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CALCULATING AND INTERPRETING THE MISLIN GENUS OF A SPECIAL CLASS OF NILPOTENT SPACES

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Abstract. We prove that there is a bijection between the Mislin genus of a circle bundle over a certain nilpotent base space M, which is constructed from a nilpotent group N of a certain specified type, and the Mislin genus of N itself.

1. Introduction

In [9] Mislin introduced the concept of the genus of a finitely generated nilpotent group N. This is the set of isomorphism classes of finitely generated nilpotent groups M such that the localizations M_p and N_p are isomorphic at every prime p. Analogously the Mislin genus of a connected nilpotent space X (of the homotopy type of a CW-complex) of finite type (see [4]) is defined as the set of homotopy types of nilpotent spaces Y of finite type such that the localizations Y_p and X_p are homotopy equivalent at every prime p.

Now it is easy to see that, given a finitely generated nilpotent group N, one may construct the Eilenberg–Mac Lane space $K(N,1)$, and there is then a bijection of Mislin genera $\mathcal{G}(N) \cong \mathcal{G}(K(N,1))$. Our object in this paper is to establish a bijection of genera $\mathcal{G}(N) \cong \mathcal{G}(X)$ for nilpotent groups N of a special class and nilpotent spaces X constructed in a more subtle way from the group N. Moreover, we will be dealing with finitely generated nilpotent groups whose Mislin genera have already been calculated. Indeed, in a series of papers ([1], [2], [5], [7]), the authors have calculated the Mislin genus of any group in a certain class $\mathcal{N}_1$ of finitely generated nilpotent groups. The class $\mathcal{N}_1$ consists of those nilpotent groups N, given in terms of the associated short exact sequence

$$TN \rightarrowtail N \twoheadrightarrow FN, \tag{1.1}$$

where TN is the torsion subgroup and FN the torsionfree quotient, by the conditions

- (a) TN and FN are commutative;
- (b) (1.1) splits on the right; and
- (c) for the associated action $\omega\colon FN \to \operatorname{Aut} TN$, the image ωFN is contained in the centre of $\operatorname{Aut} TN$.

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In fact, in the presence of (a), condition (c) is equivalent to condition

(c′) for all ξ in FN, there exists a positive integer u such that $\xi \cdot a = \omega(\xi)(a) = ua$ for all $a \in TN$.

It is shown in [6] that the genus $\mathcal{G}(N)$ of a group N in $\mathcal{N}_1$ is trivial unless FN is cyclic. Now we know (see [3], [9]) that, if $N \in \mathcal{N}_1$, then $\mathcal{G}(N)$ admits the structure of a finite abelian group (note that, since FN is commutative, the commutator subgroup of N is finite). Suppose then that FN is cyclic, generated by ξ, and let t be the order of u (see (c′)) modulo m, where m is the exponent of TN. Then the calculation of the genus yields

$$\mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\}, \tag{1.2}$$

where $(\mathbb{Z}/t)^*$ is the multiplicative group of units in the ring $\mathbb{Z}/t$. Moreover, we know how to modify the presentation of N, given by

$$TN \rightarrowtail N \twoheadrightarrow C = \langle \xi \rangle$$

with

$$\xi \cdot a = ua, \text{ for all } a \in TN,$$

to yield the other (isomorphism classes of) groups in the genus of N; namely, if ℓ is prime to t, then the group N_ℓ corresponding under the isomorphism (1.2) is simply obtained from the short exact sequence

$$TN \rightarrowtail N_\ell \twoheadrightarrow C = \langle \xi \rangle$$

by imposing the action

$$\xi \cdot a = u^\ell a, \text{ for all } a \in TN.$$

In [2] it was shown, in the special case where TN is cyclic of prime power order, that we may construct a circle bundle X over a fixed base M to correspond to our group N. It is clear that this construction generalizes to any group N in $\mathcal{N}_1$ with FN cyclic. More precisely, we prescribe the 2-type of the connected nilpotent polyhedron M by requiring that

$$\pi_1 M = C_t = \langle \eta \rangle;\ \pi_2 M = TN;\ \eta \cdot a = ua \text{ for all } a \in TN, \tag{1.3}$$

where C_t denotes the cyclic group of order t. We then let g generate the summand $\mathrm{Ext}(\mathbb{Z}/t, \mathbb{Z})$ of $H^2(M; \mathbb{Z})$, and think of g as the classifying map $g\colon M \to K(\mathbb{Z}, 2)$ for a circle bundle X over M. It is easy to see that the bundle map $q\colon X \to M$ induces isomorphisms of all homotopy groups in dimensions $n \geq 2$, while $\pi_1 X = C = \langle \xi \rangle$ and q maps ξ onto η.

If we pick ℓ prime to t, and choose ℓ' so that $\ell\ell' \equiv 1 \mod t$, then we may take $\ell' g$ to be the classifying map for a circle bundle X_ℓ over M, and we obtain a covering map $h\colon X \to X_\ell$. In this way we set up a bijective correspondence between the elements of $\mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\}$ and certain elements of the genus of X; for one may show that X_ℓ is in the genus of X and has the homotopy type of $X_{\bar{\ell}}$ if and only if $\ell \equiv \pm \bar{\ell} \mod t$.

In this generality it would not be reasonable to expect that the entire genus of X would be obtained in this way. However, we have conjectured that this would be so if M contained no non-trivial homotopy beyond that specified by (1.3), that is, if M (and thus also X) has only the two non-trivial homotopy groups π_1 and π_2. Our object in this note is to prove this conjecture:

Theorem. *If M is a connected nilpotent polyhedron with 2-type as specified in* (1.3) *and no other non-trivial homotopy groups, then the construction given above*

$$\mathcal{G}(N) \to \mathcal{G}(X) \colon N_\ell \mapsto X_\ell$$

is a bijection of genera.

Our basic tool is a result of McGibbon ([8]), which extends a theorem of Zabrodsky ([10], [11]). Essentially, McGibbon's result shows that, for spaces such as our connected nilpotent space X, whose rationalization is an H-space with only one non-vanishing homotopy group, the genus of X may (like the genus of N) be given the structure of a finite abelian group and that there is a right exact sequence, again very like that first noticed and adopted by Mislin ([9]), for specifying the genus group $\mathcal{G}(X)$. More precisely, there is a right exact sequence

$$s\text{-Equ}(X) \xrightarrow{d} (\mathbb{Z}/s)^*/\{\pm 1\} \twoheadrightarrow \mathcal{G}(X), \tag{1.4}$$

where s is a positive integer depending on X and s-Equ(X) refers to the set of homotopy classes of self-maps of X which are p-equivalences for the prime divisors p of s. Here we have not quoted McGibbon's result in its full generality in [8], being content to specialize to our situation. Thus our task is simply to use (1.4) to show that

$$\mathcal{G}(X) \cong (\mathbb{Z}/t)^*/\{\pm 1\}.$$

For we have already found a set of homotopy types of spaces X_ℓ, in the genus of X, in bijective correspondence with the elements of $(\mathbb{Z}/t)^*/\{\pm 1\}$.

In Section 2 we interpret the group N as a function of the space X; we will use this interpretation to analyse the first homomorphism of the sequence (1.4). In Section 3 we complete the analysis of the terms in (1.4) and thus achieve the proof of our main result.

2. A LEMMA ON FREE HOMOTOPY GROUPS

The following result deserves to be well-known – and perhaps is. Let A, X be pointed, path-connected topological spaces. Denote by $[A, \Omega X]$ the set of based homotopy classes of maps from A to ΩX, and let $[A, \Omega X]_{\mathrm{fr}}$ be the set of free homotopy classes of maps from A to ΩX. Then both these sets receive a group structure from the group structure of ΩX in the based homotopy category; and $\pi_1 X$ acts on $[A, \Omega X]$ by means of the rule $[g] \mapsto [\ell g \ell^{-1}]$, where ℓ is a loop, ℓ^{-1} is its reverse and $(\ell g \ell^{-1})(a) = \ell g(a) \ell^{-1}$, for all $a \in A$.

Theorem 2.1. *There is a natural isomorphism*

$$\theta \colon [A, \Omega X]_{\mathrm{fr}} \cong [A, \Omega X] \rtimes \pi_1 X, \tag{2.1}$$

where the group on the right of (2.1) *is the semidirect product for the action of* $\pi_1 X$ *on* $[A, \Omega X]$*, and* θ *is given by*

$$\theta[f] = ([f \ell^{-1}], [\ell]),$$

where $\ell = f(a_0)$*, with* a_0 *the base point of* A*.*

Proof. Let $\theta[f_i] = ([f_i \ell_i^{-1}], [\ell_i])$, for $i = 1, 2$. Then

$$\theta[f_1 f_2] = ([f_1 f_2 \ell_2^{-1} \ell_1^{-1}], [\ell_1 \ell_2]).$$

On the other hand,

$$\begin{aligned}([f_1\ell_1^{-1}],[\ell_1])([f_2\ell_2^{-1}],[\ell_2]) &= ([f_1\ell_1^{-1}\ell_1 f_2\ell_2^{-1}\ell_1^{-1}],[\ell_1\ell_2]) \\ &= ([f_1 f_2\ell_2^{-1}\ell_1^{-1}],[\ell_1\ell_2]).\end{aligned}$$

Thus θ is a homomorphism. Now define

$$\bar{\theta}\colon [A,\Omega X]\rtimes\pi_1 X\to[A,\Omega X]_{\mathrm{fr}}$$

by

$$\bar{\theta}([g],[\ell])=[g\ell].$$

It is easily verified that $\bar{\theta}$ is a two-sided inverse to θ, so that θ is an isomorphism. $\square$

We may apply this theorem to the space X constructed in the Introduction out of the group N; we will specifically suppose that M (and hence X too) has vanishing homotopy groups in dimensions $n\geq 3$. We take $A=S^1$, so that θ is an isomorphism between $[S^1,\Omega X]_{\mathrm{fr}}$ and $\pi_2X\rtimes\pi_1X$. But

$$\pi_2X\rtimes\pi_1X=TN\rtimes C=N.$$

Thus we have

Corollary 2.2. $[S^1,\Omega X]_{\mathrm{fr}}\cong N$ *and the split short exact sequence*

$$\pi_2X\rightarrowtail[S^1,\Omega X]_{\mathrm{fr}}\twoheadrightarrow\pi_1X \tag{2.2}$$

coincides with the split short exact sequence (1.1)

$$TN\rightarrowtail N\twoheadrightarrow FN.\quad\square$$

3. Proof of the main result

We first use the procedure given on p. 297 of [8][1] to calculate s in (1.4) for our particular space X. We must calculate certain integers $s_1(X)$, $s_2(X)$ and then s is the least common multiple of the product $s_1(X)s_2(X)$ and those primes which divide the exponent of the torsion in $QH^1(X;\mathbb{Z})$ and $QH^2(X;\mathbb{Z})$. (Any notational obscurities are cleared up in [8].)

We consider the homomorphisms $\sigma_1\colon\pi_1X\to PH_1(X;\mathbb{Z})/\text{torsion}$ and $\sigma_2\colon\pi_2X\to PH_2(X;\mathbb{Z})/\text{torsion}$, which are obvious quotients of the Hurewicz homomorphism. Plainly $\ker\sigma_1=\{1\}$, $\operatorname{coker}\sigma_2=\{1\}$ and $s_1(X)=1$, since there is no torsion in π_1X. As to $s_2(X)$, we must look at the lower central series, in the sense of [4, p. 34], for the action of π_1X on π_2X. Plainly the product of the exponents of the quotients of the terms of the lower central series is m itself, since, in order that N be nilpotent, it is necessary and sufficient that m divides some power of $(u-1)$. This establishes that $s_2(X)=m$. Now we know that the set of prime divisors of m is, of course, just the set of primes p such that N has p-torsion, and no other primes can enter into the torsion in the (co)homology of X. Thus $s=m$ and the semigroup s-Equ(X) of (1.4) is just the semigroup under composition of homotopy classes of self-maps of X, which are p-equivalences for all primes $p\in T(N)$, where $T(N)$ denotes the set of all primes p for which N has p-torsion.

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[1] We write s, s_i in the place of t, t_i in [8] to avoid confusion with our use of t in this part.

Finally, it remains to analyse the homomorphism d of (1.4). We consider an m-equivalence (that is, a $T(N)$-equivalence) $f\colon X \to X$ and must calculate its effect on $\pi_1 X \otimes \mathbb{Q}$. However such an f acts on the sequence (2.2), producing

$$\begin{tikzcd} TN \arrow[r, tail] \arrow[d, "\cong"] & N \arrow[r, two heads] \arrow[d, "f_*"] & FN = \pi_1 X \arrow[d, "f_*"] \\ TN \arrow[r, tail] & N \arrow[r, two heads] & FN = \pi_1 X \end{tikzcd}$$

and $f_*\colon N \to N$ is an element of $T(N)$-$\operatorname{Aut} N$. Moreover, it was proved in [7] that, to compute the image of f_* in the sequence

$$T(N)\text{-}\operatorname{Aut} N \to (\mathbb{Z}/e)^*/\{\pm 1\} \twoheadrightarrow \mathcal{G}(N),$$

which is used to yield $\mathcal{G}(N)$ an abelian group structure, it suffices to look at the determinant of $f_*\colon FN \to FN$. It follows that $f_*\colon \pi_1 X \to \pi_1 X$ is just multiplication by some integer that is congruent to 1 modulo t. Moreover, any such integer is realizable by some f (and any such f must be a $T(N)$-equivalence). For if $\ell \equiv 1 \bmod t$, then we have a commutative diagram (recall that g is of order t)

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "\ell"] & X \arrow[r, "q"] \arrow[d, "f"] & M \arrow[r, "g"] \arrow[d, equal] & K(\mathbb{Z},2) \arrow[d, "\ell"] \\ S^1 \arrow[r] & X \arrow[r, "q"] & M \arrow[r, "g"] & K(\mathbb{Z},2). \end{tikzcd}$$

Finally, we have the short exact sequence

$$K \rightarrowtail (\mathbb{Z}/m)^*/\{\pm 1\} \twoheadrightarrow \mathcal{G}(X),$$

in which K consists of the residues, modulo ± 1, which are congruent to $1 \mod t$. We claim that $\mathcal{G}(X)$ is just $(\mathbb{Z}/t)^*/\{\pm 1\}$. For consider the homomorphism

$$\rho\colon (\mathbb{Z}/m)^*/\{\pm 1\} \to (\mathbb{Z}/t)^*/\{\pm 1\}$$

which simply takes a residue class mod m and regards it as a residue class mod t. (Recall that $t \mid m$.) The kernel of ρ is of course K. On the other hand, it follows easily from Dirichlet's Theorem that ρ is surjective. Thus the quotient of $(\mathbb{Z}/m)^*/\{\pm 1\}$ by K is $(\mathbb{Z}/t)^*/\{\pm 1\}$, so that

$$\mathcal{G}(X) \cong (\mathbb{Z}/t)^*/\{\pm 1\},$$

as claimed. It therefore follows that the method of construction of spaces in the genus of X described in the Introduction produces, in fact, the entire genus and realizes the bijection $\mathcal{G}(X) \cong \mathcal{G}(N)$, as claimed.

References

[1] C. Casacuberta and P. Hilton, *Calculating the Mislin genus for a certain family of nilpotent groups*, Comm. Algebra **19** (1991), 2051–2069. MR **92j:**20032

[2] P. Hilton, *On the genus of nilpotent groups and spaces*, Israel J. Math. **54** (1986), 1–13. MR **87m:**55011

[3] P. Hilton and G. Mislin, *On the genus of a nilpotent group with finite commutator subgroup*, Math. Z. **146** (1976), 201–211. MR **53:**615

[4] P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Math. Studies vol. 15, North-Holland, Amsterdam (1975). MR **57:**17635

[5] P. Hilton and D. Scevenels, *Calculating the genus of the direct product of certain nilpotent groups*, Publ. Mat. **39** (1995), 241–261. MR **97b:**20049

[6] P. Hilton and C. Schuck, *On the structure of nilpotent groups of a certain type*, Topol. Methods Nonlinear Anal. **1** (1993), 323–327. MR **94i:**20061

[7] ———, *Calculating the genus of certain nilpotent groups*, Bull. Mex. Math. Soc. **37** (1992), 263–269. MR **95m:**20037

[8] C. A. McGibbon, *On the localization genus of a space*, in: Algebraic Topology: New Trends in Localization and Periodicity, Progress in Math. vol. 136, Birkhäuser Verlag, Basel (1996), 285–306. MR **97d:**55019

[9] G. Mislin, *Nilpotent groups with finite commutator subgroups*, Lecture Notes in Math. vol. 418, Springer-Verlag, Berlin Heidelberg New York (1974), 103–120. MR **50:**10080

[10] A. Zabrodsky, *On the genus of finite CW H-spaces*, Comment. Math. Helv. **49** (1974), 48–64. MR **50:**5792

[11] ———, *p-equivalences and homotopy type*, Lecture Notes in Math. vol. 418, Springer-Verlag, Berlin Heidelberg New York (1974), 160–171. MR **51:**14036

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