

## NEW $\Sigma_3^1$ FACTS

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ABSTRACT. We use “iterated square sequences” to show that there is an  $L$ -definable partition  $n : L\text{-Singulars} \rightarrow \omega$  such that if  $M$  is an inner model not containing  $0^\#$ :

- (a) For some  $k$ ,  $M \models \{\alpha \mid n(\alpha) \leq k\}$  is stationary.
- (b) For each  $k$  there is a generic extension of  $M$  in which  $0^\#$  does not exist and  $\{\alpha \mid n(\alpha) \leq k\}$  is non-stationary.

This result is then applied to show that if  $M$  is an inner model without  $0^\#$ , then some  $\Sigma_3^1$  sentence not true in  $M$  can be forced over  $M$ .

Assume that  $0^\#$  exists and that  $M$  is an inner model of ZFC,  $0^\# \notin M$ . Then of course  $M$  is not lightface  $\Sigma_3^1$ -correct: the true  $\Sigma_3^1$  sentence “ $0^\#$  exists” is false in  $M$ . In this article we use a result about  $L$ -definable partitions (which may be of independent interest) to show that in fact  $M$  fails to satisfy some  $\Sigma_3^1$  sentence true in a forcing extension of  $M$ . We work in Morse-Kelley class theory (though Gödel-Bernays with a satisfaction predicate for  $V$  will suffice).

**Theorem 1.** *Assume that  $0^\#$  exists. There exists an  $\omega$ -sequence of true  $\Sigma_3^1$  sentences  $\langle \varphi_n \mid n \in \omega \rangle$  such that if  $M$  is an inner model,  $0^\# \notin M$ :*

- (a)  $\varphi_n$  is false in  $M$  for some  $n$ .
- (b) For each  $n$ , some generic extension of  $M$  satisfies  $\varphi_n$ .

Moreover if  $M = L[R]$ ,  $R$  a real, then these generic extensions can be taken to be inner models of  $L[R, 0^\#]$ .

The above result is based on the next result, concerning  $L$ -definable partitions.

**Theorem 2.** *There exists an  $L$ -definable function  $n : L\text{-Singulars} \rightarrow \omega$  such that if  $M$  is an inner model,  $0^\# \notin M$ :*

- (a) For some  $k$ ,  $M \models \{\alpha \mid n(\alpha) \leq k\}$  is stationary.
- (b) For each  $k$  there is a generic extension of  $M$  in which  $0^\#$  does not exist and  $\{\alpha \mid n(\alpha) \leq k\}$  is non-stationary.

*Remark.* “Stationary in  $M$ ” means: intersects every  $M$ -definable (with parameters) closed unbounded class of ordinals.

*Proof.* We define  $n(\alpha)$ . Let  $\langle C_\alpha \mid \alpha \text{ } L\text{-singular} \rangle$  be an  $L$ -definable  $\square$ -sequence:  $C_\alpha$  is closed unbounded in  $\alpha$ , ordertype  $C_\alpha < \alpha$  and  $\bar{\alpha} \in \lim C_\alpha \rightarrow C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ . Let

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of  $C_\alpha$  denote the ordertype of  $C_\alpha$ . If  $\text{ot } C_\alpha$  is  $L$ -regular, then  $n(\alpha) = 0$ . Otherwise  $n(\alpha) = n(\text{ot } C_\alpha) + 1$ .

(a) is clear, as otherwise (using the fact that we are working in Morse–Kelley class theory) there is a closed unbounded  $C \subseteq L$ -regulars amenable to  $M$ , contradicting the Covering Theorem and the hypothesis that  $0^\#$  does not belong to  $M$ .

Now we prove (b). Fix  $n \in \omega$ . In  $M$  let  $P$  consist of closed, bounded  $p \subseteq \text{ORD}$  such that  $\alpha \in p \rightarrow \alpha$   $L$ -regular or  $n(\alpha) \geq n+1$ , ordered by  $p \leq q$  iff  $p$  end extends  $q$ .

We claim that  $P$  is  $\infty$ -distributive in  $M$ . Suppose that  $p \in P$  and  $\langle D_\alpha \mid \alpha < \kappa \rangle$  is a definable sequence of open dense subclasses of  $P$ ,  $\kappa$  regular. We wish to find  $q \leq p$ ,  $q \in D_\alpha$  for all  $\alpha < \kappa$ . Let  $C$  be the class of all strong limit cardinals  $\beta$  such that  $D_\alpha \cap V_\beta$  is dense in  $P \cap V_\beta$  for all  $\alpha < \kappa$ , a closed unbounded class of ordinals. It suffices to show that  $C \cap \{\beta \mid n(\beta) \geq n+1\}$  has a closed subset of ordertype  $\kappa+1$ , for then  $p$  can be successively extended  $\kappa$  times meeting the  $D_\alpha$ 's, to conditions with maximum in  $\{\beta \mid n(\beta) \geq n+1\}$ ; the final condition (at stage  $\kappa$ ) extends  $p$  and meets each  $D_\alpha$ .

**Lemma 3.** *Suppose  $m \geq k$ ,  $\alpha$  is regular and  $C$  is a closed set of ordertype  $\alpha^{+m}+1$ , consisting of ordinals greater than  $\alpha^{+m}$  (where  $\alpha^{+0} = \alpha$ ,  $\alpha^{+(p+1)} = (\alpha^{+p})^+$ ). Then  $C \cap \{\beta \mid n(\beta) \geq k\}$  has a closed subset of ordertype  $\alpha^{+(m-k)}+1$ .*

*Proof of Lemma 3.* By induction on  $k$ . Suppose  $k = 0$ . Let  $\beta = \max C$ . Then  $\beta$  is singular and hence singular in  $L$ . So  $C_\beta$  is defined and  $\lim(C_\beta \cap C)$  is a closed set of ordertype  $\alpha^{+m}+1$  consisting of  $L$ -singulars. So  $\lim(C_\beta \cap C) \subseteq C \cap \{\gamma \mid n(\gamma) \geq 0\}$  satisfies the lemma.

Suppose the lemma holds for  $k$  and let  $m+1 \geq k+1$ ,  $C$  a closed set of ordertype  $\alpha^{+(m+1)}+1$  consisting of ordinals greater than  $\alpha^{+(m+1)}$ . Let  $\beta = \max C$ . Then  $C_\beta$  is defined and  $D = \lim(C_\beta \cap C)$  is a closed set of ordertype  $\alpha^{+(m+1)}+1$ . Let  $\bar{\beta} = (\alpha^{+m} + \alpha^{+m} + 1)$ st element of  $D$ . Then

$$\bar{D} = \{\text{ot } C_\gamma \mid \gamma \in D, (\alpha^{+m} + 1) \text{ st element of } D \leq \gamma \leq \bar{\beta}\}$$

is a closed set of ordertype  $\alpha^{+m}+1$  consisting of ordinals greater than  $\alpha^{+m}$ . By induction there is a closed  $\bar{D}_0 \subseteq \bar{D} \cap \{\gamma \mid n(\gamma) \geq k\}$  of ordertype  $\alpha^{+(m-k)}+1$ . But then  $D_0 = \{\gamma \in D \mid \text{ot } C_\gamma \in \bar{D}_0\}$  is a closed subset of  $C \cap \{\gamma \mid n(\gamma) \geq k+1\}$  of ordertype  $\alpha^{+(m-k)}+1$ . As  $\alpha^{+(m-k)} = \alpha^{+((m+1)-(k+1))}$  we are done.  $\square$

By the lemma,  $C \cap \{\beta \mid n(\beta) \geq n\}$  has arbitrary long closed subsets for any  $n$ , for any closed unbounded  $C \subseteq \text{ORD}$ . It follows that  $P$  is  $\infty$ -distributive. Now to prove (b), we apply the forcing  $P$  to  $M$ , producing  $C$  witnessing the nonstationarity of  $\{\alpha \mid n(\alpha) \leq n\}$ , and then follow this with the forcing to code  $\langle M, C \rangle$  by a real, making  $C$  definable. Of course this will not produce  $0^\#$  as every successor to a strong limit cardinal is preserved in the coding.  $\square$

We also note that in Theorem 2 the generic extension can be formed in  $L[R, 0^\#]$  in the case  $M = L[R]$ ,  $R$  a real, using the fact that in  $L[R, 0^\#]$ , generics can be constructed for  $P$  (an ‘‘Amenable’’ forcing) and for Jensen coding (see [99, Friedman]).

*Proof of Theorem 1.* We use David’s trick (see [98, Friedman]). Let  $\varphi_n$  be the sentence:  $\exists R \forall \alpha (\text{If } L_\alpha[R] \models ZF^-, \text{ then } L_\alpha[R] \models \beta \text{ a limit cardinal} \rightarrow \beta \text{ } L\text{-regular or } n(\beta) \geq n)$ . (This is equivalent to a  $\Sigma_3^1$  sentence as it is of the form  $\exists R \psi(R)$  where  $\psi(R)$  is  $\Pi_1$  in the sense of Lévy and hence equivalent to a  $\Pi_2^1$  sentence.)

By Theorem 2(b) and cardinal collapsing (to guarantee that limit cardinals  $\beta$  are either  $L$ -regular or satisfy  $n(\beta) \geq n$ ),  $M$  has a generic extension  $L[R] \models \beta$  a limit cardinal  $\rightarrow \beta$   $L$ -regular or  $n(\beta) \geq n$  (inside  $L[S, 0^\#]$  if  $M = L[S]$ ,  $S$  a real). By David's trick we can in fact obtain  $\varphi_n$  in  $L[R]$ .  $\square$

**Question.** Can the generic extensions in Theorem 1(b) be taken to have the same cofinalities as  $M$ , in case  $M$  satisfies  $GCH$ ?

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