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A MEAN VALUE THEOREM ON BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. Let Ω be a Cartan domain of rank r and genus p and B_ν, $\nu > p-1$, the Berezin transform on Ω; the number $B_\nu f(z)$ can be interpreted as a certain invariant-mean-value of a function f around z. We show that a Lebesgue integrable function satisfying $f = B_\nu f = B_{\nu+1} f = \cdots = B_{\nu+r} f$, $\nu \geq p$, must be $\mathcal{M}$-harmonic. In a sense, this result is reminiscent of Delsarte's two-radius mean-value theorem for ordinary harmonic functions on the complex n-space $\mathbf{C}^n$, but with the role of radius r played by the quantity $1/\nu$.

Let $\Omega = G/K$ be an irreducible bounded symmetric (Cartan) domain in its Harish-Chandra realization (i.e. a circular convex domain in $\mathbf{C}^d$ centered at the origin), dm the Lebesgue measure on Ω normalized so that $m(\Omega) = 1$, and denote by $K(z,w)$ the Bergman kernel of Ω with respect to dm and by p, r its genus and rank, respectively. For $\nu \in \mathbf{R}$, it is known [FK1] that the integral $c(\nu)^{-1} := \int_\Omega K(z,z)^{1-\nu/p}\, dm(z)$ is finite if and only if $\nu > p-1$; in that case, one can consider the weighted Bergman spaces $A^2_\nu(\Omega)$ of functions analytic on Ω and square-integrable against the probability measure

$$d\mu_\nu := c(\nu) K(z,z)^{1-\nu/p}\, dm(z).$$

It can be shown that the point evaluations are continuous linear functionals on A^2_ν and the corresponding reproducing kernels are [FK1]

$$K_\nu(z,w) = K(z,w)^{\nu/p}. \tag{1}$$

The Berezin transform B_ν on Ω is the integral operator defined by

$$\begin{aligned} B_\nu f(w) :&= \int_\Omega f(z) \frac{|K_\nu(z,w)|^2}{K_\nu(w,w)}\, d\mu_\nu(z) \\ &= \int_\Omega f(z) \left[\frac{|K(z,w)|^2}{K(z,z)K(w,w)} \right]^{\nu/p} \cdot c(\nu) K(z,z)\, dm(z). \end{aligned} \tag{2}$$

The integral is easily seen to converge, for instance, for any $f \in L^\infty(\Omega, dm)$ and $\nu > p-1$; we will see below that it also converges for any $f \in L^1(\Omega, d\mu_\nu)$ if $\nu > p-1$ and for any $f \in L^1(\Omega, dm)$ if $\nu \geq p$. Let ϕ be any element of the group

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$\mathrm{Aut}(\Omega) = G$ of holomorphic automorphisms of Ω and write J_ϕ for its Jacobian. From the transformation property of the Bergman kernel

$$K(z,w) = J_\phi(z) \cdot K(\phi(z), \phi(w)) \cdot \overline{J_\phi(w)} \tag{3}$$

it follows that $(B_\nu f) \circ \phi = B_\nu(f \circ \phi)$, i.e. the operator B_ν is G-invariant. In particular, if ϕ is any holomorphic automorphism sending 0 into w, then

$$B_\nu f(w) = \int_\Omega f(\phi(z))\, d\mu_\nu(z) \tag{4}$$

since $K_\nu(z,0) \equiv 1$ owing to the circularity of Ω. Thus one can think of $B_\nu f(w)$ as a certain "mean value of f around w", and of functions f satisfying $B_\nu f = f$ as having a certain invariant mean-value property.

For functions on the complex space $\mathbf{C}^n$, it is well understood that the ordinary mean-value property is equivalent to harmonicity: if f is a harmonic function (i.e. $f \in C^\infty$ and $\Delta f = 0$), then $f(w)$ is equal to the mean value of f over any sphere or ball centered at w, for each w (Gauss's theorem); conversely, any locally integrable function f with this property must be harmonic (Koebe). The converse part can be strengthened considerably: it is enough that f have the mean value property only over sufficiently small spheres (balls) around each w, or even only for a sequence of spheres (balls) around w whose radii tend to zero. A beautiful result of Delsarte says that it even suffices that there exist two radii $r_1, r_2 > 0$, such that f has the mean value property over the two spheres $S(w, r_1)$ and $S(w, r_2)$ (balls $B(w, r_1)$ and $B(w, r_2)$) around each w and r_1/r_2 is not a quotient of two zeroes of a certain Bessel function; see Zalcman's paper [Za] or the recent survey of Netuka and Vesely [NV]. There is also a local version of the converse: if f is C^3 in a neighbourhood of a point w and has the mean-value property over spheres $S(w, r_n)$ (or balls $B(w, r_n)$) where $r_n \to 0$, then $\Delta f(w) = 0$.

The counterpart on bounded symmetric domains of harmonic functions on $\mathbf{C}^n$ are the $\mathcal{M}$-harmonic functions (the term is due to Rudin [R]). Denote by $\mathrm{Diff}_G(\Omega)$ the ring of all G-invariant linear differential operators on Ω. We say that a function f is $\mathcal{M}$-harmonic if it is annihilated by all operators in $\mathrm{Diff}_G(\Omega)$ that annihilate the constants:

$$Lf = 0 \qquad \text{if } L \in \mathrm{Diff}_G(\Omega) \text{ and } L1 = 0.$$

Clearly f is $\mathcal{M}$-harmonic if and only if $f \circ \phi$ is, for any $\phi \in G$. An analogue of Gauss and Koebe theorems for $\mathcal{M}$-harmonic functions is due to Godement: $f \in C^\infty(\Omega)$ is $\mathcal{M}$-harmonic if and only if

$$f(\phi(0)) = \int_K f(\phi(k(z)))\, dk \qquad \forall \phi \in G,\ \forall z \in \Omega = G/K, \tag{5}$$

where the integration is carried against the Haar probability measure on the compact subgroup K, the stabilizer of the origin in G.

From the Bruhat decomposition $G = KA^+K$ and Fubini's theorem, it is easy to see that if f satisfies (5), then it even satisfies

$$f(\phi(0)) = \int_\Omega f(\phi(x))\, d\lambda(x) \tag{6}$$

for any K-invariant probability measure λ on Ω; in particular $f = B_\nu f$ for every ν. It follows that every $\mathcal{M}$-harmonic function f satisfies $B_\nu f = f$, that is, has the "invariant mean-value property" mentioned above, for all ν. A natural question

is, does the converse also hold, i.e. does $B_\nu f = f$ for some ν imply that f is $\mathcal{M}$-harmonic?

This problem was first considered by Fürstenberg [Fu] who proved that for $f \in L^\infty(\Omega, dm)$, the answer is "yes" (for any ν); more generally, it even suffices that f satisfies (6) for any absolutely continuous K-invariant probability measure λ on Ω and all $\phi \in G$. For the special case of the unit disc $\Omega = \mathbf{D} \subset \mathbf{C}$ ($p = 2, r = 1$) and $\nu = 2$, a "Fourier-transform" proof has been given (in oblivion of Fürstenberg's work) in [E1], while some partial cases were also resolved by Axler and Cuckovic [AC]. For unbounded functions f, however, the answer turns out to be "no": Ahern, Flores and Rudin [AFR] showed that for Ω the unit ball in $\mathbf{C}^d$ and $f \in L^1(\Omega, dm)$, $B_p f = f$ implies that f is $\mathcal{M}$-harmonic if $d \leq 11$, but not if $d \geq 12$. Later, Arazy and Zhang [AZ] gave a proof that for any Cartan domain Ω of rank $r \geq 2$, any $q \in [1, \infty)$ and $\nu > p - 1$, there exists a spherical function f in $L^q(\Omega, dm)$ which satisfies $B_\nu f = f$ but is not $\mathcal{M}$-harmonic. (The Cartan domains of rank $r = 1$ are precisely the unit balls in $\mathbf{C}^d$, $d = 1, 2, \ldots$.)

In this paper we prove the following result in the positive direction, which is reminiscent of Delsarte's two-radius theorem above but with the role of the radius r played by the quantity $1/\nu$.

Theorem 1. *Suppose that Ω is an irreducible Cartan domain of rank r and genus p, $\nu > p - 1$, and $f \in L^1(\Omega, d\mu_\nu)$ is a function satisfying $B_\nu f = B_{\nu+1} f = \cdots = B_{\nu+r} f = f$. Then f is $\mathcal{M}$-harmonic.*

Note that $L^1(d\mu_\nu) \subset L^1(d\mu_{\nu+\alpha})$ for any $\alpha \geq 0$; in particular, $L^1(d\mu_\nu) \supset L^1(d\mu_p) = L^1(\Omega, dm)$ if $\nu \geq p$.

So, for instance, on the ball $B_\nu f = B_{\nu+1} f = f$ implies that f is $\mathcal{M}$-harmonic; this has already been noted by [AFR] (Remark 2.4).

We do not know whether the subscripts $\nu, \nu+1, \ldots, \nu+r$ can in fact be replaced by any $r+1$ distinct values $\nu > p - 1$, or whether there exist some "exceptional" $(r+1)$-tuples like the quotients of roots of the Bessel functions in Delsarte's theorem. However, as the case of the ball shows, the number of ν's cannot be reduced to r.

For completeness, we state also the following local version of the result. Again, it is reminiscent of the above-mentioned "local version" of the converse to the Gauss theorem, with a sequence of indices $\nu_j \to \infty$ in the place of the sequence of radii $r_k \to 0$.

Theorem 2. *Let w be a point in Ω and $f \in L^1(\Omega, d\mu_\alpha)$ a function which is C^∞ in a neighbourhood of w and satisfies $B_{\nu_1} f(w) = B_{\nu_2} f(w) = B_{\nu_3} f(w) = \ldots$ for some sequence $\{\nu_j\}_{j=1}^\infty, \nu_j \geq \alpha, \nu_j \to \infty$. Then $Lf(w) = 0$ for any $L \in \mathrm{Diff}_G(\Omega)$ such that $L1 = 0$.*

The proofs rely on the results from [E2] and a simple algebraic argument involving symmetric polynomials. Let us recall the salient facts from the harmonic analysis and about the Helgason-Fourier transform on Ω. Our main reference is [E2] which concentrates the necessary material from [He], [UU], [Dix], [FK1], [FK2], [AZ] and [Wa]. See also [Ka].

Employing the standard notation, let $G = NAK$ be the Iwasawa decomposition of G, $\mathfrak{a}$ the Lie algebra of A, $\rho = (\rho_1, \ldots, \rho_r) \in (\mathfrak{a}^*)^{\mathbf{C}} \simeq \mathbf{C}^r$ the half-sum of positive roots, W the Weyl group, and M the centralizer of $\mathfrak{a}$ in K; the coset space $B := K/M = G/MAN$ can be viewed as the boundary (in the sense of symmetric

spaces) of $\Omega = G/K$. Introduce the "plane waves"

$$e_{\lambda,b}(x) := e^{(-i\lambda+\rho)(A(x,b))}, \qquad x \in \Omega, b \in B, \tag{7}$$

where $A(x,b)$ is the unique element of $\mathfrak{a}$ satisfying, if $b = kM$ and $x = gK$,

$$k^{-1}g \in N \exp A(x,b)\, K$$

under the Iwasawa decomposition $G = NAK$. The Helgason-Fourier transform of $f \in C_0^\infty(\Omega)$ is a function on $\mathfrak{a}^* \times B$ ($\simeq \mathbf{R}^r \times K/M$) given by

$$\tilde{f}(\lambda, b) := \int_\Omega f(x) e_{\lambda,b}(x)\, d\mu(x)$$

where $d\mu(x) := K(x,x)\, dm(x)$ is the invariant measure on Ω. (In general, for a symmetric space $\Omega = G/K$ of noncompact type, $d\mu$ will be the image of the Haar measure on G under the projection map $g \mapsto g0$ of G onto G/K; here and below we denote by 0 the coset $eK = K$ of the unit element e of G in G/K, which in our case of the Cartan domain coincides with the origin $0 \in \Omega \subset \mathbf{C}^d$.) For any $f \in C_0^\infty(\Omega)$ we have the Fourier inversion formula

$$f(x) = \int_{\mathfrak{a}^*} \int_B \tilde{f}(\lambda, b) e_{-\lambda,b}(x) |c(\lambda)|^{-2}\, db\, d\lambda$$

and the Plancherel theorem

$$\int_\Omega |f(x)|^2\, d\mu(x) = \int_{\mathfrak{a}^*} \int_B |\tilde{f}(\lambda, b)|^2 |c(\lambda)|^{-2}\, db\, d\lambda.$$

Here db is the unique K-invariant probability measure on K/M and $d\lambda$ is a suitably normalized Lebesgue measure on $\mathfrak{a}^* \simeq \mathbf{R}^r$; $c(\lambda)$ is a certain meromorphic function on $(\mathfrak{a}^*)^{\mathbf{C}} \simeq \mathbf{C}^r$ (Harish-Chandra c-function). From the Plancherel theorem it can be deduced, in particular, that $f \mapsto \tilde{f}$ extends to a Hilbert space isomorphism of $L^2(\Omega, d\mu)$ into $L^2(\mathfrak{a}^* \times B, |c(\lambda)|^{-2} d\lambda\, db)$ whose image consists of functions $F(\lambda, b)$ which satisfy the symmetry condition $\check{F}(g, \lambda) = \check{F}(g, s\lambda)$, for all s in the Weyl group W, where

$$\check{F}(g, \lambda) := \int_B F(\lambda, g^{-1}(b)) e_{\lambda,b}(g0)\, db, \qquad g \in G, \lambda \in \mathfrak{a}^*.$$

If ϕ is an arbitrary measurable function on $\mathfrak{a}^*$ invariant under W, we associate to it the (possibly unbounded) linear operator M_ϕ (the non-euclidean Fourier multiplier) on $L^2(\Omega, d\mu)$ given by

$$\widetilde{M_\phi f}(\lambda, b) = \phi(\lambda) \tilde{f}(\lambda, b).$$

The domain of M_ϕ consists of those functions $f \in L^2(\Omega, d\mu)$ for which

$$\int_{\mathfrak{a}^*} \int_B |\phi(\lambda)|^2 |\tilde{f}(\lambda, b)|^2 |c(\lambda)|^{-2}\, db\, d\lambda < +\infty.$$

The operator M_ϕ is always closed and normal, is bounded iff ϕ is essentially bounded, and selfadjoint (possibly unbounded) if ϕ is real-valued. The set

$$\mathfrak{Z} := \{M_\phi |\ \phi \in L^\infty(\mathfrak{a}^*/W)\}$$

is precisely the set of all bounded linear operators on $L^2(\Omega, d\mu)$ which are invariant under G (i.e. $T(f \circ \phi) = (Tf) \circ \phi$, for all $\phi \in G$). Now it is immediate from the

definition that B_ν is a bounded operator on $L^\infty(\Omega, d\mu)$:

$$\begin{aligned}|B_\nu f(w)| &\le \|f\|_\infty \cdot \int_\Omega \frac{|K_\nu(z,w)|^2}{K_\nu(w,w)}\, d\mu_\nu(z)\\ &= \|f\|_\infty \cdot \frac{\langle K_\nu(\cdot,w), K_\nu(\cdot,w)\rangle_{A^2_\nu}}{K_\nu(w,w)} = \|f\|_\infty.\end{aligned}$$

A similar computation shows that B_ν is bounded as an operator on $L^1(\Omega, d\mu)$; by interpolation, B_ν is a bounded operator on $L^2(\Omega, d\mu)$. Moreover, in view of (2), B_ν is also selfadjoint. It thus follows from the invariance of B_ν that $B_\nu \in \mathfrak{Z}$, i.e. $B_\nu = M_{\psi_\nu}$ for some real-valued function ψ_ν in $L^\infty(\mathfrak{a}^*/W)$. The function ψ_ν has been computed explicitly [UU]:

$$\psi_\nu(\lambda) = \prod_{j=1}^r \frac{\Gamma(-i\lambda_j + \nu - \frac{p-1}{2})\Gamma(i\lambda_j + \nu - \frac{p-1}{2})}{\Gamma(-\rho_j + \nu - \frac{p-1}{2})\Gamma(\rho_j + \nu - \frac{p-1}{2})}. \tag{8}$$

For any $L \in \mathrm{Diff}_G(\Omega)$, an invariant differential operator on Ω, it is known that the "plane waves" (7) are eigenfunctions of L:

$$Le_{-\lambda,b} = \tilde{L}(\lambda)e_{-\lambda,b}$$

where $\tilde{L}(\lambda)$ is a polynomial in r variables[1]; that is, $L = M_{\tilde{L}}$. The correspondence $L \mapsto \tilde{L}$ sets up an isomorphism between the ring $\mathrm{Diff}_G(\Omega)$ and the ring $\mathcal{P}^W(\mathbf{C}^r)$ of all polynomials on $\mathbf{C}^r \simeq \mathfrak{a}^*$ invariant under the Weyl group W; moreover, L annihilates the constants if and only if $\tilde{L}(\lambda)$ vanishes at the point $\lambda = i\rho$. In our case of the irreducible Cartan domain Ω, the Weyl group consists simply of permutations and sign changes ($(x_j) \mapsto (\varepsilon_j x_{\sigma(j)})$, $\varepsilon_j = \pm 1$), so $\mathcal{P}^W$ is just the ring of all symmetric polynomials in $\lambda_1^2, \lambda_2^2, \dots, \lambda_r^2$. By a standard fact from algebra, the elementary symmetric polynomials

$$\begin{aligned}s_0(z_1,\dots,z_r) &= 1,\\ s_1(z_1,\dots,z_r) &= z_1 + z_2 + \cdots + z_r,\\ s_2(z_1,\dots,z_r) &= z_1z_2 + z_1z_3 + \cdots + z_{r-1}z_r,\\ &\vdots\\ s_r(z_1,\dots,z_r) &= z_1z_2\dots z_r\end{aligned}$$

are a set of free generators of the ring of all symmetric polynomials in variables $z_1, z_2, \dots, z_r$. Thus the polynomials $\tilde{D}_0, \tilde{D}_1, \dots, \tilde{D}_r$, where

$$\tilde{D}_0 := 1, \qquad \tilde{D}_k := [s_k(\lambda_1^2,\dots,\lambda_r^2) - s_k(-\rho_1^2,\dots,-\rho_r^2)], \qquad k = 1,2,\dots,r,$$

form a set of generators of $\mathcal{P}^W$, and therefore the corresponding invariant differential operators

$$D_0 = I \text{ (the identity)}, \qquad D_1, D_2, \dots, D_r,$$

($D_k = M_{\tilde{D}_k}$) form a set of generators for the ring $\mathrm{Diff}_G(\Omega)$. Note that we have chosen D_k so that they vanish on constant functions for $k > 0$; consequently, a function f on Ω will be $\mathcal{M}$-harmonic if and only if it is annihilated by the operators $D_1, D_2, \dots, D_r$.

[1]The function $\tilde{L}(\lambda)$ was denoted $\tilde{L}(i\lambda)$ in [E2].

Finally, we need also a few basics from the Jordan algebra theory of Cartan domains; our references here are [Lo] or [FK2] (condensed summaries can be found e.g. in §2 of [AZ], Chapter 1.5 of [Up], or §2 of [UU]). There can be chosen $\mathbf{R}$-linearly independent vectors $e_1, \dots, e_r$ in $\mathbf{C}^d$ (a *Jordan frame*) such that each $z \in \mathbf{C}^d$ has a *polar decomposition*

$$z = k \sum_{j=1}^{r} a_j e_j, \qquad k \in K,\ a_1 \geq a_2 \geq \cdots \geq a_r \geq 0, \tag{9}$$

with $z \in \Omega$ iff $a_1 < 1$; the r-tuple $(a_1, a_2, \dots, a_r)$ is uniquely determined by z (but k need not be). The *Jordan triple determinant* $h(z, w)$ is the polynomial in $z, \overline{w}$ on $\mathbf{C}^d \times \mathbf{C}^d$ such that for $z = w$ with the polar decomposition (9),

$$h(z, z) = \prod_{j=1}^{r} (1 - a_j^2).$$

This condition determines $h(z, w)$ uniquely, and $h(z, w)$ is independent of the choice of the Jordan frame $e_1, \dots, e_r$. For any $k \in K$, $h(k(z), k(w)) = h(z, w)$ (K-invariance), and for $0 \leq a_r \leq \cdots \leq a_1 < 1$, $0 \leq b_r \leq \cdots \leq b_1 < 1$, $h(\sum_1^r a_j e_j, \sum_1^r b_j e_j) = \prod_1^r (1 - a_j b_j)$; in particular,

$$h(z, 0) = 1, \qquad \text{for all } z \in \Omega. \tag{10}$$

In terms of $h(z, w)$, the Bergman kernel can be expressed by

$$K(z, w) = h(z, w)^{-p}. \tag{11}$$

From the transformation law (3) we thus have

$$h(\phi(z), \phi(w)) = J_\phi(z)^{1/p} h(z, w) \overline{J_\phi(w)^{1/p}}, \qquad \phi \in G,\ z, w \in \Omega. \tag{12}$$

Setting, in particular, $w = 0$ and $z = w = 0$ yields

$$|J_\phi(z)| = |h(\phi(z), \phi(0))|^p \cdot |h(\phi(0), \phi(0))|^{-p/2} \tag{13}$$

and as $h(z, w)$ is a polynomial in $z, \overline{w}$, it follows that

$$J_\phi \text{ is bounded on } \Omega, \text{ for each } \phi \in G. \tag{14}$$

From (13), (11) and (1), we also obtain the transformation formula for $d\mu_\nu$:

$$d\mu_\nu(\phi(z)) = |h(z, a)|^{-2\nu} h(a, a)^\nu \, d\mu_\nu(z), \qquad \phi \in G,\ a = \phi(0),$$

which shows that the quotients $d\mu_\nu(z)/d\mu_\nu(\phi(z))$ and (upon replacing ϕ by ϕ^{-1}) $d\mu_\nu(\phi(z))/d\mu_\nu(z)$ are bounded. In particular, $f \circ \phi \in L^1(\Omega, d\mu_\nu)$ whenever $f \in L^1(\Omega, d\mu_\nu)$, so $B_\nu f$ is well-defined by (4) for $f \in L^1(\Omega, d\mu_\nu)$ ($\supset L^1(\Omega, dm)$ if $\nu \geq p$).

Proposition 3. *For any $\nu > p - 1$, B_ν is a continuous operator from $L^1(\Omega, d\mu_\nu)$ into $C^\infty(\Omega)$ endowed with the topology of uniform convergence of all derivatives on compact subsets.*

Proof. In view of (11) the formula (2) defining B_ν can be rewritten as

$$B_\nu f(w) = \int_\Omega f(z) \frac{h(w, w)^\nu}{|h(z, w)|^{2\nu}} \, d\mu_\nu(z).$$

By the theorem on differentiation under the integral sign, it therefore suffices to show that for each compact set $K \subset \Omega$ and multiindices $(m_1, \dots, m_d), (n_1, \dots, n_d)$,

$$\sup_{z\in\Omega, w\in K} \left[\frac{\partial^{|m|}}{\partial w_1^{m_1} \dots \partial w_d^{m_d}} \frac{\partial^{|n|}}{\partial \bar{w}_1^{n_1} \dots \partial \bar{w}_d^{n_d}} \left(\frac{h(w,w)^\nu}{h(z,w)^\nu h(w,z)^\nu} \right) \right] < +\infty.$$

An easy induction argument shows that the derivative in the square brackets is equal to

$$h(w,w)^\nu \cdot \frac{\text{a polynomial in } h(w,w), h(z,w), h(w,z) \text{ and their derivatives}}{|h(z,w)|^{2(\nu+|n|+|m|)} \cdot h(w,w)^{|n|+|m|}}.$$

Since $h(z,w)$ is a polynomial in z and $\overline{w}$, it is bounded on (the closure of) $\Omega \times \Omega$, and so are its derivatives of all orders; hence, the same is true for the numerator in the last formula. It therefore suffices to show that

$$\inf_{z\in\Omega, w\in K} |h(z,w)| > 0.$$

Assume, to the contrary, that there are $z_n \in \Omega, w_n \in K$ with $h(z_n, w_n) \to 0$. Passing to a subsequence if necessary, we may assume that $w_n \to w \in K$; by continuity (recall that h is a polynomial!), $h(z_n, w) \to 0$. Pick $\phi \in G$ with $\phi(w) = 0$; from (12) and (14) it then follows that

$$h(\phi(z_n), 0) \to 0,$$

contradicting (10). This completes the proof. □

It is known that the numbers ρ_j on any Cartan domain are given by

$$\rho_j = \frac{(j-1)a + b + 1}{2}, \qquad j = 1, \dots, r,$$

where the *characteristic multiplicities* a and b are nonnegative integers satisfying

$$\frac{1}{2} r(r-1)a + rb + r = d, \qquad (r-1)a + b + 2 = p.$$

In particular, $\rho_j \le \frac{p-1}{2}$ $\forall j = 1, \dots, r$; hence, the number

$$d(\nu) := \prod_{j=1}^{r} \left[\left(\nu - \frac{p-1}{2} \right)^2 - \rho_j^2 \right]^{-1}$$

is well-defined and positive for all $\nu > p - 1$.

Proposition 4. *Let $P_\nu \in \mathrm{Diff}_G(\Omega)$ be the differential operator*

$$P_\nu = I + d(\nu) \sum_{k=1}^{r} \left(\nu - \frac{p-1}{2} \right)^{r-k} D_k.$$

Then

$$P_\nu B_\nu f = B_{\nu+1} f \tag{15}$$

for any $\nu > p - 1$ and $f \in L^1(\Omega, d\mu_\nu)$.

Proof. Consider first $f \in L^2(\Omega, d\mu)$. Then we know that $\widetilde{B_\nu f} = \psi_\nu \tilde{f}$ with ψ_ν given by (8). On the other hand, $\widetilde{P_\nu f} = \widetilde{P_\nu}\tilde{f}$ where

$$\begin{aligned}\widetilde{P_\nu} &= 1 + d(\nu)\sum_{k=1}^{r}\Big(\nu - \frac{p-1}{2}\Big)^{r-k}\tilde{D}_k \\ &= 1 + d(\nu)\sum_{k=0}^{r}\Big(\nu - \frac{p-1}{2}\Big)^{r-k}[s_k(\lambda_1^2,\dots,\lambda_r^2) - s_k(-\rho_1^2,\dots,-\rho_r^2)] \\ &= d(\nu)\prod_{j=1}^{r}\Big[\Big(\nu - \frac{p-1}{2}\Big)^2 + \lambda_j^2\Big] \\ &= \frac{\psi_{\nu+1}}{\psi_\nu},\end{aligned}$$

since $\sum_{k=0}^{r} z^{r-k}s_k(z_1,\dots,z_r) = \prod_{j=1}^{r}(z+z_j)$. Thus, $\widetilde{P_\nu B_\nu f} = \widetilde{B_{\nu+1}f}$, i.e. (15) holds for $f \in L^2(\Omega, d\mu)$.

For $z \in \Omega$, let $g_z \in G$ be the geodesic symmetry interchanging z and 0. For any invariant differential operator $L \in \mathrm{Diff}_G(\Omega)$, we can write

$$Lf(z) = (Lf)(g_z(0)) = (L(f \circ g_z))(0).$$

As the mapping $z \mapsto g_z$ from Ω into G is C^∞-smooth, it follows that any such L is a differential operator with C^∞ coefficients, and, hence, is a continuous linear map from $C^\infty(\Omega)$ into itself. In particular, this is true for P_ν, and since $L^2(\Omega, d\mu)$ is dense in $L^1(\Omega, d\mu_\nu)$, the validity of (15) for all f in the latter space follows by the preceding proposition. □

Proof of Theorem 1. By Proposition 4, $B_\nu f = B_{\nu+1}f = f$ implies that $P_\nu f = f$; similarly for $\nu+1,\dots,\nu+r-1$ in the place of ν. Thus

$$f \in \mathrm{Ker}(P_{\nu+k} - I), \qquad k = 0,1,\dots,r-1,$$

or, by the definition of P_ν (recall that $d(\nu) > 0$ for all ν!),

$$\sum_{j=1}^{r}\Big(\nu + k - \frac{p-1}{2}\Big)^{r-j} D_j f(x) = 0, \qquad k = 0,1,\dots,r-1,\ x \in \Omega.$$

Thus for each $x \in \Omega$, the polynomial

$$Q_x(t) := \sum_{j=1}^{r}\Big(t + \nu - \frac{p-1}{2}\Big)^{r-j} D_j f(x)$$

of degree $\le r-1$ has r distinct roots $t = 0,1,\dots,r-1$. It follows that Q_x vanishes identically; that is, $D_j f(x) = 0$ for all $x \in \Omega$ and $j = 1,2,\dots,r$. Thus f is $\mathcal{M}$-harmonic. □

Proof of Theorem 2. By the standard Laplace method (see e.g. [F], Theorem II.4.1), if the integral (2) defining $B_\nu f(w)$ converges absolutely for some $\nu = \nu_0$, then it converges also for all $\nu > \nu_0$ and as $\nu \to +\infty$,

$$B_\nu f(w) \asymp \sum_{k=0}^{\infty} Q_k f(w) \cdot \nu^{-k} \tag{16}$$

for any function f sufficiently smooth at w. Here Q_k are certain differential operators which depend only on the kernel $K(z,w)$, i.e. only on the domain Ω. Owing

to the G-invariance of B_ν, the operators Q_k are also G-invariant. The general formulas for Q_k in [F] are quite involved; fortunately, it is possible to describe the operators Q_k fairly explicitly using the Helgason-Fourier transform and the Unterberger-Upmeier formula (8). In particular, it was shown in [E2] that

$$Q_1, Q_3, \ldots, Q_{2r-1} \quad \text{and} \quad Q_0 = I$$

are free generators of the ring $\mathrm{Diff}_G(\Omega)$. Now from $B_{\nu_1} f(w) = B_{\nu_2} f(w) = \ldots$, $\nu_k \to +\infty$, it follows that

$$Q_k f(w) = 0 \qquad \text{for all } k > 1.$$

On the other hand, setting $f = 1$ in (16) shows that $Q_k 1 = 0\ \forall k > 1$, i.e. $Q_1, Q_2, \ldots$ annihilate the constants. Thus $Qf(w) = 0$ whenever $Q \in \mathrm{Diff}_G(\Omega)$ and $Q1 = 0$, which is what we wanted to prove. □

Remark. By (16) we may write $\lim_{\nu\to\infty} B_\nu =: B_\infty = I$; the hypothesis of Theorem 1 can thus be rewritten, formally, as

$$B_\nu f = B_{\nu+1} f = \cdots = B_{\nu+r} f = B_\infty f.$$

This makes it conceivable that perhaps even

$$B_{\nu_1} f = B_{\nu_2} f = \cdots = B_{\nu_{r+1}} f = B_{\nu_{r+2}} f \implies f \text{ is } \mathcal{M}\text{-harmonic}$$

for any $f \in L^1(\Omega, d\mu_\alpha)$ and mutually distinct parameters $\nu_1, \nu_2, \ldots, \nu_{r+2} \geq \alpha$. However, we are not going to pursue this idea any further.

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