

SUM OF CANTOR SETS: SELF-SIMILARITY AND MEASURE

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ABSTRACT. In this note it is shown that the sum of two homogeneous Cantor sets is often a uniformly contracting self-similar set and it is given a sufficient condition for such a set to be of Lebesgue measure zero (in fact, of Hausdorff dimension less than one and positive Hausdorff measure at this dimension).

1. DEFINITIONS AND RESULTS

The study of the arithmetic difference (sum) of two Cantor sets has been of great interest for the homoclinic bifurcations of dynamical systems, since the last decade. For a detailed presentation of this subject we refer the reader to [7]. About 1987, J. Palis made the following

Conjecture. *If the sum of two affine Cantor sets has positive Lebesgue measure, then it contains an interval.*

Many articles have been written on this conjecture; see [4], [5], [6] and [8].

Another subject which has been of great interest since the work of B. Mandelbrot ([3]) is the study of fractal dimensions, introduced by Hausdorff in the twenties. The most useful source of examples in this context is given by self-similar sets. Several efforts have been made ([1], [2], [9]) to solve the following

Question. *Is it true that if a self-similar set has positive Lebesgue measure, then it contains an interval?*

Our first result (Theorem A below) gives an indication that the conjecture and the question above are related to each other. We are going to state it precisely.

Let $f_i : x \in \mathbb{R} \mapsto c_i x + a_i \in \mathbb{R}$, $0 < c_i < 1$, $i = 1, \dots, r$, be affine contractions of the real line. There is a unique compact and nonempty set $A \subset \mathbb{R}$ such that $A = f_1(A) \cup \dots \cup f_r(A)$ ([2]). This set is called a *self-similar set*, and it is called a *uniformly contracting self-similar set* whenever $c_1 = \dots = c_r = c$.

Let $\mathcal{H}(2r + 1)$ be the set of all vectors $\vec{\lambda} = (\lambda, \lambda'_1, \lambda, \dots, \lambda'_r, \lambda) \in \mathbb{R}^{2r+1}$, such that $\lambda > 0$, $\lambda'_i > 0$, $i = 1, \dots, r$; and $\lambda + \lambda'_1 + \lambda + \dots + \lambda'_r + \lambda = 1$. Let $C(\vec{\lambda})$ be the self-similar set obtained, as above, using the contractions $f_i(x) = \lambda x + a_i$, with $a_0 = 0$, $a_i = \lambda + \lambda'_1 + \lambda + \dots + \lambda'_i$, $i = 1, \dots, r$. The set $C(\vec{\lambda})$ is called a *homogeneous Cantor set*. We have the following

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Theorem A. *There is a dense set $\mathcal{D} \subset \mathcal{H}(2r+1) \times \mathcal{H}(2s+1)$, such that if $(\vec{\lambda}, \vec{\gamma}) \in \mathcal{D}$, then $C(\vec{\lambda}) + C(\vec{\gamma}) = \{x + y : x \in C(\vec{\lambda}), y \in C(\vec{\gamma})\}$ is a uniformly contracting self-similar set.*

We also obtain a sufficient condition for a uniformly contracting self-similar set to be of Lebesgue measure zero. Let A be a uniformly contracting self-similar set, as above. We may assume, without loss of generality (see next section for explanation), that $0 = a_1 < a_2 < \dots < a_r = 1 - c < 1$, where c is the uniform contracting ratio. We consider the set \mathcal{A} of analytic functions f given by $f(x) = d_0 + d_1x + d_2x^2 + \dots$, where $d_k = a_i - a_j$, for some $i, j = 1, \dots, r$, for all $k = 0, 1, 2, \dots$, and $d_0 \neq 0$. Our second result is the following

Theorem B. *If $f(c) \neq 0$ for all $f \in \mathcal{A}$, then the uniformly contracting self-similar set A has Hausdorff dimension less than one and positive Hausdorff measure at this dimension. In particular, A has Lebesgue measure zero.*

Finally, we believe that Theorem A is true in the more general context of all affine Cantor sets.

2. PRELIMINARIES

Let $g_i : \mathbb{R} \rightarrow \mathbb{R}$ be given by $g_i(x) = cx + b_i$, $i = 1, \dots, r$, with $b_1 < \dots < b_r$. Consider the affine bijection $S : \mathbb{R} \rightarrow \mathbb{R}$ given by $S(x) = \frac{b_r - b_1}{1 - c}x + \frac{b_1}{1 - c}$. It is easy to verify that $f_i = S^{-1} \circ g_i \circ S$ is given by $f_i(x) = cx + a_i$, with $a_i = (1 - c)\frac{b_i - b_1}{b_r - b_1}$. Then, $0 = a_1 < a_2 < \dots < a_r = 1 - c$. The existence and uniqueness of nonempty compact sets B and A such that $B = g_1(B) \cup \dots \cup g_r(B)$ and $A = f_1(A) \cup \dots \cup f_r(A)$, was proved in ([2]). By our construction, it is clear that $S(A) = B$. Thus, we can work with A instead of B , without loss of generality.

Now, we are going to describe a useful construction of A . Let $A_0 = [0, 1]$. It is easy to see that $(f_{i_n} \circ \dots \circ f_{i_1})(A_0)$ is an interval of length c^n . The union of all such intervals is denoted by A_n and is called *n-step of the construction of A*. Moreover, $(f_{i_n} \circ \dots \circ f_{i_1})(A_0)$ is called *an interval of A_n* . Then, we have that $A = \bigcap_{n=0}^{\infty} A_n$.

Using this construction, we can characterize an element $a \in A$ by the existence of a sequence a_{i_k} , with $a_{i_k} \in \{a_1, \dots, a_r\}$, such that $a = \sum_{k=0}^{\infty} a_{i_k}c^k$. The reciprocal of this fact will be useful in the next section. It can be stated as follows. Let $\{b_1, \dots, b_r\}$ be a given finite set and let $c \in (0, 1)$. The set B of the numbers $\sum_{k=0}^{\infty} b_{i_k}c^k$, with $b_{i_k} \in \{b_1, \dots, b_r\}$, is the unique nonempty compact set satisfying the equation $B = g_1(B) \cup \dots \cup g_r(B)$, with g_i as above (see [2]). This fact will be used in the proof of Theorem A.

Another important tool we are going to use in the next section is the *strong open set condition* SOSC. We say that the self-similar set A above satisfies the SOSC if there is an open set $V \supset A$, such that $f_i(V) \cap f_j(V) = \emptyset$, whenever $i \neq j$, and $f_i(V) \subset V$, for all $i = 1, \dots, r$. An important result of ([2]) says that *if the self-similar set A satisfies the SOSC, then A has Hausdorff dimension less than one and positive Hausdorff measure at this dimension*. We are going to use this fact in the proof of Theorem B.

3. PROOFS OF THE RESULTS

3.1 Proof of Theorem A. Let us define the set \mathcal{D} . An element $(\vec{\lambda}, \vec{\gamma})$ belongs to \mathcal{D} if, and only if, there are a real number $\mu \in (0, 1)$ and positive integers p and q such that $\lambda = \mu^p$ and $\gamma = \mu^q$. Recall that $\vec{\lambda} = (\lambda, \lambda'_1, \lambda, \dots, \lambda'_r, \lambda)$ and $\vec{\gamma} = (\gamma, \gamma'_1, \gamma, \dots, \gamma'_s, \gamma)$ are elements of $\mathcal{H}(2r + 1)$ and $\mathcal{H}(2s + 1)$, respectively. For a fixed $\gamma \in (0, 1)$ the continuous decreasing function $t \mapsto \gamma^t$ assume all values in the interval $(0, 1)$, as t varies from 0 to ∞ . Therefore, an appropriate choice for the rational number $p/q > 0$, with small perturbations of $\lambda'_i, i = 1, \dots, r$, allow us to approximate an element $(\vec{\lambda}, \vec{\gamma})$ of $\mathcal{H}(2r + 1) \times \mathcal{H}(2s + 1)$ by an element of the form

$$((\gamma^{p/q}, \lambda'_1, \gamma^{p/q}, \dots, \lambda'_r, \gamma^{p/q}), (\gamma, \gamma'_1, \gamma, \dots, \gamma'_s, \gamma)).$$

Therefore, it is enough to choose $\mu = \gamma^{1/q}$ to see that this element is in \mathcal{D} .

It remains to prove that if $(\vec{\lambda}, \vec{\gamma}) \in \mathcal{D}$, then $C(\vec{\lambda}) + C(\vec{\gamma})$ is a uniformly contracting self-similar set. As we saw in the previous section, an element x of $C(\vec{\lambda})$ is given by $x = \sum_{k=0}^{\infty} a_{i_k} (\mu^p)^k$ and an element y of $C(\vec{\gamma})$ is given by $y = \sum_{k=0}^{\infty} b_{j_k} (\mu^q)^k$, because $(\vec{\lambda}, \vec{\gamma}) \in \mathcal{D}$. Then $x + y$ is given by a power series in $c = \mu^{pq}$, whose coefficients vary in the finite set of all sums of the form $\sum_{k=0}^{q-1} a_{i_k} \mu^{kp} + \sum_{k=0}^{p-1} b_{j_k} \mu^{kq}$. Thus, again by the previous section, it follows that $C(\vec{\lambda}) + C(\vec{\gamma})$ is a uniformly contracting self-similar set, with contracting ratio c .

3.2 Proof of Theorem B. Let $I = [a, b] \subset (0, 1)$ be a closed interval such that $c \in I$. We are going to be more precise about the set \mathcal{A} : an element $f \in \mathcal{A}$ is a power series $f(x) = \sum_{k=0}^{\infty} d_k x^k$, with $x \in I$ and $d_k \in D$, for all k . Here $D \subset (-1, 1)$ is the finite set $\{a_i - a_j, i, j = 1, \dots, r\}$, which is symmetric with respect to $0 \in D$. Moreover, if $f \in \mathcal{A}$, then $d_0 \neq 0$.

We are going to prove that \mathcal{A} is compact in the uniform topology. In fact, let Λ be the set of sequences $d = (d_0, d_1, d_2, \dots) \in (D \setminus \{0\}) \times D \times D \times \dots$ with the product topology. Let $\Phi : \Lambda \rightarrow \mathcal{A}$ be defined by $[\Phi(d)](x) = \sum_{k=0}^{\infty} d_k x^k$. By the definition of \mathcal{A} , it follows that Φ is onto. If $d, d' \in \Lambda$ are sufficiently near, then the distance between $\Phi(d)$ and $\Phi(d')$ in the uniform topology is bounded by $b^n \frac{2(1-c)}{1-b}$, where n is arbitrarily big. This implies that Φ is continuous and therefore \mathcal{A} is compact, as we claimed above.

The hypothesis that $f(c) \neq 0$, for all $f \in \mathcal{A}$, assures the existence of a number $\rho > 0$ such that $|f(c)| \geq \rho$, for all $f \in \mathcal{A}$, because \mathcal{A} is compact. Now we choose $n_0 > 0$ such that $\rho > c^n$, whenever $n \geq n_0$. Consider the n -step A_n of the construction of A , for some $n \geq n_0$, as described in the previous section. We are going to show that $f_i(A_n) \cap f_j(A_n) = \emptyset$, for all $i, j = 1, \dots, r, i \neq j$. In fact, suppose by contradiction that $f_i(A_n) \cap f_j(A_n) \neq \emptyset$. This means that there are intervals I, J of A_n such that $f_i(I) \cap f_j(J) \neq \emptyset$, that is, the difference between the left ends of the intervals $f_i(I)$ and $f_j(J)$ is bounded in modulus by c^{n+1} . This difference is given by

$\sum_{k=0}^{n+1} d_k c^k$, where $d_0 = a_i - a_j \neq 0$. Thus, the function $f(x) = \sum_{k=0}^{n+1} d_k x^k$ is an element of \mathcal{A} and then $|f(c)| \geq \rho > c^{n+1}$, which is a contradiction.

Let V_i be the $c\varepsilon$ -neighbourhood of $f_i(A_n)$, $i = 1, \dots, r$. The number $\varepsilon > 0$ can be chosen such that the open sets V_1, \dots, V_r are pairwise disjoint, because the sets $f_1(A_n), \dots, f_r(A_n)$ are compact and pairwise disjoint. Let V be the ε -neighbourhood of A_n . It is clear that $f_i(V) = V_i$, for all i . Moreover, $V \supset V_i$ because $A_n \supset f_i(A_n)$. Thus, the self-similar set A satisfies the strong open set condition and therefore has Hausdorff dimension less than one and positive Hausdorff measure at this dimension.

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