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CALCULATING AND INTERPRETING THE MISLIN GENUS OF A SPECIAL CLASS OF NILPOTENT SPACES

PETER HILTON AND DIRK SCEVENELS

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Abstract. We prove that there is a bijection between the Mislin genus of a circle bundle over a certain nilpotent base space M, which is constructed from a nilpotent group N of a certain specified type, and the Mislin genus of N itself.

1. Introduction

In [9] Mislin introduced the concept of the genus of a finitely generated nilpotent group N. This is the set of isomorphism classes of finitely generated nilpotent groups M such that the localizations M_p and N_p are isomorphic at every prime p. Analogously the Mislin genus of a connected nilpotent space X (of the homotopy type of a CW-complex) of finite type (see [4]) is defined as the set of homotopy types of nilpotent spaces Y of finite type such that the localizations Y_p and X_p are homotopy equivalent at every prime p.

Now it is easy to see that, given a finitely generated nilpotent group N, one may construct the Eilenberg–Mac Lane space $K(N,1)$, and there is then a bijection of Mislin genera $\mathcal{G}(N) \cong \mathcal{G}(K(N,1))$. Our object in this paper is to establish a bijection of genera $\mathcal{G}(N) \cong \mathcal{G}(X)$ for nilpotent groups N of a special class and nilpotent spaces X constructed in a more subtle way from the group N. Moreover, we will be dealing with finitely generated nilpotent groups whose Mislin genera have already been calculated. Indeed, in a series of papers ([1], [2], [5], [7]), the authors have calculated the Mislin genus of any group in a certain class $\mathcal{N}_1$ of finitely generated nilpotent groups. The class $\mathcal{N}_1$ consists of those nilpotent groups N, given in terms of the associated short exact sequence

$$TN \rightarrowtail N \twoheadrightarrow FN, \tag{1.1}$$

where TN is the torsion subgroup and FN the torsionfree quotient, by the conditions

(a) TN and FN are commutative;
(b) (1.1) splits on the right; and
(c) for the associated action $\omega\colon FN \to \operatorname{Aut} TN$, the image ωFN is contained in the centre of $\operatorname{Aut} TN$.

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In fact, in the presence of (a), condition (c) is equivalent to condition

(c′) for all ξ in FN, there exists a positive integer u such that $\xi \cdot a = \omega(\xi)(a) = ua$ for all $a \in TN$.

It is shown in [6] that the genus $\mathcal{G}(N)$ of a group N in $\mathcal{N}_1$ is trivial unless FN is cyclic. Now we know (see [3], [9]) that, if $N \in \mathcal{N}_1$, then $\mathcal{G}(N)$ admits the structure of a finite abelian group (note that, since FN is commutative, the commutator subgroup of N is finite). Suppose then that FN is cyclic, generated by ξ, and let t be the order of u (see (c′)) modulo m, where m is the exponent of TN. Then the calculation of the genus yields

$$\mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\}, \tag{1.2}$$

where $(\mathbb{Z}/t)^*$ is the multiplicative group of units in the ring $\mathbb{Z}/t$. Moreover, we know how to modify the presentation of N, given by

$$TN \rightarrowtail N \twoheadrightarrow C = \langle \xi \rangle$$

with

$$\xi \cdot a = ua, \text{ for all } a \in TN,$$

to yield the other (isomorphism classes of) groups in the genus of N; namely, if ℓ is prime to t, then the group N_ℓ corresponding under the isomorphism (1.2) is simply obtained from the short exact sequence

$$TN \rightarrowtail N_\ell \twoheadrightarrow C = \langle \xi \rangle$$

by imposing the action

$$\xi \cdot a = u^\ell a, \text{ for all } a \in TN.$$

In [2] it was shown, in the special case where TN is cyclic of prime power order, that we may construct a circle bundle X over a fixed base M to correspond to our group N. It is clear that this construction generalizes to any group N in $\mathcal{N}_1$ with FN cyclic. More precisely, we prescribe the 2-type of the connected nilpotent polyhedron M by requiring that

$$\pi_1 M = C_t = \langle \eta \rangle;\ \pi_2 M = TN;\ \eta \cdot a = ua \text{ for all } a \in TN, \tag{1.3}$$

where C_t denotes the cyclic group of order t. We then let g generate the summand $\mathrm{Ext}(\mathbb{Z}/t, \mathbb{Z})$ of $H^2(M;\mathbb{Z})$, and think of g as the classifying map $g\colon M \to K(\mathbb{Z}, 2)$ for a circle bundle X over M. It is easy to see that the bundle map $q\colon X \to M$ induces isomorphisms of all homotopy groups in dimensions $n \geq 2$, while $\pi_1 X = C = \langle \xi \rangle$ and q maps ξ onto η.

If we pick ℓ prime to t, and choose ℓ' so that $\ell\ell' \equiv 1 \mod t$, then we may take $\ell' g$ to be the classifying map for a circle bundle X_ℓ over M, and we obtain a covering map $h\colon X \to X_\ell$. In this way we set up a bijective correspondence between the elements of $\mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\}$ and certain elements of the genus of X; for one may show that X_ℓ is in the genus of X and has the homotopy type of $X_{\bar{\ell}}$ if and only if $\ell \equiv \pm\bar{\ell} \mod t$.

In this generality it would not be reasonable to expect that the entire genus of X would be obtained in this way. However, we have conjectured that this would be so if M contained no non-trivial homotopy beyond that specified by (1.3), that is, if M (and thus also X) has only the two non-trivial homotopy groups π_1 and π_2. Our object in this note is to prove this conjecture:

Theorem. *If M is a connected nilpotent polyhedron with 2-type as specified in* (1.3) *and no other non-trivial homotopy groups, then the construction given above*

$$\mathcal{G}(N) \to \mathcal{G}(X): N_\ell \mapsto X_\ell$$

is a bijection of genera.

Our basic tool is a result of McGibbon ([8]), which extends a theorem of Zabrodsky ([10], [11]). Essentially, McGibbon's result shows that, for spaces such as our connected nilpotent space X, whose rationalization is an H-space with only one non-vanishing homotopy group, the genus of X may (like the genus of N) be given the structure of a finite abelian group and that there is a right exact sequence, again very like that first noticed and adopted by Mislin ([9]), for specifying the genus group $\mathcal{G}(X)$. More precisely, there is a right exact sequence

$$s\text{-Equ}(X) \xrightarrow{d} (\mathbb{Z}/s)^*/\{\pm 1\} \twoheadrightarrow \mathcal{G}(X), \tag{1.4}$$

where s is a positive integer depending on X and s-Equ(X) refers to the set of homotopy classes of self-maps of X which are p-equivalences for the prime divisors p of s. Here we have not quoted McGibbon's result in its full generality in [8], being content to specialize to our situation. Thus our task is simply to use (1.4) to show that

$$\mathcal{G}(X) \cong (\mathbb{Z}/t)^*/\{\pm 1\}.$$

For we have already found a set of homotopy types of spaces X_ℓ, in the genus of X, in bijective correspondence with the elements of $(\mathbb{Z}/t)^*/\{\pm 1\}$.

In Section 2 we interpret the group N as a function of the space X; we will use this interpretation to analyse the first homomorphism of the sequence (1.4). In Section 3 we complete the analysis of the terms in (1.4) and thus achieve the proof of our main result.

2. A lemma on free homotopy groups

The following result deserves to be well-known – and perhaps is. Let A, X be pointed, path-connected topological spaces. Denote by $[A, \Omega X]$ the set of based homotopy classes of maps from A to ΩX, and let $[A, \Omega X]_{\text{fr}}$ be the set of free homotopy classes of maps from A to ΩX. Then both these sets receive a group structure from the group structure of ΩX in the based homotopy category; and $\pi_1 X$ acts on $[A, \Omega X]$ by means of the rule $[g] \mapsto [\ell g \ell^{-1}]$, where ℓ is a loop, ℓ^{-1} is its reverse and $(\ell g \ell^{-1})(a) = \ell g(a) \ell^{-1}$, for all $a \in A$.

Theorem 2.1. *There is a natural isomorphism*

$$\theta: [A, \Omega X]_{\text{fr}} \cong [A, \Omega X] \rtimes \pi_1 X, \tag{2.1}$$

where the group on the right of (2.1) *is the semidirect product for the action of* $\pi_1 X$ *on* $[A, \Omega X]$, *and* θ *is given by*

$$\theta[f] = ([f\ell^{-1}], [\ell]),$$

where $\ell = f(a_0)$, *with* a_0 *the base point of* A.

Proof. Let $\theta[f_i] = ([f_i \ell_i^{-1}], [\ell_i])$, for $i = 1, 2$. Then

$$\theta[f_1 f_2] = ([f_1 f_2 \ell_2^{-1} \ell_1^{-1}], [\ell_1 \ell_2]).$$

On the other hand,

$$([f_1\ell_1^{-1}],[\ell_1])([f_2\ell_2^{-1}],[\ell_2]) = ([f_1\ell_1^{-1}\ell_1 f_2\ell_2^{-1}\ell_1^{-1}],[\ell_1\ell_2]) \\ = ([f_1f_2\ell_2^{-1}\ell_1^{-1}],[\ell_1\ell_2]).$$

Thus θ is a homomorphism. Now define

$$\bar\theta\colon [A,\Omega X]\rtimes \pi_1 X \to [A,\Omega X]_{\mathrm{fr}}$$

by

$$\bar\theta([g],[\ell]) = [g\ell].$$

It is easily verified that $\bar\theta$ is a two-sided inverse to θ, so that θ is an isomorphism. □

We may apply this theorem to the space X constructed in the Introduction out of the group N; we will specifically suppose that M (and hence X too) has vanishing homotopy groups in dimensions $n \geq 3$. We take $A = S^1$, so that θ is an isomorphism between $[S^1,\Omega X]_{\mathrm{fr}}$ and $\pi_2 X \rtimes \pi_1 X$. But

$$\pi_2 X \rtimes \pi_1 X = TN \rtimes C = N.$$

Thus we have

Corollary 2.2. $[S^1,\Omega X]_{\mathrm{fr}} \cong N$ *and the split short exact sequence*

$$\pi_2 X \rightarrowtail [S^1,\Omega X]_{\mathrm{fr}} \twoheadrightarrow \pi_1 X \tag{2.2}$$

coincides with the split short exact sequence (1.1)

$$TN \rightarrowtail N \twoheadrightarrow FN. \quad \square$$

3. Proof of the main result

We first use the procedure given on p. 297 of [8][1] to calculate s in (1.4) for our particular space X. We must calculate certain integers $s_1(X)$, $s_2(X)$ and then s is the least common multiple of the product $s_1(X)s_2(X)$ and those primes which divide the exponent of the torsion in $QH^1(X;\mathbb{Z})$ and $QH^2(X;\mathbb{Z})$. (Any notational obscurities are cleared up in [8].)

We consider the homomorphisms $\sigma_1\colon \pi_1 X \to PH_1(X;\mathbb{Z})/\text{torsion}$ and $\sigma_2\colon \pi_2 X \to PH_2(X;\mathbb{Z})/\text{torsion}$, which are obvious quotients of the Hurewicz homomorphism. Plainly $\ker\sigma_1 = \{1\}$, $\operatorname{coker}\sigma_2 = \{1\}$ and $s_1(X) = 1$, since there is no torsion in $\pi_1 X$. As to $s_2(X)$, we must look at the lower central series, in the sense of [4, p. 34], for the action of $\pi_1 X$ on $\pi_2 X$. Plainly the product of the exponents of the quotients of the terms of the lower central series is m itself, since, in order that N be nilpotent, it is necessary and sufficient that m divides some power of $(u-1)$. This establishes that $s_2(X) = m$. Now we know that the set of prime divisors of m is, of course, just the set of primes p such that N has p-torsion, and no other primes can enter into the torsion in the (co)homology of X. Thus $s = m$ and the semigroup s-$\mathrm{Equ}(X)$ of (1.4) is just the semigroup under composition of homotopy classes of self-maps of X, which are p-equivalences for all primes $p \in T(N)$, where $T(N)$ denotes the set of all primes p for which N has p-torsion.

[1] We write s, s_i in the place of t, t_i in [8] to avoid confusion with our use of t in this part.

Finally, it remains to analyse the homomorphism d of (1.4). We consider an m-equivalence (that is, a $T(N)$-equivalence) $f\colon X \to X$ and must calculate its effect on $\pi_1 X \otimes \mathbb{Q}$. However such an f acts on the sequence (2.2), producing

$$\begin{array}{ccccc} TN & \rightarrowtail & N & \twoheadrightarrow & FN = \pi_1 X \\ \downarrow\cong & & \downarrow f_* & & \downarrow f_* \\ TN & \rightarrowtail & N & \twoheadrightarrow & FN = \pi_1 X \end{array}$$

and $f_*\colon N \to N$ is an element of $T(N)$-$\mathrm{Aut}\, N$. Moreover, it was proved in [7] that, to compute the image of f_* in the sequence

$$T(N)\text{-}\mathrm{Aut}\, N \to (\mathbb{Z}/e)^*/\{\pm 1\} \twoheadrightarrow \mathcal{G}(N),$$

which is used to yield $\mathcal{G}(N)$ an abelian group structure, it suffices to look at the determinant of $f_*\colon FN \to FN$. It follows that $f_*\colon \pi_1 X \to \pi_1 X$ is just multiplication by some integer that is congruent to 1 modulo t. Moreover, any such integer is realizable by some f (and any such f must be a $T(N)$-equivalence). For if $\ell \equiv 1 \bmod t$, then we have a commutative diagram (recall that g is of order t)

$$\begin{array}{ccccccc} S^1 & \longrightarrow & X & \xrightarrow{q} & M & \xrightarrow{g} & K(\mathbb{Z},2) \\ \Big\downarrow \ell & & \Big\downarrow f & & \Big\| & & \Big\downarrow \ell \\ S^1 & \longrightarrow & X & \xrightarrow{q} & M & \xrightarrow{g} & K(\mathbb{Z},2). \end{array}$$

Finally, we have the short exact sequence

$$K \rightarrowtail (\mathbb{Z}/m)^*/\{\pm 1\} \twoheadrightarrow \mathcal{G}(X),$$

in which K consists of the residues, modulo ± 1, which are congruent to $1 \mod t$. We claim that $\mathcal{G}(X)$ is just $(\mathbb{Z}/t)^*/\{\pm 1\}$. For consider the homomorphism

$$\rho\colon (\mathbb{Z}/m)^*/\{\pm 1\} \to (\mathbb{Z}/t)^*/\{\pm 1\}$$

which simply takes a residue class mod m and regards it as a residue class mod t. (Recall that $t \mid m$.) The kernel of ρ is of course K. On the other hand, it follows easily from Dirichlet's Theorem that ρ is surjective. Thus the quotient of $(\mathbb{Z}/m)^*/\{\pm 1\}$ by K is $(\mathbb{Z}/t)^*/\{\pm 1\}$, so that

$$\mathcal{G}(X) \cong (\mathbb{Z}/t)^*/\{\pm 1\},$$

as claimed. It therefore follows that the method of construction of spaces in the genus of X described in the Introduction produces, in fact, the entire genus and realizes the bijection $\mathcal{G}(X) \cong \mathcal{G}(N)$, as claimed.

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Department of Mathematical Sciences, State University of New York, Binghamton, New York 13902–6000

Department of Mathematics, University of Central Florida, Orlando, Florida 32816–1364

Departement Wiskunde, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, B-3001 Heverlee, Belgium

E-mail address: `dirk.scevenels@wis.kuleuven.ac.be`