK_q^+)u](x,y,t) = 0, \tag{46}$$

where

$$\begin{aligned} [e_q(sK_q^+)u](x,y,t) &= \sum_{n=0}^{\infty} \frac{s^n}{(q;q)_n} {K_q^+}^n u(x,y,t) \\ &= \frac{(\frac{ast}{1-q};q)_\infty}{(\frac{st}{1-q};q)_\infty} \phi^{1:1;1;2}_{1:1;1;1}\left(\begin{matrix} a : b; b'; b'', b''' \\ \frac{ast}{1-q} : c; c'; c'' \end{matrix}; x, y, t\right) t^\alpha. \end{aligned} \tag{47}$$

Now we have, using the Weisner's expansion, as explained in [25, 16]

$$[e_q(sK_q^+)u](x,y,t) = \sum_{n=0}^{\infty} A_n h_{\alpha+n}(x,y,t), \tag{48}$$

where A_n is found by putting $x = y = 0$. Rescaling suitably, we eventually arrive at

$$\begin{aligned} &\frac{(at;q)_\infty}{(t;q)_\infty} \phi^{1:1;1;2}_{1:1;1;1}\left(\begin{matrix} a : b; b'; b'', b''' \\ at : c; c'; c'' \end{matrix}; x, y, \omega t\right) \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \Phi^{(2)}\left(\begin{matrix} aq^n; b, b' \\ c, c' \end{matrix}; x, y\right) {}_3\phi_1\left(\begin{matrix} q^{-n}, b'', b''' \\ c'' \end{matrix}; q, q^n\omega\right) t^n, \end{aligned} \tag{49}$$

where $\omega = \frac{s}{1-q}$.

5.2. **Based on Model IIA.** We follow exactly the same approach as given in section 5.1 with the change that we use the other q–exponential E_q instead of e_q.

We have shown that $C_q h_\lambda = 0$, where h_λ is given by eq.(39). It therefore follows that

$$u(x,y,t) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q,c'';q)_n} \Phi^{(2)}\left(\begin{matrix} a^{-1}q^{-n}; b, b' \\ c, c' \end{matrix}; aq^n x, aq^n y\right) t^{\alpha+n} \tag{50}$$

is a solution of $C_q u(x,y,t) = 0$. Considering that $K_q^- C_q = qC_qK_q^-$, we have

$$C_q[E_q(sK_q^-)u](x,y,t) = 0, \tag{51}$$

where

(52)

$$[E_q(sK_q^-)u](x,y,t) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} s^n}{(q;q)_n} K_q^{-n} u(x,y,t)$$
$$= \frac{(\frac{\omega}{t};q)_\infty}{(\frac{a\omega}{t};q)_\infty} \sum_{k,n=0}^{\infty} \frac{(a, \frac{a\omega}{t};q)_n (a^{-1}q^{-n}, b;q)_k}{(c'', q;q)_n (\frac{q^{1-n}t}{a\omega}, c, q;q)_k}$$
$$\times \ {}_2\phi_2 \left(\begin{matrix} a^{-1}q^{-n+k}, b' \\ \frac{q^{1-n+k}t}{a\omega}, c' \end{matrix} ; q, \frac{-q^{k+1}yt}{\omega} \right) q^{\binom{k}{2}} \left(\frac{-qxt}{\omega} \right)^k t^{\alpha+n}.$$

By Weisner's expansion, we have

$$[E_q(sK_q^-)u](x,y,t) = \sum_{n=-\infty}^{\infty} B_n h_{\alpha+n}(x,y,t), \tag{53}$$

where B_n is obtained by putting $x = y = 0$. This leads to

$$\begin{aligned} &\frac{(\frac{\omega}{t};q)_\infty}{(\frac{a\omega}{t};q)_\infty} \sum_{k,n=0}^{\infty} \frac{(a, \frac{a\omega}{t};q)_n (a^{-1}q^{-n}, b;q)_k}{(c'', q;q)_n (\frac{q^{1-n}t}{a\omega}, c, q;q)_k} \\ &\times \ {}_2\phi_2 \left(\begin{matrix} a^{-1}q^{-n+k}, b' \\ \frac{q^{1-n+k}t}{a\omega}, c' \end{matrix} ; q, \frac{-q^{k+1}yt}{\omega} \right) \left(\frac{-q^{\frac{k+1}{2}}xt}{\omega} \right)^k t^n \\ &= \sum_{n=-\infty}^{\infty} \frac{\Gamma_q(\alpha+n)\Gamma_q(\gamma'')}{\Gamma_q(\alpha)\Gamma_q(\gamma''+n)\Gamma_q(q+1)} \, {}_2\phi_2 \left(\begin{matrix} aq^n, aq^{n+1} \\ q^{n+1}, c''q^n \end{matrix} ; q, \omega \right) \\ &\times \Phi^{(2)} \left(\begin{matrix} a^{-1}q^{-n}; b, b' \\ c, c' \end{matrix} ; aq^n x, aq^n y \right) t^n, \end{aligned} \tag{54}$$

where $\omega = \frac{s}{1-q}$.

We have discussed above identities corresponding to models of representation $D_q(\alpha)$. We now consider the Model IIA for the representation $\uparrow_q$ (i.e. eqs. (38) and (39) with $\lambda \in S = \{n = 0, 1, 2, \dots\}$). It can be verified that

$$u(x,y,t) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \Phi^{(2)} \left(\begin{matrix} q^{-n}; b, b' \\ c, c' \end{matrix} ; q^n x, q^n y \right) t^n \tag{55}$$

satisfies $C_q u(x,y,t) = 0$. Using the relation $\frac{1}{q} K_q^- C_q = C_q K_q^-$, we have

$$C_q[E_q(sK_q^-)u](x,y,t) = 0, \tag{56}$$

where

(57)

$$[E_q(sK_q^-)u](x,y,t) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} s^n}{(q;q)_n} K_q^{-n} u(x,y,t)$$
$$= \sum_{k,n=0}^{\infty} \frac{(a, \frac{\omega}{t};q)_n (q^{-n}, b;q)_k}{(q;q)_n (\frac{q^{1-n}t}{\omega}, c, q;q)_k}$$
$$\times \ {}_2\phi_2 \left(\begin{matrix} q^{-n+k}, b' \\ \frac{q^{1-n+k}t}{\omega}, c' \end{matrix} ; q, \frac{-q^{k+1}yt}{\omega} \right) q^{\binom{k}{2}} \left(\frac{-qxt}{\omega} \right)^k t^n.$$

Using Weisner's expansion and proceeding as in the above two cases, we get the following identity:

(58)

$$\sum_{k,n=0}^{\infty} \frac{(a, \frac{\omega}{t}; q)_n (q^{-n}, b; q)_k}{(q;q)_n (\frac{q^{1-n}t}{\omega}, c, q; q)_k}\, {}_2\phi_2 \left(\begin{matrix} q^{-n+k}, b' \\ \frac{q^{1-n+k}t}{\omega}, c' \end{matrix}; q, \frac{-q^{k+1}yt}{\omega} \right) \left(\frac{-q^{\frac{k+1}{2}}xt}{\omega} \right)^k t^n$$
$$= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, \Phi^{(2)} \left(\begin{matrix} q^{-n}; b, b' \\ c, c' \end{matrix}; q^n x, q^n y \right) {}_1\phi_0 \left(\begin{matrix} aq^n \\ - \end{matrix}; \omega \right) t^n,$$

where $\omega = \frac{s}{1-q}$.

6. Conclusion

The models and identities obtained in above sections can be generalized. For this, we need the m–fold q–integral transformation, defined as

(59)
$$\begin{aligned} h(\beta_i, \gamma_i) &= \mathcal{I}[f(u_1, \ldots, u_m)] \\ &= \Gamma_q \left(\begin{matrix} \gamma_1, \ldots, \gamma_m \\ \beta_1, \ldots, \beta_m, \gamma_1 - \beta_1, \ldots, \gamma_m - \beta_m \end{matrix} \right) \\ &\times \underbrace{\int_0^1 \cdots \int_0^1}_{m-\text{ fold}} \prod_{i=1}^{m} u_i^{\beta_i - 1} \frac{(qu_1, \ldots, qu_m; q)_\infty}{(q^{\gamma_1 - \beta_1}u_1, \ldots, q^{\gamma_m - \beta_m}u_m; q)_\infty} \\ &\times f(u_1, \ldots, u_m)\, d_q u_1 \cdots d_q u_m, \end{aligned}$$

$$\operatorname{Re}\gamma_i > \operatorname{Re}\beta_i > 0, \qquad i = 1, 2, \ldots, m.$$

Putting $z = \sum_{i=1}^{m} u_i x_i$ in Model I (*i.e.* eqs. (22) and (23)), and using

(60)
$$z\Delta_z = u_1\Delta_{u_1} + u_2\Delta_{u_2}T_{u_1} + \cdots + u_m\Delta_{u_m}T_{u_1}\cdots T_{u_{m-1}},$$

(61)
$$T_z f(z) = T_{u_1}\cdots T_{u_m} f(\sum_{i=1}^{m} u_i x_i),$$

we get the transformed $(m+1)$–variable model as

(62)
$$\begin{aligned} K_q^+ &= t[(x_1\Delta_{x_1} + \cdots + x_m\Delta_{x_m}T_{x_1}\cdots T_{x_{m-1}})T_t + t\Delta_t], \\ K_q^- &= t^{-1}T_{x_1}^{-1}\cdots T_{x_m}^{-1} \\ &\quad \times \left[\left(1 - x_1 T_t \frac{1-q^{\beta_1}}{1-q^{\gamma_1}} E_{\beta_1\gamma_1} - \cdots - x_m T_t \frac{1-q^{\beta_m}}{1-q^{\gamma_m}} E_{\beta_m\gamma_m} \right) \right. \\ &\quad \left. \times \left(x_1\Delta_{x_1} + \cdots + x_m\Delta_{x_m}T_{x_1}\cdots T_{x_{m-1}} \right) - t\Delta_t \right], \\ K_q^0 &= t\Delta_t, \end{aligned}$$

with the basis functions as

(63)
$$\begin{aligned} h_\lambda &= \phi_{-:1;\ldots;1}^{1:1;\ldots;1} \left(\begin{matrix} q^\lambda : b_1; \ldots; b_m \\ - : c_1; \ldots; c_m \end{matrix}; x_1, \ldots, x_m \right) t^\lambda \\ &= \sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(q^\lambda; q)_{k_1+\cdots+k_m} (b_1;q)_{k_1}\cdots(b_m;q)_{k_m}}{(c_1,q;q)_{k_1}\cdots(c_m,q;q)_{k_m}}\, x_1^{k_1}\cdots x_m^{k_m}\, t^\lambda,\ \lambda \in S, \end{aligned}$$

where $b_i = q^{\beta_i}, c_i = q^{\gamma_i}$.

Using the same techniques as given in section 5, the above model, and similar other $(m+1)$–variable models, can be exploited for obtaining identities.

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