

**FINITE TIME BLOW-UP FOR THE INHOMOGENEOUS
EQUATION $u_t = \Delta u + a(x)u^p + \lambda\phi$ IN R^d**

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ABSTRACT. We consider the inhomogeneous equation

$$\begin{aligned}u_t &= \Delta u + a(x)u^p + \lambda\phi(x) \text{ in } R^d, t \in (0, T), \\u(x, 0) &= f(x),\end{aligned}$$

where $a, \phi \geq 0$, $\lambda > 0$ and $f \geq 0$, and give criteria on p, d, a , and ϕ which determine whether for all λ and all f the solution blows up in finite time or whether for λ and f sufficiently small, the solution exists for all time.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we investigate the existence and nonexistence of nonnegative global solutions for the equation

$$(1.1) \quad \begin{aligned}u_t &= \Delta u + a(x)u^p + \lambda\phi(x) \text{ in } R^d, t \in (0, T), \\u(x, 0) &= f,\end{aligned}$$

where $0 \leq a, \phi \in C^\alpha(R^d)$, a and ϕ grow no faster than polynomially, $p > 1$, $\lambda > 0$, and $f \geq 0$.

Under these conditions, it is well known [4] that there exists a unique, bounded, nonnegative solution to (1.1), defined on a maximal time interval $[0, T^*)$, and that if $T^* < \infty$, then $\lim_{t \rightarrow T^*} \sup_{x \in R^d} u(x, t) = \infty$.

In a recent paper [5], we studied the same question for the homogeneous equation ($\lambda = 0$) with nonzero initial data ($f \geq 0$). It was shown that if $d \geq 2$ and $a(x)$ behaves on the order $|x|^m$ as $|x| \rightarrow \infty$, then if $1 < p \leq 1 + \frac{2+m}{d}$, every solution to the homogeneous equation with nonzero initial data blows up in finite time, while if $p > 1 + \frac{(2+m)^+}{d}$, then for sufficiently small initial data, there exist global solutions. (Results were also obtained in one dimension, but the formulas are a little different; in particular, if $d = 1$ and $1 < p \leq 2$, then all solutions blow up in finite time even if a has compact support.)

Our initial interest in (1.1) stemmed from its importance in the study of large deviations for super Brownian motion in the case that $p = 2$, $a(x) = 1$, and ϕ has

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compact support [2]. In any case, the study of (1.1) seems a natural follow-up to the study of the homogeneous case described above.

Note that if $u(x, t)$ solves (1.1), then by the maximum principle, for $\epsilon > 0$, the function $v(x, t) \equiv u(x, t + \epsilon)$ is larger than the solution to the homogeneous equation with initial data $u(x, \epsilon) \geq 0$. Thus, if d, p , and m fall in the blowup range for the homogeneous problem, they must also fall in the blowup range for the inhomogeneous one, for any choice of ϕ and λ . We will prove the following theorem.

Theorem. (a) Let $d \leq 2$ or let $d \geq 3$ and

$$\int_{\mathbb{R}^d} \frac{\phi(y)}{|y|^{d-2}} dy = \infty.$$

Then for any $a \geq 0$, any $\lambda > 0$, and any $f \geq 0$, the solution to (1.1) blows up in finite time.

(b) Let $d \geq 3$.

1-i. Assume that

$$a(x) \geq c|x|^m, \text{ for large } |x| \text{ and } c > 0,$$

that

$$\phi(x) \geq c|x|^{-q}, \text{ for large } |x| \text{ and } c > 0, \text{ where } q \in (2, d),$$

and that

$$1 < p < 1 + \frac{2+m}{q-2}.$$

Then for any $\lambda > 0$ and any $f \geq 0$, the solution to (1.1) blows up in finite time.

1-ii. Assume that

$$a(x) \geq c|x|^m, \text{ for large } |x| \text{ and } c > 0,$$

that

$$0 \leq \phi(x) \leq c|x|^{-d}, \text{ for some } c > 0,$$

and that

$$1 < p \leq 1 + \frac{2+m}{d-2}.$$

Then for every $\lambda > 0$ and any $f \geq 0$, the solution to (1.1) blows up in finite time.

2. Assume that

$$0 \leq a(x) \leq c|x|^m, \text{ for large } |x| \text{ and } c > 0,$$

that

$$(1.2) \quad 0 \leq \phi(x) \leq c|x|^{-q}, \text{ for large } |x| \text{ and } c > 0, \text{ where } q \in (2, d),$$

and that

$$p > 1 + \frac{(2+m)^+}{q-2}.$$

Then for sufficiently small $\lambda > 0$ and for sufficiently small $f \geq 0$, the solution to (1.1) exists for all time. More specifically, for sufficiently small $\lambda > 0$ and each $\epsilon > 0$, there exists a constant $c > 0$ such that if

$$(1.3) \quad 0 \leq f(x) \leq c(1 + |x|)^{-\frac{(2+m)^+}{p-1} - \epsilon},$$

then the solution to (1.1) exists for all time.

Furthermore, if $f \equiv 0$, then $U(x) \equiv \lim_{t \rightarrow \infty} u(x, t)$ exists, $U \in C^2(R^d)$, and $\Delta U + a(x)U^p + \lambda\phi(x) = 0$ in R^d .

Remark 1. Note that if $\phi(x) \leq c|x|^{-d}$ and $m > -2$, then the critical case, $p = 1 + \frac{2+m}{d-2}$, belongs to the blow-up regime. However, when $c_1|x|^{-q} \leq \phi(x) \leq c_2|x|^{-q}$, for large $|x|$, with $q \in (2, d)$, and $m > -2$, we have been unable to prove that blow-up occurs in the critical case, $p = 1 + \frac{2+m}{q-2}$. We expect that blow-up does occur in this case as we are unaware of examples where blow-up has been proven not to occur in the critical case [3].

Remark 2. Note that if $p > 1 + \frac{(2+m)^+}{q-2}$, then $q - 2 > \frac{(2+m)^+}{p-1}$; thus, in particular, global existence occurs in part (b-2) when $\lambda > 0$ and $c > 0$ are sufficiently small and $f(x) \leq c(1 + |x|)^{2-q}$.

Remark 3. A referee has brought to my attention the paper [6] which includes the result of this paper in the case that $a(x) \equiv 1$, $\phi(x) \leq c|x|^{-d-\epsilon}$, for large $|x|$ and some $\epsilon > 0$, and $f(x) \leq c_1(1 + |x|)^{-d-\epsilon}$. More specifically, it is proved in [6] that if ϕ and f satisfy the above bounds, and $p > \frac{d}{d-2}$, then for sufficiently small $\lambda > 0$ and sufficiently small $c_1 > 0$, the solution to (1.1) exists for all time, while if $\phi \not\equiv 0$ and $p \leq \frac{d}{d-2}$, then for all $\lambda > 0$ and all $f \geq 0$, the solution to (1.1) blows up in finite time.

In the sequel, we will use the notation $P(t, x, y) = (4\pi t)^{-\frac{d}{2}} \exp(-\frac{|y-x|^2}{4t})$.

2. PROOF OF THE THEOREM

Proof of part a. Assume to the contrary that $T^* = \infty$. Let w denote the solution to (1.1) when $a(x) \equiv 0$. By the maximum principle, $u(x, t) \geq w(x, t) = \int_0^t \int_{R^d} P(s, x, y)\phi(y)dyds$. Thus, under the condition in (a), it follows that

$$(2.1) \quad \lim_{t \rightarrow \infty} u(x, t) = \infty, \text{ for all } x \in R^d.$$

Choose a smooth bounded domain $D \in R^d$ and a $\delta > 0$ such that $\inf_{x \in D} a(x) \geq \delta$. Let $\mu > 0$ denote the principal eigenvalue of $-\Delta$ in D , and let ψ , normalized by $\int_D \psi(x)dx = 1$, denote the corresponding positive eigenfunction. Define

$$F(t) = \int_D u(x, t)\psi(x)dx.$$

Using (1.1), integrating by parts, and using Jensen's inequality, we obtain

$$(2.2) \quad \begin{aligned} F'(t) &= \int_D u_t(x, t)\psi(x)dx = \int_D (\Delta u(x, t) + a(x)u^p(x, t) + \lambda\phi(x))\psi(x)dx \\ &\geq -\mu F(t) + \delta F^p(t) + \lambda \int_D \phi(x)\psi(x)dx. \end{aligned}$$

By (2.1), it follows that $\lim_{t \rightarrow \infty} F(t) = \infty$. Since the right-hand side of (2.2) is positive and increasing as a function of F for $F > (\frac{\mu}{\delta})^{\frac{1}{p-1}}$, it follows that $F(t)$ blows up in finite time, contradicting the assumption that $T^* = \infty$. \square

Proof of part b-1. We may assume that $m > -2$ since otherwise there is nothing to prove.

We begin with a very simple proof which works in the case that $1 < p < 1 + \frac{2+m}{q-2}$ and

$$(2.3) \quad \phi(x) \geq c|x|^{-q}, \text{ for large } |x|, \text{ with } q \in (2, d] \text{ and } c > 0.$$

Assume to the contrary that $T^* = \infty$. Define $D_n = \{n < |x| < 2n\}$, let $\mu_n > 0$ denote the principal eigenvalue for $-\Delta$ in D_n , and let ψ_n , normalized by $\int_{D_n} \psi_n(x)dx = 1$, denote the corresponding positive eigenfunction. Note that by scaling, we have $\mu_n = \frac{\mu_1}{n^2}$ and $\psi_n(x) = n^{-d}\psi_1(\frac{x}{n})$. We will assume that n is sufficiently large so that $a(x) \geq c|x|^m$, for $|x| \geq n$ and some $c > 0$, and so that (2.3) holds for $|x| \geq n$. Define

$$F_n(t) = \int_{D_n} u(x, t)\psi_n(x)dx.$$

Using (1.1), integrating by parts, and using Jensen's inequality, we obtain

$$(2.4) \quad F'_n(t) \geq -\frac{\mu_1}{n^2}F_n(t) + c_1n^mF_n^p(t) + \lambda \int_D \phi(x)\psi_n(x)dx,$$

where $c_1 > 0$. By elementary calculus, we have

$$(2.5) \quad -\frac{\mu_1}{n^2}F_n(t) + c_1n^mF_n^p(t) \geq -\gamma n^{\frac{-m-2p}{p-1}}, t \geq 0,$$

for some $\gamma > 0$. Also, by (2.3), we have

$$(2.6) \quad \int_{D_n} \phi(x)\psi_n(x)dx = \int_{D_1} \phi(nx)\psi_1(x)dx \geq c_2n^{-q},$$

for some $c_2 > 0$. By assumption, $p < 1 + \frac{2+m}{q-2}$, or equivalently, $-q > \frac{-m-2p}{p-1}$. Thus, it follows from (2.4), (2.5) and (2.6) that for sufficiently large n , the right-hand side of (2.4) is positive and bounded away from 0 in t . Consequently, it follows that $F_n(t)$ blows up in finite time, contradicting the assumption that $T^* = \infty$.

We now turn to a more involved proof which works in all cases under consideration in the theorem. In the sequel, c will denote a positive constant whose value may change from expression to expression. We assume to the contrary that $T^* = \infty$ and define F_n as above. Ignoring the last term on the right-hand side of (2.4), we have

$$(2.7) \quad F'_n(t) \geq -\frac{\mu_1}{n^2}F_n(t) + c_1n^mF_n^p(t).$$

The function $g(z) = -\frac{\mu_1}{n^2}z + c_1n^mz^p$ is positive and increasing for $z > cn^{\frac{-2-m}{p-1}}$, for an appropriate $c > 0$. Therefore, if for large n , we find a t_n and a γ_n satisfying

$$(2.8) \quad F_n(t_n) \geq \gamma_n n^{\frac{-2-m}{p-1}} \text{ and } \lim_{n \rightarrow \infty} \gamma_n = \infty,$$

it will follow from (2.7) that for n sufficiently large, $F_n(t)$ blows up in finite time, contradicting the assumption that $T^* = \infty$. Thus, to complete the proof, it remains to demonstrate (2.8).

We begin with two lemmas.

Lemma 1. (i) Let $\phi(x) \geq c|x|^{-q}$, for large $|x|$ and some $c > 0$, where $q < d$. Then for any $r \geq 0$, there exists a $c_1 > 0$ such that

$$\int_{|x| \geq r} P(t, 0, x)\phi(x)dx \geq c_1(1+t)^{-\frac{q}{2}}.$$

(ii) Let $\phi(x) \geq c|x|^{-d}$ for large $|x|$ and some $c > 0$. Then for any $r \geq 0$, there exists a $c_1 > 0$ such that

$$\int_{|x| \geq r} P(t, 0, x)\phi(x)dx \geq c_1(1+t)^{-\frac{d}{2}} \log(2+t).$$

(iii) Let $\phi(x) \geq 0$. Then for any $r \geq 0$, there exists a $c_1 > 0$ such that

$$\int_{|x| \geq r} P(t, 0, x)\phi(x)dx \geq c_1(1+t)^{-\frac{d}{2}}.$$

Proof. We leave this exercise in advanced calculus to the reader. \square

Lemma 2. (i) Let $\phi(x) \geq c|x|^{-q}$ for large $|x|$ and some $c > 0$, where $q \in (2, d)$. Then there exists a $c_1 > 0$ such that

$$\int_0^t \int_{R^d} P(s, x, y)\phi(y)dyds \geq c_1|x|^{2-q} \exp\left(-\frac{|x|^2}{t}\right), \text{ for } |x| \geq 1 \text{ and } t \geq 2.$$

(ii) Let $\phi(x) \geq 0$. Then there exists a $c_1 > 0$ such that

$$\int_0^t \int_{R^d} P(s, x, y)\phi(y)dyds \geq c_1|x|^{2-d} \exp\left(-\frac{|x|^2}{t}\right), \text{ for } |x| \geq 1 \text{ and } t \geq 2.$$

Proof. Define

$$(2.9) \quad \begin{aligned} \gamma &= q, \text{ if } \phi \text{ satisfies the condition in part (i) of Lemma 2,} \\ \gamma &= d, \text{ if } \phi \text{ satisfies the condition in part (ii) of the Lemma 2.} \end{aligned}$$

Using Lemma 1 and the fact that $P(s, x, y) \geq c \exp\left(-\frac{|x|^2}{2s}\right)P\left(\frac{s}{2}, 0, y\right)$, we have

$$\begin{aligned} \int_0^t \int_{R^d} P(s, x, y)\phi(y)dyds &\geq c \int_0^t ds \exp\left(-\frac{|x|^2}{2s}\right) \int_{R^d} P\left(\frac{s}{2}, 0, y\right)\phi(y)dy \\ &\geq c \int_0^t \exp\left(-\frac{|x|^2}{2s}\right)(1+s)^{-\frac{\gamma}{2}} ds. \end{aligned}$$

Making the change of variables, $r = \frac{|x|^2}{s}$, on the right-hand side above, we obtain

$$(2.10) \quad \int_0^t \int_{R^d} P(s, x, y)\phi(y)dyds \geq c|x|^2 \int_{\frac{|x|^2}{t}}^{\infty} \exp\left(-\frac{r}{2}\right)(r+|x|^2)^{-\frac{\gamma}{2}} r^{\frac{\gamma}{2}-2} dr.$$

As in the statement of the lemma, we assume that $|x| \geq 1$ and $t \geq 2$. If in addition, $\frac{|x|^2}{t} \leq 1$, then from (2.10), it is clear that the statement of the lemma holds. Now consider the case that $\frac{|x|^2}{t} \geq 1$. Since $r^{\frac{\gamma}{2}-2} \exp\left(-\frac{r}{2}\right) \geq c \exp\left(-\frac{2r}{3}\right)$, for $r \geq 1$, we have

$$\begin{aligned} \int_0^t \int_{R^d} P(s, x, y)\phi(y)dyds &\geq c|x|^2 \int_{\frac{|x|^2}{t}}^{\frac{3|x|^2}{2t}} \exp\left(-\frac{2r}{3}\right)(r+|x|^2)^{-\frac{\gamma}{2}} dr \\ &\geq c|x|^{2-\gamma} \exp\left(-\frac{|x|^2}{t}\right). \end{aligned}$$

\square

As is well known, the solution $u(x, t)$ to (1.1) may be obtained as the pointwise limit of successive iterations $\{u_n(x, t)\}_{n=0}^{\infty}$, where $u_0 = 0$ and

$$(2.11) \quad u_{n+1}(x, t) = \int_0^t \int_{R^d} P(t-s, x, y) (a(y)u_n^p(y, s) + \lambda\phi(y)) dyds.$$

By construction, the sequence $\{u_n\}_{n=0}^\infty$ is nondecreasing; thus, in particular,

$$(2.12) \quad u(x, t) \geq u_2(x, t).$$

Also, by Lemma 2,

$$(2.13) \quad u_1(x, t) \geq c_1 \lambda |x|^{2-\gamma} \exp\left(-\frac{|x|^2}{t}\right), \text{ for } |x| \geq 1 \text{ and } t \geq 2,$$

where γ is as in (2.9).

By assumption, we can choose an $r \geq 1$ and a $c > 0$ such that $a(y) \geq c|y|^m$, for $|y| \geq r$. Using this, and substituting (2.13) into (2.11), we obtain

$$(2.14) \quad u_2(x, t) \geq c \lambda^p \int_2^t \int_{|y| \geq r} P(t-s, x, y) |y|^{p(2-\gamma)+m} \exp\left(-\frac{p|y|^2}{s}\right) dy ds, \text{ for } t > 2.$$

We have

$$(2.15) \quad \begin{aligned} P(t-s, x, y) \exp\left(-\frac{p|y|^2}{s}\right) &\geq c(t-s)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2(t-s)}\right) \exp\left(-\frac{(s+2p(t-s))}{2s(t-s)}|y|^2\right) \\ &= c(t-s)^{-\frac{d}{2}} \left(\frac{s(t-s)}{s+2p(t-s)}\right)^{\frac{d}{2}} \exp\left(-\frac{|x|^2}{2(t-s)}\right) P\left(\frac{s(t-s)}{2(s+2p(t-s))}, 0, y\right). \end{aligned}$$

Recall that we are assuming that either $\gamma = q \in (2, d)$ and $1 < p < 1 + \frac{2+m}{q-2}$, or that $\gamma = d$ and $1 < p \leq 1 + \frac{2+m}{d-2}$. It then follows that the exponent $p(2-\gamma) + m$ appearing in (2.14) satisfies $p(2-\gamma) + m > -d$, except when $\gamma = d$ and $p = 1 + \frac{2+m}{d-2}$, in which case $p(2-\gamma) + m = -d$. Substituting (2.15) into (2.14) and using Lemma 1 along with the above observation concerning the exponent $p(2-\gamma) + m$, we obtain

$$(2.16) \quad \begin{aligned} u_2(x, t) &\geq c \int_2^t \exp\left(-\frac{|x|^2}{2(t-s)}\right) (t-s)^{-\frac{d}{2}} \left(\frac{s(t-s)}{s+2p(t-s)}\right)^{\frac{d}{2} + \frac{p}{2}(2-\gamma) + \frac{m}{2}} ds, \text{ for } t > 2, \\ &\text{if } \gamma = q \in (2, d) \text{ and } 1 < p < 1 + \frac{2+m}{q-2}, \text{ or if } \gamma = d \text{ and } 1 < p < 1 + \frac{2+m}{d-2}, \end{aligned}$$

and

$$(2.17) \quad \begin{aligned} u_2(x, t) &\geq c \int_2^t \exp\left(-\frac{|x|^2}{2(t-s)}\right) (t-s)^{-\frac{d}{2}} \log\left(\frac{s(t-s)}{s+2p(t-s)}\right) ds, \text{ for } t > 2, \\ &\text{if } q = d \text{ and } p = 1 + \frac{2+m}{d-2}. \end{aligned}$$

Substituting $s = ut$ in (2.16) and integrating up to $\frac{t}{2}$ instead of up to t in order to replace the term $\exp\left(-\frac{|x|^2}{2(t-s)}\right)$ by $\exp\left(-\frac{|x|^2}{t}\right)$, we obtain

$$(2.18) \quad \begin{aligned} u_2(x, t) &\geq ct^{\frac{p}{2}(2-\gamma) + \frac{m}{2} + 1} \exp\left(-\frac{|x|^2}{t}\right), \text{ for } t > 5, \\ &\text{if } \gamma = q \in (2, d) \text{ and } 1 < p < 1 + \frac{2+m}{q-2}, \text{ or if } \gamma = d \text{ and } 1 < p < 1 + \frac{2+m}{d-2}. \end{aligned}$$

Applying this same procedure to (2.17) gives

$$(2.19) \quad \begin{aligned} u_2(x, t) &\geq ct^{-\frac{d}{2}+1}(\log t) \exp\left(-\frac{|x|^2}{t}\right), \text{ for } t > 5, \\ &\text{if } \gamma = d \text{ and } p = 1 + \frac{2+m}{d-2}. \end{aligned}$$

From (2.9), (2.12), (2.18), and (2.19), we conclude that

$$(2.20) \quad \begin{aligned} u(x, n^2) &\geq cn^{p(2-q)+m+2}, \text{ for } |x| \leq n, \text{ if } \phi \text{ satisfies the condition in Lemma 2(i)} \\ &\text{or if } \phi \text{ satisfies the condition in Lemma 2(ii) and } 1 < p < 1 + \frac{2+m}{d-2}, \end{aligned}$$

and that

$$(2.21) \quad \begin{aligned} u(x, n^2) &\geq cn^{-d+2} \log n \text{ for } |x| \leq n, \text{ if } \phi \text{ satisfies the condition in Lemma 2(ii)} \\ &\text{and } p = 1 + \frac{2+m}{d-2}. \end{aligned}$$

With the estimates (2.20) and (2.21), we can verify (2.8) and thereby conclude the proof. If ϕ, p, d , and m satisfy the conditions of (2.20), then letting $t_n = n^2$, we have $F_n(t_n) \geq cn^{p(2-q)+m+2}$. The condition on p, d, q and m in (2.20) insures that $p(2-q) + m + 2 > \frac{-2-m}{p-1}$; therefore (2.8) holds. If ϕ, p, d , and m satisfy the conditions of (2.21), then letting $t_n = n^2$, we have $F_n(t_n) \geq cn^{-d+2} \log n$. The condition on p, d, q and m in (2.21) gives $\frac{-2-m}{p-1} = -d + 2$; therefore (2.8) holds. This completes the proof of part b-1. \square

Proof of part b-2. We employ a technique used in [2] for the case $p = 2$ and $a(x) = 1$. We will prove that the following inequality holds: there exist a $c_1 > 0$ and, for all $\delta > 0$, a $C_\delta > 0$ such that

$$(2.22) \quad c_1(1 + |x|)^{2-q} \leq \int_{R^d} \frac{(1 + |y|)^{-q}}{|x - y|^{d-2}} dy \leq C_\delta(1 + |x|)^{2-q+\delta}, \text{ where } q \in (2, d].$$

We will now use (2.22) to prove global existence. Then we will return to prove (2.22). Let

$$(2.23) \quad v(x, t) \equiv v(x) = \mu \int_{R^d} \frac{(1 + |y|)^{-q}}{|x - y|^{d-2}} dy, \text{ where } \mu > 0.$$

Since $a(x) \leq c_1(1 + |x|)^m$ and $\phi(x) \leq c_2(1 + |x|)^{-q}$, for some $q \in (2, d]$, where $c_1, c_2 > 0$, and since the Green's function for $-\Delta$ on R^d satisfies $G(x, y) = \frac{\gamma_d}{|x-y|^{d-2}}$, for an appropriate constant $\gamma_d > 0$, it follows from (2.23), from the right-hand inequality in (2.22), and from the fact that $p > 1 + \frac{(2+m)^+}{q-2}$, that there exists an $\delta > 0$ such that

$$(2.24) \quad \Delta v - v_t + a(x)v^p + \lambda\phi \leq \left(-\frac{\mu}{\gamma_d} + c_1 C_\delta^p \mu^p + \lambda c_2 \right) (1 + |x|)^{-q}.$$

By first choosing μ sufficiently small so that $c_1 C_\delta^p \mu^p \leq \frac{1}{2} \frac{\mu}{\gamma_d}$, and then choosing λ sufficiently small, we can make the right-hand side of (2.24) negative. Thus, by the maximum principle and by the left-hand inequality in (2.22), it follows that for

sufficiently small $\lambda > 0$ and for sufficiently small $c > 0$, the solution $u(x, t)$ to (1.1) with $f(x) \leq c(1 + |x|)^{2-q}$ satisfies

$$(2.25) \quad u(x, t) \leq v(x),$$

and is thus a global solution. We will now show how to increase the exponent $(2-q)$ appearing in the bound for f above to the exponent $-\frac{(2+m)^+}{p-1} - \varepsilon$ appearing in (1.3) in the statement of the theorem.

We argue as follows. By assumption, $p > 1 + \frac{(2+m)^+}{q-2} = \frac{q+\max(m,-2)}{q-2}$. Note that the right-hand side of the above inequality is decreasing for $q \in (2, d]$. Define $q_0 < q$ by $p = \frac{q_0+\max(m,-2)}{q_0-2}$. Then

$$(2.26) \quad 2 - q_0 = -\frac{(2+m)^+}{p-1}.$$

Let $q_1 \in (q_0, q)$ and note that q_1 satisfies $p > 1 + \frac{(2+m)^+}{q_1-2}$. Thus, by the above results, it follows that if $\phi(x) \leq c|x|^{-q_1}$, for large $|x|$ and some $c > 0$, then for sufficiently small $\lambda > 0$ and sufficiently small $c_1 > 0$, the solution to (1.1) with $f(x) \leq c_1(1 + |x|)^{2-q_1}$ is global. But then, by the maximum principal, the same holds true if we decrease ϕ so that it satisfies the original bound, $\phi(x) \leq c|x|^{-q}$. Thus we have proven that (1.1) possesses a global solution if $\lambda > 0$ and $c_1 > 0$ are sufficiently small and $f(x) \leq c_1(1 + |x|)^{2-q_1}$. Since q_1 can be chosen arbitrarily close to q_0 , it follows from (2.26) that a global solution exists if $f(x) \leq c_1(1 + |x|)^{-\frac{(2+m)^+}{p-1} - \epsilon}$, for some $\epsilon > 0$ and for $c_1 > 0$ sufficiently small. This completes the proof of global existence.

We now return to prove (2.22). We emphasize that c will denote a positive constant whose value may change from expression to expression. For the right-hand inequality, we will show that, for any $\delta \in (0, q-2)$,

$$\int_{R^d} \frac{(1 + |x|)^{q-2-\delta}}{|x-y|^{d-2}(1 + |y|)^q} dy$$

is bounded in x . We split the integral into three parts. Let $D_1 = \{y \in R^d : |x-y| < \frac{|x|}{2}\}$, $D_2 = \{y \in R^d : |x-y| > \frac{|x|}{2}, |x-y| < \frac{|y|}{2}\}$, and $D_3 = \{y \in R^d : |x-y| > \frac{|x|}{2}, |x-y| > \frac{|y|}{2}\}$. We have

$$\begin{aligned} \int_{D_1} \frac{(1 + |x|)^{q-2-\delta}}{|x-y|^{d-2}(1 + |y|)^q} dy &\leq c(1 + |x|)^{-2-\delta} \int_{D_1} |x-y|^{2-d} dy \\ &\leq c(1 + |x|)^{-2-\delta} |x|^2 \leq c, \end{aligned}$$

where the first inequality uses the fact that $|y| \geq \frac{|x|}{2}$ on D_1 . Also,

$$\int_{D_2} \frac{(1 + |x|)^{q-2-\delta}}{|x-y|^{d-2}(1 + |y|)^q} dy \leq c \int_{D_2} |x-y|^{-d-\delta} dy \leq c.$$

Finally, we consider $\int_{D_3} \frac{(1 + |x|)^{q-2-\delta}}{|x-y|^{d-2}(1 + |y|)^q} dy$. It is clear that

$$\int_{D_3 \cap \{|x-y| \leq 1\}} \frac{(1 + |x|)^{q-2-\delta}}{|x-y|^{d-2}(1 + |y|)^q} dy \leq c.$$

Letting $D_4 = D_3 \cap \{|x - y| > 1\}$, and noting that $q - d - \delta < 0$, we have

$$\int_{D_4} \frac{(1 + |x|)^{q-2-\delta}}{|x - y|^{d-2}(1 + |y|)^q} dy \leq c \int_{D_4} \frac{|x - y|^{q-d-\delta}}{(1 + |y|)^q} dy \leq c \int_{D_4} \frac{|y|^{q-d-\delta}}{(1 + |y|)^q} dy \leq c.$$

We now prove the left-hand inequality in (2.22). Since $|y| < \frac{3}{2}|x|$ on D_1 , we have

$$(2.27) \quad \int_{R^d} \frac{(1 + |y|)^{-q}}{|x - y|^{d-2}} dy \geq c(1 + |x|)^{-q} \int_{D_1} |x - y|^{2-d} dy \geq c(1 + |x|)^{-q}|x|^2.$$

Since it is clear that $\int_{R^d} \frac{(1 + |y|)^{-q}}{|x - y|^{d-2}} dy$ is bounded away from zero on $\{|x| < 1\}$, the left-hand inequality in (2.22) follows from (2.27)

It remains to prove the final statement of the theorem. Since the initial condition in (1.1) is 0, it follows by comparison that $u(x, t)$ is monotone in t . Thus $U(x) \equiv \lim_{t \rightarrow \infty} u(x, t)$ exists. By (2.22), (2.23), and (2.25), it follows that for any $\delta > 0$, $U(x) \leq C_\delta(1 + |x|)^{2-q+\delta}$. Using the integral equation for $u(x, t)$, namely (2.11) with u_n and u_{n+1} replaced by u , and letting $t \rightarrow \infty$, gives

$$U(x) = \int_{R^d} G(x, y)(aU^p + \lambda\phi)(y)dy,$$

where $G(x, y) = \gamma_d \frac{1}{|y-x|^{d-2}}$ is the Green's function for Δ on R^d . Note that from the above bound on U , along with the conditions on a, ϕ , and q , and the assumption that $p > 1 + \frac{(2+m)^+}{q-2}$, it follows that $(aU^p + \lambda\phi)(y) \leq c(1 + |y|)^{-q}$, and $q > 2$. It then follows from standard theory [1] that $U \in C^2(R^d)$ and $\Delta U + a(x)U^p + \lambda\phi(x) = 0$ in R^d . \square

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