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THE NECESSARY AND SUFFICIENT CONDITIONS FOR THE GLOBAL STABILITY OF TYPE-K LOTKA–VOLTERRA SYSTEM

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ABSTRACT. This paper provides necessary and sufficient conditions for the type-K Lotka-Volterra system to have a globally asymptotically stable positive steady state. The generalization of such a result is given.

1. INTRODUCTION

Consider the Lotka-Volterra system

$$\dot{x} = \operatorname{diag}(x)(r + Mx), \quad x \in R^n_+, \quad r \in R^n, \tag{S}$$

where

$$M = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}, \tag{T}$$

A is a $k \times k$ matrix with nonnegative off-diagonal elements, D is an $(n-k)\times(n-k)$ matrix with the same property and $B \geq 0, C \geq 0$. We call such matrix M a type-K matrix and system (S) a type-K monotone system. There are many papers about the global behavior of monotone systems (S). Some interesting results are due to Takeuchi and Adachi. Let $N = \{1, 2, \ldots, n\}$ and let Q be a subset of N. In [1], Takeuchi and Adachi proved that if M is stable, then (S) has a unique nonnegative steady state $\bar{x}$ with $\bar{x}_q = 0$ for $q \in Q$ and $\bar{x}_r > 0$ for $r \in N\backslash Q$, which attracts all solutions with initial conditions in $\{x \in R^n_+ : x_r > 0 \quad \text{for} \quad r \in N\backslash Q\}$. They also proved that $\bar{x}$ is globally asymptotically stable relative to $\{x \in R^n_+ : x_r > 0 \quad \text{for} \quad r \in N\backslash Q\}$ if and only if

$$r_q + \sum_{j=1}^{n} m_{q_j}\bar{x}_j \leq 0 \quad \text{for all} \quad q \in Q.$$

Concerning system (S), as pointed out by Smith [2, p. 872], one of the most interesting problems is whether or not the groups $I = \{1, 2, \ldots, k\}$ and $J = \{k+1, \ldots, n\}$ can coexist. The results of Hirsch [3], [4] suggest that the coexistence must take the form of a positive steady state which should be asymptotically stable. Therefore, providing conditions to guarantee that (S) has a globally asymptotically stable steady state is of great interest. Smith [2, Theorem 4.1] gave one set of

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sufficient conditions for the existence of such a steady state which can only be applied to nonobligate systems ($r > 0$). The present authors generalized Smith's result to obligate systems in [5].

The goal of the present paper is to provide the necessary and sufficient conditions for (S) to have a globally asymptotically stable steady state in the positive orthant. Obviously, the essential condition for (S) to have a globally asymptotically stable steady state is that M is stable. Actually, in this paper, we shall consider the more general system

$$(\mathrm{S}^*) \qquad \dot{x} = \mathrm{diag}(x) f(x), \quad x \in R^n_+,$$

where $Df(x)$ is a type-K matrix as in (T). We call such system (S^*) a general type-K monotone system. Under the suitable condition

$$(\mathrm{C}) \qquad Df(x) \leq_K M, \qquad \text{for all} \quad x \in R^n_+,$$

where M is a type-K matrix as in (T) and stable, we shall present necessary and sufficient conditions for (S^*) to have a globally asymptotically stable steady state. " $\leq_K$ " will be defined in the next section. Let $s(M) = \max\ \mathrm{Re}\ \lambda$, where λ runs through the eigenvalues of M. M is stable, that is, $s(M) < 0$ if and only if the principal minors of M^+ alternate in sign as follows:

$$(-1)^k \begin{vmatrix} a^+_{11} & \dots & a^+_{1k} \\ \vdots & & \vdots \\ a^+_{k1} & \dots & a^+_{kk} \end{vmatrix} > 0, \quad 1 \leq k \leq n,$$

where $a^+_{ij} = |a_{ij}|$ for $i \neq j$ and $a^+_{ii} = a_{ii}$.

2. The main result

In this section, we will agree on some notation and establish some conventions. Then, the main result will be stated.

Let $R^n_+ = \{x \in R^n : x_i \geq 0, 1 \leq i \leq n\}$ denote the nonnegative orthant and $\mathrm{Int}\, R^n_+ = \{x \in R^n_+ : x_i > 0, 1 \leq i \leq n\}$ denote its interior. $x > 0$ means $x_i > 0$ for all i.

In this paper, K is a proper cone in R^n which is a nonempty closed convex subset of R^n with the property $K \cap (-K) = \{0\}$. If the partial order relation is generated by a cone K, we write $x \leq_K y$ whenever $y - x \in K$. Then $K = R^k_+ \times (-R^{n-k}_+)$ is a cone. Let $x^1 \in R^k, x^2 \in R^{n-k}$; it is convenient to write $x = (x^1, x^2) \in R^n$. Then $x \leq_K y$, where $x = (x^1, x^2)$, $y = (y^1, y^2)$ implies $x^1 \leq y^1, x^2 \geq y^2$. For two $n \times n$ type-K matrices M_1 and M_2, $M_1 \geq_K M_2$ if and only if $A_1 \geq A_2$, $B_1 \geq B_2$, $C_1 \geq C_2$, $D_1 \geq D_2$.

Let $N = \{1, 2, \cdots, n\}, I = \{1, 2, \cdots, k\}$ and $J = \{k+1, \cdots, n\}$, where n is the dimension of our Euclidean space R^n. If L, P are nonempty sets of N such that $L \supset I$ and $P \supset J$, then $\bar{L} = N \backslash L$ and $\bar{P} = N \backslash P$ denote their complementary sets in N. u will always represent vectors in $\sharp L$-dimensional Euclidean space $R^{\sharp L}_+$, v in $\sharp\bar{L}$-dimensional Euclidean space $R^{\sharp\bar{L}}_+$, w in $R^{\sharp P}_+$ and z in $R^{\sharp\bar{P}}_+$. $\sharp L$ represents the cardinality of L. $\sharp\bar{L}$, $\sharp P$ and $\sharp\bar{P}$ have the same meanings respectively.

Without loss of generality, we may assume $L = \{1, 2, \cdots, k, k+1, \cdots, l\}$, $k < l < n$, and $P = \{n-p+1, \cdots, k+1, \cdots n\}$, $p > n-k$. Let $x = (u, v)$ and

$f(x) = (f_L(u,v), f_{\bar{L}}(u,v))$. Then we can rewrite the system (S*) as

$$(\mathrm{S}_1) \qquad \begin{cases} \dot{u} = \mathrm{diag}(u) f_L(u,v), \\ \dot{v} = \mathrm{diag}(v) f_{\bar{L}}(u,v), \end{cases} \qquad u \in R_+^{\sharp L}, \quad v \in R_+^{\sharp \bar{L}}.$$

Similarly, let $x = (z,w)$ and $f(x) = (f_{\bar{P}}(z,w), f_P(z,w))$. Then we can rewrite the system (S*) as

$$(\mathrm{S}_2) \qquad \begin{cases} \dot{z} = \mathrm{diag}(z) f_{\bar{P}}(z,w), \\ \dot{w} = \mathrm{diag}(w) f_P(z,w), \end{cases} \qquad z \in R_+^{\sharp \bar{P}}, \quad w \in R_+^{\sharp P}.$$

Setting $v = 0$ and $z = 0$ in (S$_1$) and (S$_2$) respectively, we obtain two subsystems

$$(\mathrm{S}_L) \qquad \dot{u} = \mathrm{diag}(u) f_L(u,0), \qquad u \in R_+^{\sharp L},$$

and

$$(\mathrm{S}_P) \qquad \dot{w} = \mathrm{diag}(w) f_P(0,w), \qquad w \in R_+^{\sharp P}.$$

Because $Df(u,v)$ and $Df(z,w)$ are type-K matrices, $Df_L(u,0)$ is a type-K_1 submatrix of $Df(u,0)$ and $Df_P(0,w)$ is a type-K_2 submatrix of $Df(0,w)$, where $K_1 = R_+^k \times (-R_+^{l-k})$ and $K_2 = R_+^{p+k-n} \times (-R_+^{n-k})$.

We write $\phi_t(x)$ for the unique solution $x(t)$ of (S*) satisfying $x(0) = x$; $\phi_t^L(u)$, $\phi_t^P(w)$ are the unique solutions $u(t)$ of (S$_L$) and $w(t)$ of (S$_P$) respectively, satisfying $u(0) = u$ and $w(0) = w$. $(\phi_t^L(u))_I$ consists of components $\{\phi_t^L(u)\}_i$ of $\phi_t^L(u)$ for all $i \in I$, and $(\phi_t^P(w))_J$ is defined similarly.

Similar to the result of Takeuchi and Adachi, we have proved the following theorem [7, Theorem 4.1] for the type-K system (S*).

Theorem 2.1. *Assume that the system* (S*) *is type-K monotone and the condition* (C) *holds. If there exists a nonnegative steady state $c \in R_+^n$ with $c_Q > 0$, $c_{\bar{Q}} = 0$ where we agree on $c = 0$ if $Q = \varnothing$, then c attracts all solutions with initial conditions in $\{x \in R_+^n : x_Q > 0\}$ if and only if $f(c) \le 0$.*

Our main result is as follows.

Theorem A. *Assume that the condition* (C) *holds. Then the type-K monotone system* (S*) *has a unique positive steady state which is globally asymptotically stable relative to* $\mathrm{Int}\, R_+^n$ *if and only if* (S*) *satisfies the following conditions:*

(1) *There exists $L \subset N$ with $L \supset I$ such that* (S$_L$) *has a positive steady state u^0 and $f_{\bar{L}}(u^0, 0) > 0$.*

(2) *There exists $P \subset N$ with $P \supset J$ such that* (S$_P$) *has a positive steady state w^0 and $f_{\bar{P}}(0, w^0) > 0$.*

3. The proof of the main result

It is essential to establish some preliminary results before giving the proof of Theorem A.

Theorem 3.1 (Kamke Theorem)**.** *Assume that the system* (S*) *is type-K monotone, and $x(t), y(t)$ are the solutions of* (S*) *defined on $a \le t \le b$ with $x(a) \le_K y(a)$. Then $x(t) \le_K y(t)$ for all $t \in [a,b]$.*

Theorem 3.2. *Let the system* (S*) *be a type-K monotone system and let $f(x) \ge_K 0$ for some $x \in R_+^n$. Then $\{\phi_t(x)\}_i$ is nondecreasing if $i \in I$ and $\{\phi_t(x)\}_j$ is nonincreasing if $j \in J$ for all $t \ge 0$ for which the solution exists. A similar result holds if $f(x) \le_K 0$.*

Theorem 3.1 is extended in a natural way from cooperative systems to type-K monotone systems (see [2, Theorem 2.4]). Theorem 3.2 is a criterion for the monotonicity of every component of a solution which is generalized from a cooperative system given by Selgrade [6] to a type-K monotone system ([2, p.862]).

The proof of Theorem A (Necessity). Since the condition (C) holds, every solution of the type-K monotone system (S^*) is bounded by Proposition 4.4 in [7]. Let $p = (p_1, p_2)$ be a positive steady state of (S^*) which is globally asymptotically stable in $\mathrm{Int}\, R^n_+$. Then the subsystems (S_I) and (S_J) have positive steady states x^0_1 and x^0_2, respectively, and $x^0_1 \geq p_1$, $x^0_2 \geq p_2$ by Proposition 3.3 in [2].

We claim that $J_1 = \{j \in J : f_j(x^0_1, 0) > 0\} \neq \varnothing$. Otherwise, $f_J(x^0_1, 0) \leq 0$, which implies $f(x^0_1, 0) \leq 0$. Then $(x^0_1, 0)$ and p are both globally asymptotically stable relative to $\mathrm{Int}\, R^n_+$ by Theorem 2.1 and the assumption. This is a contradiction and proves our claim.

Set $I_1 = I \cup J_1$. We consider the subsystem

$$(\mathrm{S}_{I_1}) \qquad \dot{x}_{I_1} = \mathrm{diag}(x_{I_1}) f_{I_1}(x_{I_1}, 0).$$

Obviously, $f_{I_1}(x^0_1, 0) \leq_{K_1} 0$ where $K_1 = R^k_+ \times (-R^{\# J_1}_+)$. Then there exists sufficiently small $v \in \mathrm{Int}\, R^{\# J_1}_+$ such that $f_I(x^0_1, v, 0) \leq 0$ because of $\partial f_I / \partial x_J \leq 0$ and $f_{J_1}(x^0_1, v, 0) > 0$ because of the continuity of f. It follows that $\phi^{I_1}_t(x^0_1, v)$ is type-K_1 nonincreasing from Theorem 3.2. Choose a small v such that $(x^0_1, v) \geq_{K_1} (p_I, p_{J_1})$. Then $\phi^{I_1}_t(x^0_1, v) \geq_{K_1} \phi^{I_1}_t(p_{I_1})$ by Theorem 3.1, that is, $x^0_1 \geq \{\phi^{I_1}_t(x^0_1, v)\}_I \geq \{\phi^{I_1}_t(p_{I_1})\}_I$ and $v \leq \{\phi^{I_1}_t(x^0_1, v)\}_{J_1} \leq \{\phi^{I_1}_t(p_{I_1})\}_{J_1}$ for all $t > 0$.

It is easy to prove that $f_{I_1}(p_{I_1}, 0) \geq_{K_1} 0$. In fact, $f_I(p_{I_1}, 0) \geq f_I(p_{I_1}, p_{\bar{I}_1}) = 0$ because of $\partial f_I / \partial x_J \leq 0$ and $f_{J_1}(p_{I_1}, 0) \leq f_{J_1}(p_{I_1}, p_{\bar{I}_1}) = 0$ because of $\partial f_{J_1} / \partial x_{\bar{I}_1} \geq 0$. Then $\phi^{I_1}_t(p_{I_1})$ is type-K_1 nondecreasing by Theorem 3.2. Then $\{\phi^{I_1}_t(p_{I_1})\}_I \geq p_I$ and $\{\phi^{I_1}_t(p_{I_1})\}_{J_1} \leq p_{J_1}$ for $t > 0$. Thus we have $x^0_1 \geq \{\phi^{I_1}_t(x^0_1, v)\}_I \geq p_I$ and $v \leq \{\phi^{I_1}_t(x^0_1, v)\}_{J_1} \leq p_{J_1}$. It follows from the type-K_1 increase of $\phi^{I_1}_t(x^0_1, v)$ that

$$\lim_{t \to +\infty} \phi^{I_1}_t(x^0_1, v) = u_{I_1}.$$

Clearly, $u_{I_1} > 0$, that is, (S_{I_1}) has a positive steady state u_{I_1}.

We claim that $f_{\bar{I}_1}(u_{I_1}, 0) \leq 0$ is impossible. If not, then $(u_{I_1}, 0)$ is globally asymptotically stable by Theorem 2.1, contradicting the assumption. Thus, there exists $J_2 = \{j \in \bar{I}_1 : f_j(u_{I_1}, 0) > 0\} \neq \varnothing$. If $J_2 = \bar{I}_1$, then we choose $L = I_1$. Otherwise, setting $I_2 = I_1 \cup J_2$, we consider the subsystem

$$(\mathrm{S}_{I_2}) \qquad \dot{x}_{I_2} = \mathrm{diag}(x_{I_2}) f_{I_2}(x_{I_2}, 0).$$

Since $f_j(u_{I_1}, 0_{J_2}, 0) > 0$ for $j \in J_2$ and $f_j(u_{I_1}, 0_{J_2}, 0) = 0$ for $j \in I_1$, we have $f_{I_2}(u_{I_1}, 0_{J_2}, 0) \leq_{K_2} 0$, where $K_2 = R^k_+ \times (-R^{\#(J_1 \cup J_2)}_+)$. It is easy to see that $f_j(p_{I_1}, p_{J_2}, 0) \leq f_j(p_{I_1}, p_{J_2}, p_{\bar{I}_2}) = 0$ for $j \in J_1 \cup J_2$ because of $\partial f_j / \partial x_{\bar{I}_2} \geq 0$ for $j \in J_1 \cup J_2$ and $f_j(p_{I_1}, p_{J_2}, 0) \geq f_j(p_{I_1}, p_{J_2}, p_{\bar{I}_2}) = 0$ for $j \in I$. Then $f_{I_2}(p_{I_1}, p_{J_2}, 0) \geq_{K_2} 0$. The same reasoning can conclude that (S_{I_2}) has a positive steady state u_{I_2}.

Similarly, $f_{\bar{I}_2}(u_{I_2}, 0) \leq 0$ cannot hold. Then there exists $J_3 = \{j \in \bar{I}_2 : f_j(u_{I_2}, 0) > 0\} \neq \varnothing$. This process must stop at at most the $(n-k-1)$th-step. We assume the process stops at the mth-step. Let $L = I \cup J_1 \cup \cdots \cup J_m$. Then the subsystem (S_L) has a positive steady state u^0 such that $f_{\bar{L}}(u^0, 0) > 0$. Clearly, $L \supset I$.

The necessity of (2) can be proved by using the same method.

(Sufficiency) In fact, the main idea to prove the sufficiency is taken from [5], [7].

First, we prove that if (S^*) has a positive steady state $\bar{x}$, then $\bar{x}$ is globally asymptotically stable in $\operatorname{Int} R_+^n$.

Since $\bar{x}$ is a positive steady state, for any $x \in R_+^n$, we have

$$f(x) - f(\bar{x}) = \left[\int_0^1 Df(sx + (1-s)\bar{x})ds\right](x - \bar{x}).$$

The condition (C) shows that

$$\int_0^1 Df(sx + (1-s)\bar{x})ds \leq_K M,$$

which implies that

$$\operatorname{diag}(x)f(x) \leq_K \operatorname{diag}(x)M(x - \bar{x}) \quad \text{for} \quad x \geq_K \bar{x}, \quad x \geq 0,$$

and

$$\operatorname{diag}(x)f(x) \geq_K \operatorname{diag}(x)M(x - \bar{x}) \quad \text{for} \quad x \leq_K \bar{x}, \quad x \geq 0.$$

Make a Lotka–Volterra system

$$\text{(S)} \qquad \dot{x} = \operatorname{diag}(x)(r + Mx), \quad x \in R_+^n \quad \text{and} \quad r \in R^n,$$

where $r = -M\bar{x}$. Obviously, $\bar{x}$ is a positive steady state for (S) and attracts all points in $\operatorname{Int} R_+^n$ by the result of Takeuchi and Adachi [1] mentioned in the introduction. Hence it follows that

$$\lim_{t\to+\infty} \phi_t(x) = \bar{x} \quad \text{for} \quad x \in \operatorname{Int} R_+^n \quad \text{with} \quad x \geq_K \bar{x} \quad \text{or} \quad x \leq_K \bar{x}$$

from standard differential inequality arguments.

For any $x \in \operatorname{Int} R_+^n$, there exist $y \geq_K \bar{x}$ and $y > 0$, $z \leq_K \bar{x}$ and $z > 0$ such that $z \leq_K x \leq_K y$. Applying Theorem 3.1, we have

$$\phi_t(z) \leq_K \phi_t(x) \leq_K \phi_t(y) \quad \text{for} \quad t \geq 0.$$

Because

$$\lim_{t\to+\infty} \phi_t(z) = \lim_{t\to+\infty} \phi_t(y) = \bar{x},$$

we have established that

$$\lim_{t\to+\infty} \phi_t(x) = \bar{x} \quad \text{for all} \quad x \in \operatorname{Int} R_+^n.$$

Since $Df(\bar{x}) \leq_K M$, we have $s(Df(\bar{x})) \leq s(M) < 0$ from Perron-Frobenius theory (see [2, Thm.2.2]) which implies that $\bar{x}$ is asymptotically stable. Thus $\bar{x}$ is globally asymptotically stable.

Next, we prove that $(0, w^0) \leq_K (u^0, 0)$.

Since u^0 and w^0 are positive steady states of (S_L) and (S_P) respectively, we deduce that

$$\lim_{t\to+\infty} \phi_t^L(u) = u^0 \quad \text{for} \quad u \in \operatorname{Int} R_+^{\#L},$$

and

$$\lim_{t\to+\infty} \phi_t^P(w) = w^0 \quad \text{for} \quad w \in \operatorname{Int} R_+^{\#P}$$

from the above result. It is easy to choose $u \in \operatorname{Int} R_+^{\#L}$ and $w \in \operatorname{Int} R_+^{\#P}$ such that $(0, w) \leq_K (u, 0)$. Then

$$(0, \phi_t^P(w)) \leq_K (\phi_t^L(u), 0) \qquad \text{for} \quad t \geq 0$$

by Theorem 3.1. Therefore, we conclude that

$$(0, w^0) \leq_K (u^0, 0).$$

Finally, we prove that (S*) has a unique positive steady state which is globally asymptotically stable in $\operatorname{Int} R_+^n$.

Since $(0, w^0) \leq_K (u^0, 0)$, we have $(0, w^0) \leq_K (z, w^0) \leq_K (u^0, 0)$ for sufficiently small $z > 0$. From condition (2), for above z, it is easy to obtain that $f(z, w^0) \geq_K 0$ from the continuity and type-K monotonicity of f. Then $\phi_t(z, w^0)$ is type-K nondecreasing for $t \geq 0$ by Theorem 3.2. We deduce that $(0, w^0) \leq_K \phi_t(z, w^0) \leq_K (u^0, 0)$ for all $t \geq 0$ from Theorem 3.1. Hence, $\phi_t(z, w^0)$ is bounded. Consequently, $\phi_t(z, w^0)$ converges to some point $\bar{x}$ as $t \longrightarrow +\infty$ and $(0, w^0) \leq_K \bar{x} \leq_K (u^0, 0)$. It is clear that $\bar{x}_I \geq \{(z, w^0)\}_I > 0$. Similarly, from the condition (1), for sufficiently small v, we have $\phi_t(u^0, v) \longrightarrow \tilde{x}$ as $t \longrightarrow +\infty$, $(0, w^0) \leq_K \tilde{x} \leq_K (u^0, 0)$ and

$$\tilde{x}_J \geq \{(u^0, v)\}_J > 0.$$

Choose $z > 0, v > 0$ to be small enough such that $(z, w^0) \leq_K (u^0, v)$. Then $\phi_t(z, w^0) \leq_K \phi_t(u^0, v)$ for $t > 0$. So $\bar{x} \leq_K \tilde{x}$, namely, $(\bar{x}_I, \bar{x}_J) \leq_K (\tilde{x}_I, \tilde{x}_J)$. This means

$$\tilde{x}_I \geq \bar{x}_I > 0, \quad \bar{x}_J \geq \tilde{x}_J > 0.$$

Hence, we conclude that $\bar{x} > 0$ and $\tilde{x} > 0$. From the first paragraph of the proof of sufficiency, it follows that $\bar{x} = \tilde{x}$ which is globally asymptotically stable in $\operatorname{Int} R_+^n$. The proof is completed. □

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