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MAXIMAL BENNEQUIN NUMBERS AND KAUFFMAN POLYNOMIALS OF POSITIVE LINKS

TOSHIFUMI TANAKA

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Abstract. By using results of Yamada and of Yokota, concerning link diagrams and link polynomials, we give some relationships between maximal Bennequin numbers and Kauffman polynomials of positive links.

1. Introduction

A *contact structure* in a smooth 3-dimensional manifold M is a global differential 1-form η such that $\eta \wedge (d\eta) \neq 0$ everywhere on M. A *contact distribution* is the subbundle of TM on which the contact structure vanishes. The *standard contact structure* in 3-space is the differential 1-form $dz - ydx$. A *Legendrian link* is a C^∞-embedding of disjoint circles in 3-space with the standard contact structure, which are everywhere tangent to the contact distribution. We call the composition of a Legendrian link L and the (x, z)-projection of 3-space, the *front* of L. The front of a Legendrian link may have cusps because the fibers of the contact distribution are parallel to the y-axis. Generically, the only singularities of a front are cusps and transverse double points [4]. We assume that all fronts are generic in this paper. A *Legendrian isotopy* between Legendrian links L_0 and L_1 is an ambient isotopy between L_0 and L_1, with each level Legendrian. It is known that two Legendrian links are Legendrian isotopic if and only if their fronts are related by a sequence of the Legendrian version of Reidemeister moves in Figure 1.

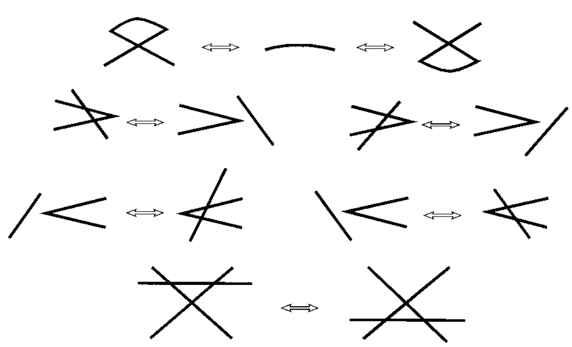

Figure 1

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The *Bennequin number* denoted by $\beta(L)$ of an oriented Legendrian link L is defined, using the (x, z)-projection by the following equation:

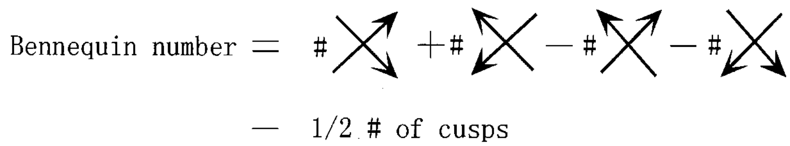

FIGURE 2

The Bennequin number is a Legendrian isotopy invariant of Legendrian links.

Let D be a diagram of an oriented link L and $\omega(D)$ be the writhe of D. The *Kauffman polynomial* $F_{(x,y)}(L) \in \mathbf{Z}[x^{\pm}, y^{\pm}]$ is defined by $a^{-\omega(D)} \bigwedge_{(x,y)}(D)$, where $\bigwedge_{(x,y)}(D)$ is a regular isotopy invariant of diagrams of L with the properties in Figure 3. The Kauffman polynomial is a topological invariant of oriented links.

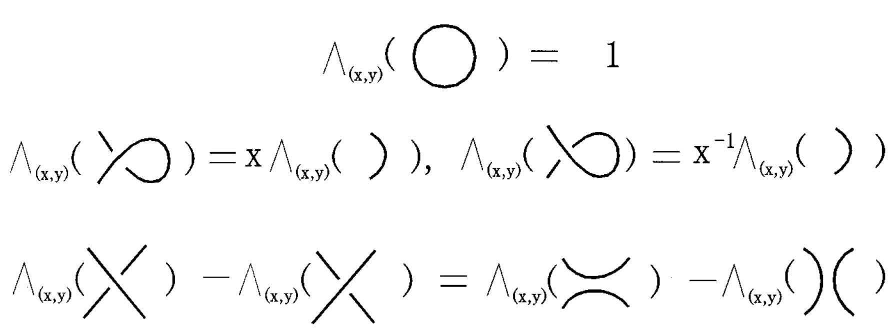

FIGURE 3

In this paper, we consider the relationship between the degree of the Kauffman polynomial of an oriented link L and the maximal value, denoted by $B(L)$, of the Bennequin numbers for Legendrian links ambient isotopic to L. An upper bound on $B(L)$ in terms of the Kauffman polynomial is given in [7], [8] as follows.

Theorem 1. *Let L be a Legendrian link. Then the following inequality holds:*

$$B(L) < -\text{max-deg}_x F_{(x,y)}(L).$$

Remark. Our proof of Theorem 1 seems different from that given in [7], [8].

By using Theorem 1 (see Figure 9), we can show that almost all the knots L of 9 or fewer crossings in the knot table of Rolfsen [1] satisfy the equality

$$B(L) + 1 = -\text{max-deg}_x F_{(x,y)}(L).$$

So we consider the following problem.

Problem. Does the above equality hold for any oriented link?

An oriented link is said to be *positive* if L has a diagram with no negative crossings. Our main result is the following:

Theorem 2. *Let L be a positive link and let D be any positive diagram of L. Then the following equality holds:*

$$B(L) + 1 = \omega(D) - S(D) + 1 = -\text{max-deg}_x F_{(x,y)}(L),$$

where $\omega(D)$ and $S(D)$ are the writhe and number of Seifert circles of D respectively.

Remark. Rudolph defined the above invariant for any knot [6]. He shows that if a knot K has non-negative Bennequin invariant, then K is not slice. He also shows that if K is a non-trivial *strongly quasipositive knot* [6], then $B(K) \geq 0$. From the fact above and Theorem 1, we know that any non-trivial strongly quasipositive knot has negative degree in x of the Kauffman polynomial.

2. Proofs

Proof of Theorem 1. Let L be a Legendrian link in $\mathbf{R}^3$. We denote the front of L by F_L. We can convert F_L into a diagram of the link with the same (topological) isotopy type by using the local deformations as in Figure 4. (Round the cusps and make the strand with the smaller slope overcross at each double point.) We denote the resultant diagram by D_L. We now flatten D_L in a small neighberhood of each crossing as in Figure 5. We denote the resultant diagram by $\widehat{D_L}$.

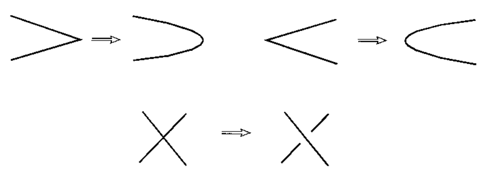

Figure 4

Figure 5

Let ψ be a height function from the (x, z)-plane to the x-axis. Note that $\widehat{D_L}$ lies in general position relative to ψ. In fact, there is no local maximal point or minimal point in $F_L - \{\text{small neighborhood of a cusp}\}$ because F_L has no vertical tangents. Thus we can assume that a point p is a local maximal (or local minimal) point in $\widehat{D_L}$ if and only if p corresponds to the cusp in F_L. Let $C(F_L)$ be the number of cusps in F_L and $b(\widehat{D_L})$ be the number of local maxima in $\widehat{D_L}$ with respect to ψ. Now we obtain that $1/2C(F_L) = b(\widehat{D_L})$.

Then by using Lemma 1 in [5], we know that

$$\text{max-deg}_x \bigwedge_{(x,y)} (D_L) \leq 1/2C(F_L) - 1.$$

So,

$$-\text{max-deg}_x F_{(x,y)}(L) = \omega(D_L) - \text{max-deg}_x \bigwedge_{(x,y)}(L)$$
$$\geq \omega(D_L) - 1/2C(F_L) + 1$$
$$= \beta(L) + 1 > \beta(L).$$

This completes the proof. □

Proof of Theorem 2. It is known that, by using so-called "*bunching deformations*" [3] as in [5] (see Figure 7) that every positive link diagram of a link L can be transformed into a positive diagram D_L of L which satisfies the following properties:

i) each crossing is oriented downward relative to ψ as in Figure 6;

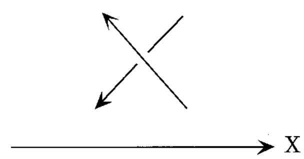

FIGURE 6

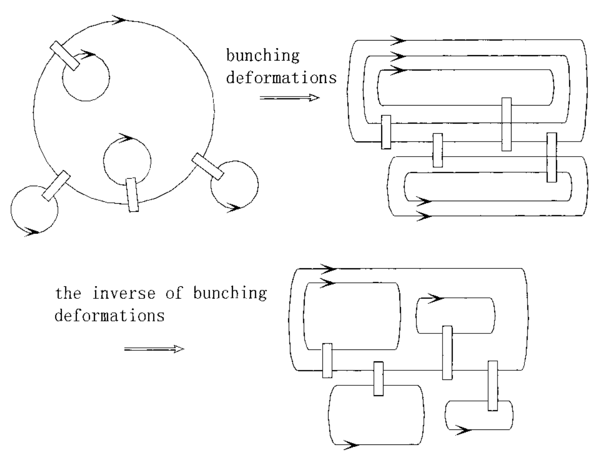

FIGURE 7

ii) each Seifert circle has exactly 1 local maximum and 1 local minimum relative to ψ.

We now construct the front D_+ of a Legendrian link L_+ which has the same topological isotopy type as in Figure 8. (Make the neighborhoods of local maxima and local minima into cusps and reinsert the crossings which connected Seifert circles.)

Let $S(D_+)$ be the number of Seifert circles in D_+. Then by using the equality concerning the Kauffman polynomial below the Main Theorem of [5], we obtain that

$$\text{max-deg}_x F_{(x,y)}(L) = -\omega(D_+) + S(D_+) - 1$$
$$= -\omega(D_+) + 1/2C(D_+) - 1 = -\beta(L_+) - 1.$$

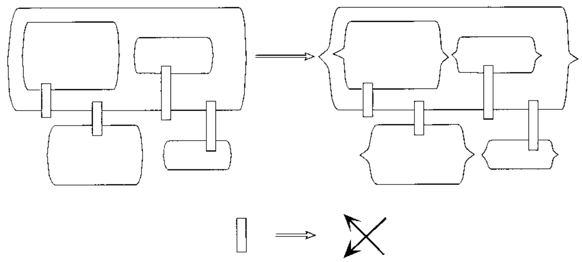

FIGURE 8

Now the result follows from Theorem 1 at once.

Let $n(K)$ be the max-degree in x of the Kauffman polynomial of a knot K. We have calculated the Bennequin invariants of the knots with 9 or fewer crossings; the results are tabulated in Figure 9. (All knots are oriented as in [2]. We know the marked knots are positive. The round bracket (x) means that the number is at least x.)

knot	B(K)	−n(K)	6_2	−1	0	7_5^*	3	4	8_4	−7	−6	8_{10}	−2	−1
3_1^*	1	2	6_3	−4	−3	7_6	−1	0	8_5	1	2	8_{11}	−1	0
4_1	−3	−2	7_1^*	5	6	7_7	−5	−4	8_6	−1	0	8_{12}	−5	−4
5_1^*	3	4	7_2	−10	−9	8_1	−7	−6	8_7	−2	−1	8_{13}	−2	−1
5_2	−8	−7	7_3^*	3	4	8_2	−11	−10	8_8	−6	−5	8_{14}	−9	−8
6_1	−5	−4	7_4	−10	−9	8_3	−5	−4	8_9	−5	−4	8_{15}^*	3	4

8_{16}	−2	−1	9_1^*	7	8	9_7^*	3	4	9_{13}^*	3	4	9_{19}	−6	−5
8_{17}	−5	−4	9_2	−12	−11	9_8	−3	−2	9_{14}	−7	−6	9_{20}	−12	−11
8_{18}	−5	−4	9_3^*	5	6	9_9^*	5	6	9_{15}	−10	−9	9_{21}	−1	0
8_{19}	(−14)	−10	9_4^*	3	4	9_{10}^*	3	4	9_{16}	−16	−15	9_{22}	−3	−2
8_{20}	−2	−1	9_5^*	1	2	9_{11}	1	2	9_{17}	−8	−7	9_{23}	−14	−13
8_{21}	(−1)	2	9_6^*	5	6	9_{12}	−1	0	9_{18}^*	3	4	9_{24}	−5	−4

9_{25}	−1	0	9_{31}	−2	−1	9_{37}	−5	−4	9_{43}	1	2	9_{49}	(−13)	−11
9_{26}	−2	−1	9_{32}	−2	−1	9_{38}^*	3	4	9_{44}	−6	−5			
9_{27}	−5	−4	9_{33}	−5	−4	9_{39}	−10	−9	9_{45}	−10	−9			
9_{28}	−9	−8	9_{34}	−6	−5	9_{40}	−2	−1	9_{46}	(−8)	−6			
9_{29}	−8	−7	9_{35}	−12	−11	9_{41}	−7	−6	9_{47}	−7	−6			
9_{30}	−5	−4	9_{36}	1	2	9_{42}	(−6)	−2	9_{48}	(−9)	−7			

FIGURE 9

Acknowledgements

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Graduate School of Mathematics, Kyushu University, Hakozaki 6-10-1, Higashiku, Fukuoka, 812-8581 Japan

E-mail address: ttanaka@math.kyushu-u.ac.jp