

A MEAN VALUE THEOREM ON BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. Let Ω be a Cartan domain of rank r and genus p and B_ν , $\nu > p-1$, the Berezin transform on Ω ; the number $B_\nu f(z)$ can be interpreted as a certain invariant-mean-value of a function f around z . We show that a Lebesgue integrable function satisfying $f = B_\nu f = B_{\nu+1} f = \cdots = B_{\nu+r} f$, $\nu \geq p$, must be \mathcal{M} -harmonic. In a sense, this result is reminiscent of Delsarte's two-radius mean-value theorem for ordinary harmonic functions on the complex n -space \mathbf{C}^n , but with the role of radius r played by the quantity $1/\nu$.

Let $\Omega = G/K$ be an irreducible bounded symmetric (Cartan) domain in its Harish-Chandra realization (i.e. a circular convex domain in \mathbf{C}^d centered at the origin), dm the Lebesgue measure on Ω normalized so that $m(\Omega) = 1$, and denote by $K(z, w)$ the Bergman kernel of Ω with respect to dm and by p, r its genus and rank, respectively. For $\nu \in \mathbf{R}$, it is known [FK1] that the integral $c(\nu)^{-1} := \int_\Omega K(z, z)^{1-\nu/p} dm(z)$ is finite if and only if $\nu > p-1$; in that case, one can consider the weighted Bergman spaces $A_\nu^2(\Omega)$ of functions analytic on Ω and square-integrable against the probability measure

$$d\mu_\nu := c(\nu)K(z, z)^{1-\nu/p} dm(z).$$

It can be shown that the point evaluations are continuous linear functionals on A_ν^2 and the corresponding reproducing kernels are [FK1]

$$(1) \quad K_\nu(z, w) = K(z, w)^{\nu/p}.$$

The Berezin transform B_ν on Ω is the integral operator defined by

$$(2) \quad \begin{aligned} B_\nu f(w) &:= \int_\Omega f(z) \frac{|K_\nu(z, w)|^2}{K_\nu(w, w)} d\mu_\nu(z) \\ &= \int_\Omega f(z) \left[\frac{|K(z, w)|^2}{K(z, z)K(w, w)} \right]^{\nu/p} \cdot c(\nu)K(z, z) dm(z). \end{aligned}$$

The integral is easily seen to converge, for instance, for any $f \in L^\infty(\Omega, dm)$ and $\nu > p-1$; we will see below that it also converges for any $f \in L^1(\Omega, d\mu_\nu)$ if $\nu > p-1$ and for any $f \in L^1(\Omega, dm)$ if $\nu \geq p$. Let ϕ be any element of the group

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$\text{Aut}(\Omega) = G$ of holomorphic automorphisms of Ω and write J_ϕ for its Jacobian. From the transformation property of the Bergman kernel

$$(3) \quad K(z, w) = J_\phi(z) \cdot K(\phi(z), \phi(w)) \cdot \overline{J_\phi(w)}$$

it follows that $(B_\nu f) \circ \phi = B_\nu(f \circ \phi)$, i.e. the operator B_ν is G -invariant. In particular, if ϕ is any holomorphic automorphism sending 0 into w , then

$$(4) \quad B_\nu f(w) = \int_\Omega f(\phi(z)) d\mu_\nu(z)$$

since $K_\nu(z, 0) \equiv 1$ owing to the circularity of Ω . Thus one can think of $B_\nu f(w)$ as a certain “mean value of f around w ”, and of functions f satisfying $B_\nu f = f$ as having a certain invariant mean-value property.

For functions on the complex space \mathbf{C}^n , it is well understood that the ordinary mean-value property is equivalent to harmonicity: if f is a harmonic function (i.e. $f \in C^\infty$ and $\Delta f = 0$), then $f(w)$ is equal to the mean value of f over any sphere or ball centered at w , for each w (Gauss’s theorem); conversely, any locally integrable function f with this property must be harmonic (Koebe). The converse part can be strengthened considerably: it is enough that f have the mean value property only over sufficiently small spheres (balls) around each w , or even only for a sequence of spheres (balls) around w whose radii tend to zero. A beautiful result of Delsarte says that it even suffices that there exist two radii $r_1, r_2 > 0$, such that f has the mean value property over the two spheres $S(w, r_1)$ and $S(w, r_2)$ (balls $B(w, r_1)$ and $B(w, r_2)$) around each w and r_1/r_2 is not a quotient of two zeroes of a certain Bessel function; see Zalcman’s paper [Za] or the recent survey of Netuka and Vesely [NV]. There is also a local version of the converse: if f is C^3 in a neighbourhood of a point w and has the mean-value property over spheres $S(w, r_n)$ (or balls $B(w, r_n)$) where $r_n \rightarrow 0$, then $\Delta f(w) = 0$.

The counterpart on bounded symmetric domains of harmonic functions on \mathbf{C}^n are the \mathcal{M} -harmonic functions (the term is due to Rudin [R]). Denote by $\text{Diff}_G(\Omega)$ the ring of all G -invariant linear differential operators on Ω . We say that a function f is \mathcal{M} -harmonic if it is annihilated by all operators in $\text{Diff}_G(\Omega)$ that annihilate the constants:

$$Lf = 0 \quad \text{if } L \in \text{Diff}_G(\Omega) \text{ and } L1 = 0.$$

Clearly f is \mathcal{M} -harmonic if and only if $f \circ \phi$ is, for any $\phi \in G$. An analogue of Gauss and Koebe theorems for \mathcal{M} -harmonic functions is due to Godement: $f \in C^\infty(\Omega)$ is \mathcal{M} -harmonic if and only if

$$(5) \quad f(\phi(0)) = \int_K f(\phi(k(z))) dk \quad \forall \phi \in G, \forall z \in \Omega = G/K,$$

where the integration is carried against the Haar probability measure on the compact subgroup K , the stabilizer of the origin in G .

From the Bruhat decomposition $G = KA^+K$ and Fubini’s theorem, it is easy to see that if f satisfies (5), then it even satisfies

$$(6) \quad f(\phi(0)) = \int_\Omega f(\phi(x)) d\lambda(x)$$

for any K -invariant probability measure λ on Ω ; in particular $f = B_\nu f$ for every ν . It follows that every \mathcal{M} -harmonic function f satisfies $B_\nu f = f$, that is, has the “invariant mean-value property” mentioned above, for all ν . A natural question

is, does the converse also hold, i.e. does $B_\nu f = f$ for some ν imply that f is \mathcal{M} -harmonic?

This problem was first considered by Fürstenberg [Fu] who proved that for $f \in L^\infty(\Omega, dm)$, the answer is “yes” (for any ν); more generally, it even suffices that f satisfies (6) for any absolutely continuous K -invariant probability measure λ on Ω and all $\phi \in G$. For the special case of the unit disc $\Omega = \mathbf{D} \subset \mathbf{C}$ ($p = 2, r = 1$) and $\nu = 2$, a “Fourier-transform” proof has been given (in oblivion of Fürstenberg’s work) in [E1], while some partial cases were also resolved by Axler and Cuckovic [AC]. For unbounded functions f , however, the answer turns out to be “no”: Ahern, Flores and Rudin [AFR] showed that for Ω the unit ball in \mathbf{C}^d and $f \in L^1(\Omega, dm)$, $B_p f = f$ implies that f is \mathcal{M} -harmonic if $d \leq 11$, but not if $d \geq 12$. Later, Arazy and Zhang [AZ] gave a proof that for any Cartan domain Ω of rank $r \geq 2$, any $q \in [1, \infty)$ and $\nu > p - 1$, there exists a spherical function f in $L^q(\Omega, dm)$ which satisfies $B_\nu f = f$ but is not \mathcal{M} -harmonic. (The Cartan domains of rank $r = 1$ are precisely the unit balls in \mathbf{C}^d , $d = 1, 2, \dots$)

In this paper we prove the following result in the positive direction, which is reminiscent of Delsarte’s two-radius theorem above but with the role of the radius r played by the quantity $1/\nu$.

Theorem 1. *Suppose that Ω is an irreducible Cartan domain of rank r and genus p , $\nu > p - 1$, and $f \in L^1(\Omega, d\mu_\nu)$ is a function satisfying $B_\nu f = B_{\nu+1} f = \dots = B_{\nu+r} f = f$. Then f is \mathcal{M} -harmonic.*

Note that $L^1(d\mu_\nu) \subset L^1(d\mu_{\nu+\alpha})$ for any $\alpha \geq 0$; in particular, $L^1(d\mu_\nu) \supset L^1(d\mu_p) = L^1(\Omega, dm)$ if $\nu \geq p$.

So, for instance, on the ball $B_\nu f = B_{\nu+1} f = f$ implies that f is \mathcal{M} -harmonic; this has already been noted by [AFR] (Remark 2.4).

We do not know whether the subscripts $\nu, \nu + 1, \dots, \nu + r$ can in fact be replaced by any $r + 1$ distinct values $\nu > p - 1$, or whether there exist some “exceptional” $(r + 1)$ -tuples like the quotients of roots of the Bessel functions in Delsarte’s theorem. However, as the case of the ball shows, the number of ν ’s cannot be reduced to r .

For completeness, we state also the following local version of the result. Again, it is reminiscent of the above-mentioned “local version” of the converse to the Gauss theorem, with a sequence of indices $\nu_j \rightarrow \infty$ in the place of the sequence of radii $r_k \rightarrow 0$.

Theorem 2. *Let w be a point in Ω and $f \in L^1(\Omega, d\mu_\alpha)$ a function which is C^∞ in a neighbourhood of w and satisfies $B_{\nu_1} f(w) = B_{\nu_2} f(w) = B_{\nu_3} f(w) = \dots$ for some sequence $\{\nu_j\}_{j=1}^\infty, \nu_j \geq \alpha, \nu_j \rightarrow \infty$. Then $Lf(w) = 0$ for any $L \in \text{Diff}_G(\Omega)$ such that $L1 = 0$.*

The proofs rely on the results from [E2] and a simple algebraic argument involving symmetric polynomials. Let us recall the salient facts from the harmonic analysis and about the Helgason-Fourier transform on Ω . Our main reference is [E2] which concentrates the necessary material from [He], [UU], [Dix], [FK1], [FK2], [AZ] and [Wa]. See also [Ka].

Employing the standard notation, let $G = NAK$ be the Iwasawa decomposition of G , \mathfrak{a} the Lie algebra of A , $\rho = (\rho_1, \dots, \rho_r) \in (\mathfrak{a}^*)^{\mathbf{C}} \simeq \mathbf{C}^r$ the half-sum of positive roots, W the Weyl group, and M the centralizer of \mathfrak{a} in K ; the coset space $B := K/M = G/MAN$ can be viewed as the boundary (in the sense of symmetric

spaces) of $\Omega = G/K$. Introduce the “plane waves”

$$(7) \quad e_{\lambda,b}(x) := e^{(-i\lambda+\rho)(A(x,b))}, \quad x \in \Omega, b \in B,$$

where $A(x, b)$ is the unique element of \mathfrak{a} satisfying, if $b = kM$ and $x = gK$,

$$k^{-1}g \in N \exp A(x, b) K$$

under the Iwasawa decomposition $G = NAK$. The Helgason-Fourier transform of $f \in C_0^\infty(\Omega)$ is a function on $\mathfrak{a}^* \times B (\simeq \mathbf{R}^r \times K/M)$ given by

$$\tilde{f}(\lambda, b) := \int_{\Omega} f(x)e_{\lambda,b}(x) d\mu(x)$$

where $d\mu(x) := K(x, x) dm(x)$ is the invariant measure on Ω . (In general, for a symmetric space $\Omega = G/K$ of noncompact type, $d\mu$ will be the image of the Haar measure on G under the projection map $g \mapsto g0$ of G onto G/K ; here and below we denote by 0 the coset $eK = K$ of the unit element e of G in G/K , which in our case of the Cartan domain coincides with the origin $0 \in \Omega \subset \mathbf{C}^d$.) For any $f \in C_0^\infty(\Omega)$ we have the Fourier inversion formula

$$f(x) = \int_{\mathfrak{a}^*} \int_B \tilde{f}(\lambda, b)e_{-\lambda,b}(x)|c(\lambda)|^{-2} db d\lambda$$

and the Plancherel theorem

$$\int_{\Omega} |f(x)|^2 d\mu(x) = \int_{\mathfrak{a}^*} \int_B |\tilde{f}(\lambda, b)|^2 |c(\lambda)|^{-2} db d\lambda.$$

Here db is the unique K -invariant probability measure on K/M and $d\lambda$ is a suitably normalized Lebesgue measure on $\mathfrak{a}^* \simeq \mathbf{R}^r$; $c(\lambda)$ is a certain meromorphic function on $(\mathfrak{a}^*)^{\mathbf{C}} \simeq \mathbf{C}^r$ (Harish-Chandra c -function). From the Plancherel theorem it can be deduced, in particular, that $f \mapsto \tilde{f}$ extends to a Hilbert space isomorphism of $L^2(\Omega, d\mu)$ into $L^2(\mathfrak{a}^* \times B, |c(\lambda)|^{-2} d\lambda db)$ whose image consists of functions $F(\lambda, b)$ which satisfy the symmetry condition $\check{F}(g, \lambda) = F(g, s\lambda)$, for all s in the Weyl group W , where

$$\check{F}(g, \lambda) := \int_B F(\lambda, g^{-1}(b))e_{\lambda,b}(g0) db, \quad g \in G, \lambda \in \mathfrak{a}^*.$$

If ϕ is an arbitrary measurable function on \mathfrak{a}^* invariant under W , we associate to it the (possibly unbounded) linear operator M_ϕ (the non-euclidean Fourier multiplier) on $L^2(\Omega, d\mu)$ given by

$$\widetilde{M_\phi f}(\lambda, b) = \phi(\lambda)\tilde{f}(\lambda, b).$$

The domain of M_ϕ consists of those functions $f \in L^2(\Omega, d\mu)$ for which

$$\int_{\mathfrak{a}^*} \int_B |\phi(\lambda)|^2 |\tilde{f}(\lambda, b)|^2 |c(\lambda)|^{-2} db d\lambda < +\infty.$$

The operator M_ϕ is always closed and normal, is bounded iff ϕ is essentially bounded, and selfadjoint (possibly unbounded) if ϕ is real-valued. The set

$$\mathfrak{Z} := \{M_\phi | \phi \in L^\infty(\mathfrak{a}^*/W)\}$$

is precisely the set of all bounded linear operators on $L^2(\Omega, d\mu)$ which are invariant under G (i.e. $T(f \circ \phi) = (Tf) \circ \phi$, for all $\phi \in G$). Now it is immediate from the

definition that B_ν is a bounded operator on $L^\infty(\Omega, d\mu)$:

$$\begin{aligned} |B_\nu f(w)| &\leq \|f\|_\infty \cdot \int_\Omega \frac{|K_\nu(z, w)|^2}{K_\nu(w, w)} d\mu_\nu(z) \\ &= \|f\|_\infty \cdot \frac{\langle K_\nu(\cdot, w), K_\nu(\cdot, w) \rangle_{A_\nu^2}}{K_\nu(w, w)} = \|f\|_\infty. \end{aligned}$$

A similar computation shows that B_ν is bounded as an operator on $L^1(\Omega, d\mu)$; by interpolation, B_ν is a bounded operator on $L^2(\Omega, d\mu)$. Moreover, in view of (2), B_ν is also selfadjoint. It thus follows from the invariance of B_ν that $B_\nu \in \mathfrak{Z}$, i.e. $B_\nu = M_{\psi_\nu}$ for some real-valued function ψ_ν in $L^\infty(\mathfrak{a}^*/W)$. The function ψ_ν has been computed explicitly [UU]:

$$(8) \quad \psi_\nu(\lambda) = \prod_{j=1}^r \frac{\Gamma(-i\lambda_j + \nu - \frac{\nu-1}{2})\Gamma(i\lambda_j + \nu - \frac{\nu-1}{2})}{\Gamma(-\rho_j + \nu - \frac{\nu-1}{2})\Gamma(\rho_j + \nu - \frac{\nu-1}{2})}.$$

For any $L \in \text{Diff}_G(\Omega)$, an invariant differential operator on Ω , it is known that the “plane waves” (7) are eigenfunctions of L :

$$Le_{-\lambda, b} = \tilde{L}(\lambda)e_{-\lambda, b}$$

where $\tilde{L}(\lambda)$ is a polynomial in r variables¹; that is, $L = M_{\tilde{L}}$. The correspondence $L \mapsto \tilde{L}$ sets up an isomorphism between the ring $\text{Diff}_G(\Omega)$ and the ring $\mathcal{P}^W(\mathbf{C}^r)$ of all polynomials on $\mathbf{C}^r \simeq \mathfrak{a}^*$ invariant under the Weyl group W ; moreover, L annihilates the constants if and only if $\tilde{L}(\lambda)$ vanishes at the point $\lambda = i\rho$. In our case of the irreducible Cartan domain Ω , the Weyl group consists simply of permutations and sign changes $((x_j) \mapsto (\varepsilon_j x_{\sigma(j)}), \varepsilon_j = \pm 1)$, so \mathcal{P}^W is just the ring of all symmetric polynomials in $\lambda_1^2, \lambda_2^2, \dots, \lambda_r^2$. By a standard fact from algebra, the elementary symmetric polynomials

$$\begin{aligned} s_0(z_1, \dots, z_r) &= 1, \\ s_1(z_1, \dots, z_r) &= z_1 + z_2 + \dots + z_r, \\ s_2(z_1, \dots, z_r) &= z_1z_2 + z_1z_3 + \dots + z_{r-1}z_r, \\ &\vdots \\ s_r(z_1, \dots, z_r) &= z_1z_2 \dots z_r \end{aligned}$$

are a set of free generators of the ring of all symmetric polynomials in variables z_1, z_2, \dots, z_r . Thus the polynomials $\tilde{D}_0, \tilde{D}_1, \dots, \tilde{D}_r$, where

$$\tilde{D}_0 := 1, \quad \tilde{D}_k := [s_k(\lambda_1^2, \dots, \lambda_r^2) - s_k(-\rho_1^2, \dots, -\rho_r^2)], \quad k = 1, 2, \dots, r,$$

form a set of generators of \mathcal{P}^W , and therefore the corresponding invariant differential operators

$$D_0 = I \text{ (the identity)}, \quad D_1, D_2, \dots, D_r,$$

($D_k = M_{\tilde{D}_k}$) form a set of generators for the ring $\text{Diff}_G(\Omega)$. Note that we have chosen D_k so that they vanish on constant functions for $k > 0$; consequently, a function f on Ω will be \mathcal{M} -harmonic if and only if it is annihilated by the operators D_1, D_2, \dots, D_r .

¹The function $\tilde{L}(\lambda)$ was denoted $\tilde{L}(i\lambda)$ in [E2].

Finally, we need also a few basics from the Jordan algebra theory of Cartan domains; our references here are [Lo] or [FK2] (condensed summaries can be found e.g. in §2 of [AZ], Chapter 1.5 of [Up], or §2 of [UU]). There can be chosen \mathbf{R} -linearly independent vectors e_1, \dots, e_r in \mathbf{C}^d (a *Jordan frame*) such that each $z \in \mathbf{C}^d$ has a *polar decomposition*

$$(9) \quad z = k \sum_{j=1}^r a_j e_j, \quad k \in K, \quad a_1 \geq a_2 \geq \dots \geq a_r \geq 0,$$

with $z \in \Omega$ iff $a_1 < 1$; the r -tuple (a_1, a_2, \dots, a_r) is uniquely determined by z (but k need not be). The *Jordan triple determinant* $h(z, w)$ is the polynomial in z, \bar{w} on $\mathbf{C}^d \times \mathbf{C}^d$ such that for $z = w$ with the polar decomposition (9),

$$h(z, z) = \prod_{j=1}^r (1 - a_j^2).$$

This condition determines $h(z, w)$ uniquely, and $h(z, w)$ is independent of the choice of the Jordan frame e_1, \dots, e_r . For any $k \in K$, $h(k(z), k(w)) = h(z, w)$ (K -invariance), and for $0 \leq a_r \leq \dots \leq a_1 < 1$, $0 \leq b_r \leq \dots \leq b_1 < 1$, $h(\sum_1^r a_j e_j, \sum_1^r b_j e_j) = \prod_1^r (1 - a_j b_j)$; in particular,

$$(10) \quad h(z, 0) = 1, \quad \text{for all } z \in \Omega.$$

In terms of $h(z, w)$, the Bergman kernel can be expressed by

$$(11) \quad K(z, w) = h(z, w)^{-p}.$$

From the transformation law (3) we thus have

$$(12) \quad h(\phi(z), \phi(w)) = J_\phi(z)^{1/p} h(z, w) \overline{J_\phi(w)^{1/p}}, \quad \phi \in G, \quad z, w \in \Omega.$$

Setting, in particular, $w = 0$ and $z = w = 0$ yields

$$(13) \quad |J_\phi(z)| = |h(\phi(z), \phi(0))|^p \cdot |h(\phi(0), \phi(0))|^{-p/2}$$

and as $h(z, w)$ is a polynomial in z, \bar{w} , it follows that

$$(14) \quad J_\phi \text{ is bounded on } \Omega, \text{ for each } \phi \in G.$$

From (13), (11) and (1), we also obtain the transformation formula for $d\mu_\nu$:

$$d\mu_\nu(\phi(z)) = |h(z, a)|^{-2\nu} h(a, a)^\nu d\mu_\nu(z), \quad \phi \in G, \quad a = \phi(0),$$

which shows that the quotients $d\mu_\nu(z)/d\mu_\nu(\phi(z))$ and (upon replacing ϕ by ϕ^{-1}) $d\mu_\nu(\phi(z))/d\mu_\nu(z)$ are bounded. In particular, $f \circ \phi \in L^1(\Omega, d\mu_\nu)$ whenever $f \in L^1(\Omega, d\mu_\nu)$, so $B_\nu f$ is well-defined by (4) for $f \in L^1(\Omega, d\mu_\nu)$ ($\supset L^1(\Omega, dm)$ if $\nu \geq p$).

Proposition 3. *For any $\nu > p - 1$, B_ν is a continuous operator from $L^1(\Omega, d\mu_\nu)$ into $C^\infty(\Omega)$ endowed with the topology of uniform convergence of all derivatives on compact subsets.*

Proof. In view of (11) the formula (2) defining B_ν can be rewritten as

$$B_\nu f(w) = \int_\Omega f(z) \frac{h(w, w)^\nu}{|h(z, w)|^{2\nu}} d\mu_\nu(z).$$

By the theorem on differentiation under the integral sign, it therefore suffices to show that for each compact set $K \subset \Omega$ and multiindices $(m_1, \dots, m_d), (n_1, \dots, n_d)$,

$$\sup_{z \in \Omega, w \in K} \left[\frac{\partial^{|m|}}{\partial w_1^{m_1} \dots \partial w_d^{m_d}} \frac{\partial^{|n|}}{\partial \bar{w}_1^{n_1} \dots \partial \bar{w}_d^{n_d}} \left(\frac{h(w, w)^\nu}{h(z, w)^\nu h(w, z)^\nu} \right) \right] < +\infty.$$

An easy induction argument shows that the derivative in the square brackets is equal to

$$h(w, w)^\nu \cdot \frac{\text{a polynomial in } h(w, w), h(z, w), h(w, z) \text{ and their derivatives}}{|h(z, w)|^{2(\nu+|n|+|m|)} \cdot h(w, w)^{|n|+|m|}}.$$

Since $h(z, w)$ is a polynomial in z and \bar{w} , it is bounded on (the closure of) $\Omega \times \Omega$, and so are its derivatives of all orders; hence, the same is true for the numerator in the last formula. It therefore suffices to show that

$$\inf_{z \in \Omega, w \in K} |h(z, w)| > 0.$$

Assume, to the contrary, that there are $z_n \in \Omega, w_n \in K$ with $h(z_n, w_n) \rightarrow 0$. Passing to a subsequence if necessary, we may assume that $w_n \rightarrow w \in K$; by continuity (recall that h is a polynomial!), $h(z_n, w) \rightarrow 0$. Pick $\phi \in G$ with $\phi(w) = 0$; from (12) and (14) it then follows that

$$h(\phi(z_n), 0) \rightarrow 0,$$

contradicting (10). This completes the proof. □

It is known that the numbers ρ_j on any Cartan domain are given by

$$\rho_j = \frac{(j-1)a + b + 1}{2}, \quad j = 1, \dots, r,$$

where the *characteristic multiplicities* a and b are nonnegative integers satisfying

$$\frac{1}{2}r(r-1)a + rb + r = d, \quad (r-1)a + b + 2 = p.$$

In particular, $\rho_j \leq \frac{p-1}{2} \forall j = 1, \dots, r$; hence, the number

$$d(\nu) := \prod_{j=1}^r \left[\left(\nu - \frac{p-1}{2} \right)^2 - \rho_j^2 \right]^{-1}$$

is well-defined and positive for all $\nu > p - 1$.

Proposition 4. *Let $P_\nu \in \text{Diff}_G(\Omega)$ be the differential operator*

$$P_\nu = I + d(\nu) \sum_{k=1}^r \left(\nu - \frac{p-1}{2} \right)^{r-k} D_k.$$

Then

$$(15) \quad P_\nu B_\nu f = B_{\nu+1} f$$

for any $\nu > p - 1$ and $f \in L^1(\Omega, d\mu_\nu)$.

Proof. Consider first $f \in L^2(\Omega, d\mu)$. Then we know that $\widetilde{B_\nu f} = \psi_\nu \widetilde{f}$ with ψ_ν given by (8). On the other hand, $\widetilde{P_\nu f} = \widetilde{P_\nu} \widetilde{f}$ where

$$\begin{aligned} \widetilde{P_\nu} &= 1 + d(\nu) \sum_{k=1}^r \left(\nu - \frac{p-1}{2}\right)^{r-k} \widetilde{D}_k \\ &= 1 + d(\nu) \sum_{k=0}^r \left(\nu - \frac{p-1}{2}\right)^{r-k} [s_k(\lambda_1^2, \dots, \lambda_r^2) - s_k(-\rho_1^2, \dots, -\rho_r^2)] \\ &= d(\nu) \prod_{j=1}^r \left[\left(\nu - \frac{p-1}{2}\right)^2 + \lambda_j^2 \right] \\ &= \frac{\psi_{\nu+1}}{\psi_\nu}, \end{aligned}$$

since $\sum_{k=0}^r z^{r-k} s_k(z_1, \dots, z_r) = \prod_{j=1}^r (z + z_j)$. Thus, $\widetilde{P_\nu B_\nu f} = \widetilde{B_{\nu+1} f}$, i.e. (15) holds for $f \in L^2(\Omega, d\mu)$.

For $z \in \Omega$, let $g_z \in G$ be the geodesic symmetry interchanging z and 0 . For any invariant differential operator $L \in \text{Diff}_G(\Omega)$, we can write

$$Lf(z) = (Lf)(g_z(0)) = (L(f \circ g_z))(0).$$

As the mapping $z \mapsto g_z$ from Ω into G is C^∞ -smooth, it follows that any such L is a differential operator with C^∞ coefficients, and, hence, is a continuous linear map from $C^\infty(\Omega)$ into itself. In particular, this is true for P_ν , and since $L^2(\Omega, d\mu)$ is dense in $L^1(\Omega, d\mu_\nu)$, the validity of (15) for all f in the latter space follows by the preceding proposition. □

Proof of Theorem 1. By Proposition 4, $B_\nu f = B_{\nu+1} f = f$ implies that $P_\nu f = f$; similarly for $\nu + 1, \dots, \nu + r - 1$ in the place of ν . Thus

$$f \in \text{Ker}(P_{\nu+k} - I), \quad k = 0, 1, \dots, r - 1,$$

or, by the definition of P_ν (recall that $d(\nu) > 0$ for all $\nu!$),

$$\sum_{j=1}^r \left(\nu + k - \frac{p-1}{2}\right)^{r-j} D_j f(x) = 0, \quad k = 0, 1, \dots, r - 1, \quad x \in \Omega.$$

Thus for each $x \in \Omega$, the polynomial

$$Q_x(t) := \sum_{j=1}^r \left(t + \nu - \frac{p-1}{2}\right)^{r-j} D_j f(x)$$

of degree $\leq r - 1$ has r distinct roots $t = 0, 1, \dots, r - 1$. It follows that Q_x vanishes identically; that is, $D_j f(x) = 0$ for all $x \in \Omega$ and $j = 1, 2, \dots, r$. Thus f is \mathcal{M} -harmonic. □

Proof of Theorem 2. By the standard Laplace method (see e.g. [F], Theorem II.4.1), if the integral (2) defining $B_\nu f(w)$ converges absolutely for some $\nu = \nu_0$, then it converges also for all $\nu > \nu_0$ and as $\nu \rightarrow +\infty$,

$$(16) \quad B_\nu f(w) \asymp \sum_{k=0}^\infty Q_k f(w) \cdot \nu^{-k}$$

for any function f sufficiently smooth at w . Here Q_k are certain differential operators which depend only on the kernel $K(z, w)$, i.e. only on the domain Ω . Owing

to the G -invariance of B_ν , the operators Q_k are also G -invariant. The general formulas for Q_k in [F] are quite involved; fortunately, it is possible to describe the operators Q_k fairly explicitly using the Helgason-Fourier transform and the Unterberger-Upmeyer formula (8). In particular, it was shown in [E2] that

$$Q_1, Q_3, \dots, Q_{2r-1} \quad \text{and} \quad Q_0 = I$$

are free generators of the ring $\text{Diff}_G(\Omega)$. Now from $B_{\nu_1}f(w) = B_{\nu_2}f(w) = \dots$, $\nu_k \rightarrow +\infty$, it follows that

$$Q_k f(w) = 0 \quad \text{for all } k > 1.$$

On the other hand, setting $f = 1$ in (16) shows that $Q_k 1 = 0 \forall k > 1$, i.e. Q_1, Q_2, \dots annihilate the constants. Thus $Qf(w) = 0$ whenever $Q \in \text{Diff}_G(\Omega)$ and $Q1 = 0$, which is what we wanted to prove. \square

Remark. By (16) we may write $\lim_{\nu \rightarrow \infty} B_\nu =: B_\infty = I$; the hypothesis of Theorem 1 can thus be rewritten, formally, as

$$B_\nu f = B_{\nu+1} f = \dots = B_{\nu+r} f = B_\infty f.$$

This makes it conceivable that perhaps even

$$B_{\nu_1} f = B_{\nu_2} f = \dots = B_{\nu_{r+1}} f = B_{\nu_{r+2}} f \implies f \text{ is } \mathcal{M}\text{-harmonic}$$

for any $f \in L^1(\Omega, d\mu_\alpha)$ and mutually distinct parameters $\nu_1, \nu_2, \dots, \nu_{r+2} \geq \alpha$. However, we are not going to pursue this idea any further.

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