

## LACUNARY SETS BASED ON LORENTZ SPACES

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**ABSTRACT.** A new lacunary set for compact abelian groups is introduced; this is called a  $\Lambda(p, q)$  set. This set is defined in terms of the Lorentz spaces and is shown to be a generalization of  $\Lambda(p)$  sets and Sidon sets. A number of functional-analytic statements about  $\Lambda(p, q)$  sets are established by making use of the structural similarities between  $L^p$  spaces and Lorentz spaces. These statements are analogous to several well-known properties of a set which are equivalent to the definition of a  $\Lambda(p)$  set. Some general set-theoretic and arithmetic properties of  $\Lambda(p, q)$  sets are also developed; these properties extend known results on the structure of  $\Lambda(p)$  sets. Open problems and directions for further research are outlined.

### 1. INTRODUCTION

Throughout this paper  $G$  denotes an infinite compact abelian group and  $\Gamma$  its discrete dual group. If  $X \subseteq L^1$  and  $E \subseteq \Gamma$ , let  $X_E = \{f \in X : \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin E\}$ . For  $p \in (1, \infty)$  recall that a subset  $E$  of  $\Gamma$  is called a  $\Lambda(p)$  set if  $L_E^p = L_E^1$ . The set  $E$  is called a Sidon set if  $L_E^\infty \subseteq \{f \in L^1 : \hat{f} \in \ell^1(\Gamma)\}$ . Sidon sets and  $\Lambda(p)$  sets are the most widely studied types of lacunary sets for compact abelian groups; two standard references on the theory of these sets are [10] and [12]. In this paper we introduce and study a new type of lacunary set which is defined in terms of the Lorentz spaces,  $L(p, q)$ .

**Definition 1.1.** Let  $p \in (1, \infty)$  and  $q \in [1, \infty)$ . A subset  $E$  of  $\Gamma$  is called a  $\Lambda(p, q)$  set if  $L(p, q)_E = L_E^1$ .

What motivates the notation for  $\Lambda(p, q)$  sets is the notation for Lorentz spaces. These spaces are a two-parameter family of function spaces which are closely related to the  $L^p$  spaces. In particular, they are intermediate to the  $L^p$  spaces in the sense that whenever  $1 \leq q < p < r \leq \infty$ ,

$$(1) \quad L^\infty \subset \bigcup_{t>p} L^t \subseteq L(p, q) \subset L^p \subset L(p, r) \subseteq \bigcap_{s<p} L^s \subset L^1.$$

Furthermore, each  $L^p$  space is itself a Lorentz space as  $L^p = L(p, p)$ . It follows from this that every  $\Lambda(p)$  set is also a  $\Lambda(p, p)$  set. In [6, Section 3] two types of lacunary sets based on Lorentz spaces are introduced; these are called  $\Lambda_1(p, q)$  and  $\Lambda_2(p, q)$  sets. We shall see that these sets are also examples of  $\Lambda(p, q)$  sets.

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There are a number of well-known functional-analytic properties of a set which are equivalent to the definition of a  $\Lambda(p)$  set (see [10, 37.9] and [12, 5.3]). Since the Lorentz spaces are generalizations of the  $L^p$  spaces, it is not surprising that similar characterizations of  $\Lambda(p, q)$  sets may be obtained by straightforward modifications of known results for  $\Lambda(p)$  sets. An easy, yet important, theorem states that if  $p > 2$ , then the class of  $\Lambda(p)$  sets is closed under the formation of finite unions. We will prove a similar result for  $\Lambda(p, q)$  sets and discuss some related questions concerning unions. In [1], [3], [7], and [8] one finds a number of results on arithmetic and general set-theoretic properties of  $\Lambda(p)$  sets. These properties deal with the structural nature of  $\Lambda(p)$  sets. We will establish analogous properties for  $\Lambda(p, q)$  sets.

## 2. PRELIMINARIES

Let  $\lambda$  denote the normalized Haar measure on  $G$  and let  $\|\cdot\|_p$  denote the usual  $p$ -norm where  $p \in [1, \infty]$ . Let  $T$  denote the set of trigonometric polynomials on  $G$  and  $M$  denote the set of complex bounded Borel measures on  $G$ . For the reader's convenience, we shall give the definition and state some basic properties of Lorentz spaces; further details on these spaces are found in [6], [11], and [13].

Let  $f$  be a complex-valued measurable function on  $G$  which is finite almost everywhere. The distribution function  $\lambda_f$  of  $f$  is defined by

$$\lambda_f(y) = \lambda\{x \in G : |f(x)| > y\} \quad \text{for } y \geq 0.$$

The non-increasing rearrangement of  $f$  is the function

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\} \quad \text{for } t \geq 0.$$

The Lorentz space  $L(p, q)$  is defined as the set of equivalence classes of functions  $f$  such that  $\|f\|_{p,q}^* < \infty$ , where

$$\|f\|_{p,q}^* = \begin{cases} \left( \frac{q}{p} \int_0^1 [t^{1/p} f^*(t)]^q \frac{dt}{t} \right)^{1/q} & \text{if } 1 \leq p, q < \infty, \\ \sup_{t \in (0, \infty)} t^{1/p} f^*(t) & \text{if } 1 \leq p \leq \infty, q = \infty. \end{cases}$$

Since  $\lambda_{f^*} = \lambda_f$ , it follows that  $\|f\|_{p,p}^* = \|f\|_p$  and hence  $L(p, p) = L^p$  for all  $p \in [1, \infty]$ . The function  $f \mapsto \|f\|_{p,q}^*$  is a quasi-norm for  $L(p, q)$ , but is not generally a norm. However,  $L(p, q)$  does have a norm which is related to  $\|\cdot\|_{p,q}^*$ . To define this norm, consider a function  $f$  and its averaging function  $f^{**}$  where

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds \quad \text{for } t > 0.$$

Then  $L(p, q)$  can be taken as the set of equivalence classes of functions  $f$  such that  $\|f\|_{(p,q)} < \infty$ , where

$$\|f\|_{(p,q)} = \begin{cases} \left( \int_0^\infty [t^{1/p} f^{**}(t)]^q \frac{dt}{t} \right)^{1/q} & \text{if } 1 \leq p, q < \infty, \\ \sup_{t \in (0, \infty)} t^{1/p} f^{**}(t) & \text{if } 1 \leq p \leq \infty, q = \infty. \end{cases}$$

If  $p = q \in \{1, \infty\}$  or if  $p \in (1, \infty)$  and  $q \in [1, \infty)$ , then  $L(p, q)$  is a Banach space with norm  $\|\cdot\|_{(p,q)}$ . The quasi-norm and norm are related by the inequality

$$(2) \quad \left(\frac{p}{q}\right)^{1/q} \|f\|_{p,q}^* \leq \|f\|_{(p,q)} \leq p' \left(\frac{p}{q}\right)^{1/q} \|f\|_{p,q}^*$$

where  $p \in (1, \infty)$  and  $q \in [1, \infty]$ . Note that  $p'$  is the index conjugate to  $p$  and  $\left(\frac{p}{q}\right)^{1/q} = 1$  if  $q = \infty$ . The most useful inequalities for the quasi-norm and norm are as follows. For  $p \in [1, \infty]$  and  $1 \leq q_1 < q_2 \leq \infty$ ,

$$(3) \quad \|f\|_{p,q_2}^* \leq \|f\|_{p,q_1}^*$$

and, for  $p \in (1, \infty)$ ,

$$(4) \quad \|f\|_{(p,q_2)} \leq \left(\frac{q_1}{p}\right)^{(q_1^{-1}-q_2^{-1})} \|f\|_{(p,q_1)}.$$

If  $1 < p_1 < p_2 < \infty$  and  $q \in [1, \infty)$ , then

$$(5) \quad \|f\|_{p_1,q}^* \leq \left(\frac{p_2}{p_2 - p_1}\right)^{1/q} \|f\|_{p_2,\infty}^*.$$

A consequence of this last inequality is the proper inclusion

$$(6) \quad L(p_2, q_2) \subset L(p_1, q_1)$$

whenever  $1 < p_1 < p_2 < \infty$  and  $q_1, q_2 \in [1, \infty]$ . As in [6, p. 368] define a total ordering on the set  $J = (1, \infty) \times [1, \infty)$  by  $(r, s) > (p, q)$  if and only if either  $r > p$  or  $r = p$  and  $s < q$ . Using this ordering and the inclusions in (1) and (6), it follows that

$$(7) \quad L(r, s) \subset L(p, q) \text{ if and only if } (r, s) > (p, q).$$

### 3. BASIC PROPERTIES OF $\Lambda(p, q)$ SETS

If  $E$  is a finite subset of  $\Gamma$ , then  $E$  is a  $\Lambda(p, q)$  set for all  $(p, q) \in J$  since  $L(p, q)_E = L_E^1 = T_E$ . In contrast, the group  $\Gamma$  itself is not a  $\Lambda(p, q)$  set for any  $(p, q) \in J$  since (1) shows that each such space  $L(p, q)$  is always a proper subset of  $L^1$ . Our first results give simple relationships between  $\Lambda(p, q)$  sets,  $\Lambda(p)$  sets, and Sidon sets.

**Theorem 3.1.** *Let  $(p, q), (r, s) \in J$  and let  $E \subset \Gamma$ .*

- (a) *If  $E$  is a  $\Lambda(p, q)$  set, then  $E$  is a  $\Lambda(r, s)$  set whenever  $(p, q) > (r, s)$ .*
- (b) *If  $E$  is a  $\Lambda(p)$  set, then  $E$  is a  $\Lambda(r, s)$  set whenever  $(p, p) > (r, s)$ .*
- (c) *If  $E$  is a  $\Lambda(p, q)$  set, then  $E$  is a  $\Lambda(r)$  set whenever  $(p, q) > (r, r)$ .*

*Proof.* This is evident from (7) and the definition of  $\Lambda(p)$  sets and  $\Lambda(p, q)$  sets.  $\square$

**Theorem 3.2.** *Each Sidon set is a  $\Lambda(p, q)$  set for all  $(p, q) \in J$ .*

*Proof.* If  $E$  is a Sidon set, then [10, 37.10] shows that  $E$  is a  $\Lambda(r)$  set for each  $r \in (1, \infty)$ . Given  $(p, q) \in J$ , if  $r > p$ , then  $(r, r) > (p, q)$  and thus  $E$  is a  $\Lambda(p, q)$  set by Theorem 3.1(b).  $\square$

**Corollary 3.3.** *Each infinite subset of  $\Gamma$  contains an infinite set which is a  $\Lambda(p, q)$  set for every  $(p, q) \in J$ .*

*Proof.* By [10, 37.18] each infinite subset of  $\Gamma$  contains an infinite Sidon set; hence the result follows from Theorem 3.2.  $\square$

Theorem 3.2 and Corollary 3.3 are generalizations of classical results about Sidon sets and  $\Lambda(p)$  sets. The following result is also such a generalization.

**Theorem 3.4.** *There exists a subset of  $\Gamma$  which is a  $\Lambda(p, q)$  set for all  $(p, q) \in J$  but is not a Sidon set.*

*Proof.* By [12, 5.14]  $\Gamma$  contains an infinite subset  $E$  which is a  $\Lambda(r)$  set for all  $r \in (1, \infty)$  but is not a Sidon set. If  $(p, q) \in J$ , then  $(r, r) > (p, q)$  whenever  $r > p$ , and thus  $E$  is a  $\Lambda(p, q)$  set by Theorem 3.1(b).  $\square$

In [6, Section 3] the following pair of lacunary sets based on Lorentz spaces are introduced.

**Definition 3.5.** Let  $E \subset \Gamma$  and let  $(p, q) \in J$ .

(a) The set  $E$  is a  $\Lambda_1(p, q)$  set if there exists  $s \in (1, p)$  such that  $L(p, q)_E = L(s, q)_E$ .

(b) The set  $E$  is a  $\Lambda_2(p, q)$  set if there exists  $r \in (q, \infty)$  such that  $L(p, q)_E = L(p, r)_E$ .

As shown for  $\Lambda(p, q)$  sets, each finite subset of  $\Gamma$  is a  $\Lambda_1(p, q)$  set and a  $\Lambda_2(p, q)$  set for every  $(p, q) \in J$ . As well,  $\Gamma$  itself is not a  $\Lambda_1(p, q)$  set or a  $\Lambda_2(p, q)$  set for any  $(p, q) \in J$ . The  $\Lambda_2(p, q)$  sets are used in [6] to establish a characterization theorem for Lorentz-improving measures [6, 3.4]. Some inclusions between  $\Lambda_2(p, q)$  sets,  $\Lambda_1(p, q)$  sets, and  $\Lambda(p, q)$  sets are given in the following result.

**Theorem 3.6.** *Let  $E \subset \Gamma$  and let  $(p, q) \in J$ .*

(a) *If  $E$  is a  $\Lambda_1(p, q)$  set, then  $E$  is a  $\Lambda(r, s)$  set whenever  $(r, s) < (u, u)$  and  $1 < u < p$ .*

(b) *If  $E$  is a  $\Lambda(p, q)$  set, then  $E$  is a  $\Lambda_1(r, s)$  set whenever  $(p, q) \geq (r, s)$ .*

(c) *If  $E$  is a  $\Lambda(p, q)$  set, then  $E$  is a  $\Lambda_2(r, s)$  set whenever  $(p, q) \geq (r, s)$ .*

*Proof.* (a) If  $E$  is a  $\Lambda_1(p, q)$  set, then [6, 3.1(b)] shows that  $E$  is a  $\Lambda(u)$  set for every  $u \in (1, p)$ . By Theorem 3.1(b)  $E$  is also a  $\Lambda(r, s)$  set whenever  $(r, s) < (u, u)$ .

(b) Assume  $E$  is a  $\Lambda(p, q)$  set and let  $(p, q) \geq (r, s)$ . Then for each  $w \in (1, r)$ ,

$$L_E^1 \subseteq L(p, q)_E \subseteq L(r, s)_E \subseteq L(w, s)_E \subseteq L_E^1$$

and thus  $E$  is a  $\Lambda_1(r, s)$  set.

(c) If  $E$  is a  $\Lambda(p, q)$  set and  $(p, q) \geq (r, s)$ , then

$$L_E^1 \subseteq L(p, q)_E \subseteq L(r, s)_E \subseteq L(r, w)_E \subseteq L_E^1$$

for each  $w \in (s, \infty)$ . This shows  $E$  is a  $\Lambda_2(r, s)$  set.  $\square$

#### 4. EQUIVALENT STATEMENTS FOR $\Lambda(p, q)$ SETS

There are a number of functional-analytic properties of a subset  $E$  of  $\Gamma$  which are equivalent to the definition of a  $\Lambda(p, q)$  set. These properties are analogous to some well-known and useful statements about a set which are equivalent to the definition of a  $\Lambda(p)$  set (see [10, 37.7 and 37.9] and [12, 5.3]). The first of two results gives a characterization of  $\Lambda(p, q)$  sets in terms of norms and quasi-norms, and is almost a full generalization of [10, 37.7].

**Theorem 4.1.** *Let  $E \subset \Gamma$ , let  $(p, q), (r, s) \in J$ , and assume  $r > p$ . The following assertions are equivalent:*

- (i)  $E$  is a  $\Lambda(r, s)$  set;
- (ii)  $L(r, s)_E = L(p, q)_E$ ;
- (iii) there exists a constant  $k$  such that  $\|f\|_{r,s}^* \leq k\|f\|_{p,q}^*$  for every  $f \in T_E$ ;
- (iv) there exists a constant  $k$  such that  $\|f\|_{r,s}^* \leq k\|f\|_1$  for every  $f \in T_E$ ;
- (v)  $M_E = L(r, s)_E$ .

*Proof.* We will just outline the proof of each implication; the reader who is interested in the details of the original proof for  $\Lambda(p)$  sets can refer to [10, 37.7].

(i  $\Rightarrow$  ii) If  $E$  is a  $\Lambda(r, s)$  set where  $r > p$ , then  $(r, s) > (p, q)$  and hence  $L(r, s)_E = L_E^1$  and  $L(r, s)_E \subseteq L(p, q)_E$  by (7). Since  $L(p, q)_E \subseteq L_E^1$ , assertion (ii) follows trivially.

(ii  $\Rightarrow$  iii) If  $L(r, s)_E = L(p, q)_E$  where  $r > p$ , then  $(r, s) > (p, q)$  and thus the mapping  $f \mapsto f$  of  $L(r, s)_E$  into  $L(p, q)_E$  is surjective and continuous. Since  $L(r, s)_E$  and  $L(p, q)_E$  are Banach spaces [5, 14.2], it follows from the open mapping theorem that there exists a constant  $k$  such that

$$\|f\|_{(r,s)} \leq k \|f\|_{(p,q)} \text{ for all } f \in L(p, q)_E.$$

Assertion (iii) is now clear as a result of this inequality, (2), and the fact that  $T_E \subseteq L(p, q)_E$ .

(iii  $\Rightarrow$  iv) Since  $L(r, s)_E \subseteq L(p, q)_E$  and  $T_E$  is dense in both of these spaces, it follows from (iii) that  $L(r, s)_E = L(p, q)_E$ . Let  $p_1$  and  $p_2$  satisfy  $p < p_1 < p_2 < r$ . Then  $L(r, s)_E \subseteq L_E^{p_2} \subseteq L_E^{p_1} \subseteq L(p, q)_E$  and hence  $L_E^{p_2} = L_E^{p_1}$ . By [10, 37.7]  $L_E^{p_2} = L_E^1$  and thus  $L(r, s)_E = L(p, q)_E = L_E^1$ . By the open mapping theorem there exists a constant  $k$  such that  $\|f\|_{(r,s)} \leq k\|f\|_1$  for all  $f \in L_E^1$ . Combining this inequality with those in (2) and assertion (iii) yields (iv).

(iv  $\Rightarrow$  v) From (1) and the Radon-Nikodym theorem,  $L(r, s)_E$  may be regarded as a subset of  $M_E$ . We will show that  $M_E \subseteq L(r, s)_E$ . Let  $\mu \in M_E$  and let  $h \in T$ . Then  $\mu * h \in T_E$  and from (iv),

$$\|\mu * h\|_{r,s}^* \leq k \|\mu * h\|_1 \leq k \|\mu\| \|h\|_1.$$

It follows from this inequality and (2) that

$$\sup\{\|\mu * h\|_{(r,s)} : h \in T, \|h\|_1 \leq 1\} < \infty.$$

A straightforward modification of [10, 35.11] shows there exists a function  $g \in L(r, s)$  such that  $d\mu = g d\lambda$ . Since  $\hat{\mu}$  vanishes off of  $E$ ,  $\hat{g}$  does also and thus  $g \in L(r, s)_E$ . This proves  $\mu \in L(r, s)_E$  which establishes (v).

(v  $\Rightarrow$  i) If  $L(r, s)_E = M_E$ , then  $L_E^1 \subseteq M_E = L(r, s)_E \subseteq L_E^1$  and hence  $L(r, s)_E = L_E^1$  which gives (i).  $\square$

The proof of Theorem 4.1 shows that assertions (i), (iv), and (v) there are equivalent under the weaker hypothesis  $(r, s) > (p, q)$ . However, the full equivalence of assertions (i)–(v) has not been established for pairs  $(r, s), (p, q)$  where  $p = r$  and  $q < s$ . It is to this extent that Theorem 4.1 is not a complete generalization of [10, 37.7].

The next result gives additional properties of a set which are equivalent to the definition of a  $\Lambda(p, q)$  set. These properties complement those in Theorem 4.1 in that they characterize  $\Lambda(p, q)$  sets in terms of the dual of a Lorentz space. This result is almost a full generalization of the equivalences for  $\Lambda(p)$  sets as found in [10, 37.9].

**Theorem 4.2.** *Let  $E \subset \Gamma$  and let  $(p, q) \in J$ . The following assertions are equivalent:*

- (i)  $E$  is a  $\Lambda(p, q)$  set;
- (ii) for each  $g \in L(p', q')$  there exists an  $h \in L^\infty$  such that  $\hat{g}(\gamma) = \hat{h}(\gamma)$  for all  $\gamma \in E$ ;
- (iii) for each  $g \in L(p', q')$  there exists a continuous function  $h$  such that  $\hat{g}(\gamma) = \hat{h}(\gamma)$  for all  $\gamma \in E$ .

*If  $p > 2$ , then statements (i)–(iii) above are equivalent to the assertion:*

- (iv) for each  $g \in L(p', q')$ ,  $\sum_{\gamma \in E} |\hat{g}(\gamma)|^2 < \infty$ .

*Proof.* Again we will just outline the proof of each implication and refer the interested reader to [10, 37.9] for the original details concerning  $\Lambda(p)$  sets.

(i  $\Rightarrow$  ii) This follows very closely the proof that (i) implies (ii) in [10, 37.9] except that Hölder’s inequality for Lorentz spaces [13, 3.5] is used in place of the standard Hölder inequality.

(ii  $\Rightarrow$  iii) Assume (ii) and let  $g \in L(p', q')$ . A factorization theorem for Lorentz spaces ([5, 14.3] and [10, 32.33(d)]) shows there exist an  $f \in L^1$  and a  $g_1 \in L(p', q')$  such that  $g = f * g_1$ . Now (ii) implies there is an  $h_1 \in L^\infty$  such that  $\hat{h}_1(\gamma) = \hat{g}_1(\gamma)$  for all  $\gamma \in E$ . Let  $h = f * h_1$  and note from [9, 20.16] that  $h$  is continuous. One easily checks that  $\hat{g}(\gamma) = \hat{h}(\gamma)$  for all  $\gamma \in E$  and this establishes (iii).

(iii  $\Rightarrow$  i) The proof of this implication follows very closely the proof that (iii  $\Rightarrow$  i) in [10, 37.9]. For convenience and completeness we outline the argument here for Lorentz spaces. Using [10, 35.7(c)] it is straightforward to verify that  $L_{E^c}(p', q')$  is a closed two-sided ideal in  $L(p', q')$  where  $E^c$  denotes the complement of  $E$  in  $\Gamma$ . As well, one checks that for a function  $g \in L(p', q')$ ,  $g + L_{E^c}(p', q') = h + L_{E^c}(p', q')$  precisely when  $\hat{g}(\gamma) = \hat{h}(\gamma)$  for all  $\gamma \in E$ . There exists a constant  $k$  such that, for each  $g \in L(p', q')$ , there is some continuous function  $h$  which satisfies  $h + L_{E^c}(p', q') = g + L_{E^c}(p', q')$  and  $\|h\|_u \leq k \|g\|_{(p', q')}$ . Here  $\|\cdot\|_u$  denotes the uniform norm. Now let  $f \in T_E$ , let  $g \in L(p', q')$ , and consider the function  $h$  above. Since  $\hat{h}(\gamma) = \hat{g}(\gamma)$  for all  $\gamma \in E$ , it follows from [10, 34.33(ii)] that

$$\int_G \bar{f}(s)(g(s) - h(s))d\lambda(s) = 0.$$

We see that

$$\left| \int_G \bar{f}(s)g(s)d\lambda(s) \right| \leq \|h\|_u \|f\|_1 \leq k \|g\|_{(p', q')} \|f\|_1.$$

From this inequality and the duality result in [2, (2.5), p. 9], it follows that  $\|f\|_{(p, q)} \leq k \|f\|_1$ . Assertion (i) is now evident from (2) and Theorem 4.1.

For the final implication assume that  $p > 2$ .

(iii  $\Rightarrow$  iv) If  $g \in L(p', q')$ , consider a continuous function  $h$  such that  $\hat{g}(\gamma) = \hat{h}(\gamma)$  for all  $\gamma \in E$ . Then  $\sum_{\gamma \in E} |\hat{g}(\gamma)|^2 = \sum_{\gamma \in E} |\hat{h}(\gamma)|^2 \leq \|h\|_2^2 < \infty$  which yields (iv).

(iv  $\Rightarrow$  i) Following [10, 37.9] suppose that  $\sum_{\gamma \in E} |\hat{g}(\gamma)|^2 < \infty$  for each  $g \in L(p', q')$ . Since  $p > 2$ , (1) shows that  $L^2 \subset L(p', q')$  and thus the mapping  $f \mapsto f$  from  $L^2$  into  $L(p', q')$  is continuous. By [10, 28.43] there exists an  $h \in L^2$  such that  $\hat{h}(\gamma) = \hat{g}(\gamma)$  for all  $\gamma \in E$  and  $\hat{h}$  vanishes off of  $E$ . It follows from the proof of (iii  $\Rightarrow$  i) and (2) that there exists a constant  $k$  such that  $\|f\|_{p, q}^* \leq k \|f\|_2$  for all  $f \in T_E$ . Theorem 4.1 now shows that  $E$  is a  $\Lambda(p, q)$  set which establishes (i).  $\square$

Theorem 4.2 is not a complete generalization of [10, 37.9] as the equivalence of assertions (i)–(iv) has not been established for  $(p, q)$  where  $p = 2$  and  $q < 2$ .

## 5. SET-THEORETIC AND ARITHMETIC PROPERTIES

An important problem in the theory of lacunary sets is the so-called “union problem”. The problem is to determine whether a particular type of lacunary set is closed under finite unions. The following instances of this problem are well-known. If  $p > 2$  and if  $E_1$  and  $E_2$  are  $\Lambda(p)$  sets, then  $E_1 \cup E_2$  is also a  $\Lambda(p)$  set [10, 37.21]. If  $p \in (1, \infty)$ , then  $E_1 \cup E_2$  is a  $\Lambda(p)$  set whenever  $E_1$  is a  $\Lambda(p)$  set consisting of non-negative integers and  $E_2$  is a  $\Lambda(p)$  set consisting of negative integers [14, 4.4]. Lastly, if  $E_1$  and  $E_2$  are Sidon sets, then so is  $E_1 \cup E_2$  [12, 3.5]. These three examples motivate the following results for  $\Lambda(p, q)$  sets.

**Theorem 5.1.** *Let  $(p, q) \in J$  and assume  $p > 2$ . If  $E_1$  and  $E_2$  are  $\Lambda(p, q)$  sets, then  $E_1 \cup E_2$  is a  $\Lambda(p, q)$  set.*

*Proof.* The argument is similar to that for  $\Lambda(p)$  sets found in [10, 37.21]. Assume  $E_1$  and  $E_2$  are  $\Lambda(p, q)$  sets and let  $E = E_1 \cup E_2$ ; we may assume the union is disjoint. By Theorem 4.1 and (2) there exist constants  $k_j$ ,  $j = 1, 2$ , such that  $\|f\|_{(p,q)} \leq k_j \|f\|_2$  for each  $f \in T_{E_j}$ . Let  $k = \max\{k_1, k_2\}$  and let  $f \in T_E$ . Since  $f$  can be written  $f = f_1 + f_2$  where  $f_j \in T_{E_j}$ , it follows from the Peter-Weyl theorem that  $\|f\|_2^2 = \|f_1\|_2^2 + \|f_2\|_2^2$  and thus  $\|f_j\|_2 \leq \|f\|_2$ . Now  $\|f\|_{(p,q)} \leq \|f_1\|_{(p,q)} + \|f_2\|_{(p,q)} \leq 2k \|f\|_2$ , hence Theorem 4.1 and (2) imply that  $E$  is a  $\Lambda(p, q)$  set.  $\square$

An obvious consequence of Theorem 5.1 is that a finite union of  $\Lambda(p, q)$  sets is itself a  $\Lambda(p, q)$  set. However, this is a best possible result in the sense that one cannot generally replace the word “finite” with “infinite” as  $\Gamma$  is not a  $\Lambda(p, q)$  set for any  $(p, q) \in J$ . Our next result is a generalization of [14, 4.4] where the analysis takes place on the circle group and its dual, the integers.

**Theorem 5.2.** *Let  $E_1$  be a set of non-negative integers and let  $E_2$  be a set of negative integers. If  $E_1$  and  $E_2$  are  $\Lambda(p, q)$  sets for some  $(p, q) \in J$ , then the set  $E = E_1 \cup E_2$  is also a  $\Lambda(p, q)$  set.*

*Proof.* Choose  $(r, s) \in J$  with  $p > r$  and hence  $(p, q) > (r, s)$ . By Theorem 3.1(a) both  $E_1$  and  $E_2$  are  $\Lambda(r, s)$  sets. It follows from Theorem 4.1 and (2) that there are constants  $k_j$ ,  $j = 1, 2$ , with  $\|f_j\|_{(p,q)} \leq k_j \|f_j\|_{(r,s)}$  for all  $f_j \in T_{E_j}$ . Let  $f \in T_E$  and write  $f = f_1 + f_2$  where  $f_j \in T_{E_j}$ . If  $1 < r_1 < r < r_2 < \infty$ , then  $L^{r_2} \subseteq L(r, s) \subseteq L^{r_1}$ , and thus the M. Riesz theorem [4, 12.9.1] shows there are constants  $c_1$  and  $c_2$  such that  $\|f_j\|_{r_m} \leq c_m \|f_j\|_{r_m}$  where  $j, m = 1, 2$ . Interpolate via [11, p. 264] to conclude there exists a constant  $k$  such that  $\|f_j\|_{(r,s)} \leq k \|f_j\|_{(r,s)}$ . We see that

$$\|f\|_{(p,q)} \leq (k_1 + k_2) \|f\|_{(r,s)} \leq (k_1 + k_2) \|f\|_{(r,s)}.$$

Since  $p > r$ , it follows from Theorem 4.1 that  $E$  is a  $\Lambda(p, q)$  set.  $\square$

If  $E \subseteq \Gamma$  and  $\tau \in \Gamma$ , the translate of  $E$  by  $\tau$  is the set  $\tau E = \{\tau\gamma : \gamma \in E\}$ . Our next result shows that  $\Lambda(p, q)$  sets are translation-invariant.

**Theorem 5.3.** *If  $E$  is a  $\Lambda(p, q)$  set for some  $(p, q) \in J$ , then  $\tau E$  is a  $\Lambda(p, q)$  set for each  $\tau \in \Gamma$ .*

*Proof.* By (1) we need only verify the containment  $L^1_{\tau E} \subseteq L(p, q)_{\tau E}$ . If  $f \in L^1_{\tau E}$  and  $\gamma \in \Gamma$ , then

$$\begin{aligned} (\tau^{-1}f)^\wedge(\gamma) &= \int_G \tau^{-1}(s)f(s)\overline{\gamma}(s)d\lambda(s) \\ &= \int_G f(s)\overline{(\tau\gamma)}(s)d\lambda(s) = \hat{f}(\tau\gamma). \end{aligned}$$

If  $\gamma \notin E$ , then  $\tau\gamma \notin \tau E$  and thus  $(\tau^{-1}f)^\wedge$  vanishes off of  $E$ . Since  $f$  and  $\tau^{-1}f$  have equal distribution functions, we see that  $\tau^{-1}f \in L^1_E$  and, as  $E$  is a  $\Lambda(p, q)$  set,  $f \in L(p, q)_{\tau E}$ . □

If  $E$  is a  $\Lambda(p, q)$  set and  $F$  is a finite subset of  $\Gamma$ , then  $FE = \{\tau\gamma : \tau \in F, \gamma \in E\}$  is also a  $\Lambda(p, q)$  set. As mentioned above, one cannot replace  $F$  with an arbitrary infinite subset of  $\Gamma$  as  $\Gamma$  is not a  $\Lambda(p, q)$  set.

We now consider arithmetic and geometric properties of  $\Lambda(p, q)$  sets. It is known that Sidon sets and  $\Lambda(p)$  sets for  $p > 2$  do not contain large parts of certain sets that are themselves generalized arithmetic progressions [12, Chapter 6]. This result for  $\Lambda(p)$  sets was generalized and extended for all  $p > 1$  in [8].

**Definition 5.4** ([8, 1.1 and 1.2]). A subset  $P$  of  $\Gamma$  is called a pseudo-parallelepiped of dimension  $N$  if  $P = \prod_{i=1}^N \{\gamma_i, \tau_i\}$  where  $\gamma_i, \tau_i \in \Gamma$ ,  $1 \leq i \leq N$ . A parallelepiped  $P$  of dimension  $N$  is a pseudo-parallelepiped of dimension  $N$  such that  $P$  consists of  $2^N$  elements.

As remarked in [8, p. 144], pseudo-parallelepipeds are generalizations of arithmetic progressions of integers. Our first result is an easy generalization of [8, 1.2].

**Theorem 5.5.** *If  $E$  is a  $\Lambda(p, q)$  set for some  $(p, q) \in J$ , then there exists an integer  $N$  such that  $E$  does not contain any parallelepipeds of dimension  $N$ .*

*Proof.* This follows immediately from [8, 1.2] by noting from Theorem 3.1(c) that  $E$  is a  $\Lambda(r)$  set for each  $r \in (1, p)$ . □

The next theorem extends the result of [8, 2.4] which gives an upper bound on the cardinality of the intersection of a  $\Lambda(p)$  set and a generalized arithmetic progression. If  $F$  is a finite subset of  $\Gamma$ , let  $|F|$  denote the cardinality of  $F$ .

**Theorem 5.6.** *Let  $E$  be a  $\Lambda(p, q)$  set for some  $(p, q) \in J$ . There exist constants  $c > 0$  and  $\epsilon \in (0, 1)$  such that if  $S$  is an arithmetic progression in  $\Gamma$  of length  $N$ , then  $|E \cap S| \leq cN^\epsilon$ .*

*Proof.* This follows easily from [8, 2.3 and 2.4] since  $E$  is a  $\Lambda(r)$  set for each  $r \in (1, p)$ . □

A consequence of Theorem 5.6 is that, like  $\Lambda(p)$  sets, a  $\Lambda(p, q)$  set cannot contain a generalized arithmetic progression of arbitrary length. For  $p \in (1, 2]$  it is not known whether the union of two  $\Lambda(p)$  sets is itself a  $\Lambda(p)$  set. However [8, 2.11] shows that  $E_1 \cup E_2$  does not contain parallelepipeds of arbitrarily large dimension if  $E_1$  and  $E_2$  are  $\Lambda(p)$  sets for  $p > 1$ . It is not difficult to see that this result is also true for  $\Lambda(p, q)$  sets for all  $(p, q) \in J$ . Further connections between parallelepipeds,  $\Lambda(p)$  sets, and  $\Lambda(p, q)$  sets are also seen in [8, 4.1].

There is an interesting dichotomy in regards to the question of whether one class of  $\Lambda(p)$  sets is contained in another. If  $p \in (1, 2)$ , then each  $\Lambda(p)$  set is also a

$\Lambda(p + \epsilon)$  set for some  $\epsilon > 0$  (see [1, Main Theorem] and [7, Main Theorem]). On the other hand, it is shown in [3, Theorem 2] that, for each  $p > 2$ , there exists a  $\Lambda(p)$  subset of the integers which is not a  $\Lambda(r)$  set for any  $r > p$ . Both of these results have consequences in our study of  $\Lambda(p, q)$  sets.

**Theorem 5.7.** *If  $E$  is a  $\Lambda(p_1, q_1)$  set for some  $(p_1, q_1) \in J$  and  $p_1 \in (1, 2)$ , then  $E$  is a  $\Lambda(p, q)$  set for some  $(p, q) > (p_1, q_1)$ .*

*Proof.* Consider first the case where  $q_1 \in [1, p_1)$ . By (1),  $L(p_1, q_1) \subset L^{p_1}$  and thus  $E$  is a  $\Lambda(p_1)$  set. From [7, Main Theorem]  $E$  is  $\Lambda(p_1 + \epsilon)$  set for some  $\epsilon > 0$  and hence the result follows easily by just letting  $p = q = p_1 + \epsilon$ . Suppose now that  $q_1 > p_1$ . By (7),  $L(p_1, q_1) \subset L^r$  for all  $r \in (1, p_1)$  and thus  $E$  is a  $\Lambda(r)$  set for each such  $r$ . We will show that  $E$  is a  $\Lambda(s)$  set for some  $s > p_1$ . The conclusion will then follow from the case above.

As motivated by [7], for  $r \in (1, p_1]$  and  $n \geq 2$ , let  $s$  be defined by the equation

$$(8) \quad \frac{1}{s} = \frac{1}{r} + \left( \frac{2-r}{4r} \right) c(n) \text{ where } c(n) = \frac{\log(1 - \frac{1}{n^2})}{\log(2n^2)}.$$

Note that  $s > r$  since  $c(n) < 0$ . Rearrange (8) so that  $s = s(r, n) = \frac{4r}{4 + (2-r)c(n)}$ .

By letting  $k(r, r/2; E) = \inf\{\|f\|_r : f \in T_E, \|f\|_{r/2} \leq 1\}$  we see from [7, Remark, p. 7] that for each  $r \in (1, p_1)$ , if  $n \geq 4(k(r, r/2; E))^r$ , then  $E$  is a  $\Lambda(s)$  set. For  $n \geq 2$ , let  $r_n = p_1 - \frac{1}{2}[s(p_1, n) - p_1]$ . It is clear that  $r_n < p_1$ . Since  $c(n) \rightarrow 0^-$  as  $n \rightarrow \infty$ , it follows that  $r_n > 1$  for all sufficiently large  $n$ . Consequently for these such  $n$ ,  $E$  is a  $\Lambda(s)$  set where  $s = s(r_n, n)$ . A straightforward calculation shows that  $s(r_n, n) > p_1$  if  $n$  is sufficiently large. This completes the proof.  $\square$

**Theorem 5.8.** *For each  $(p_1, q_1) \in J$  where  $q_1 \geq p_1 > 2$ , there exists a  $\Lambda(p_1, q_1)$  set in  $\Gamma$  which is not a  $\Lambda(p, q)$  set for any  $(p, q) \in J$  such that  $p > p_1$ .*

*Proof.* Let  $(p_1, q_1) \in J$  where  $q_1 \geq p_1 > 2$ . By [3, Theorem 1] there exists a subset  $E$  of  $\Gamma$  which is a  $\Lambda(p_1)$  set but is not a  $\Lambda(r)$  set for any  $r > p_1$ . It follows from (1) that  $E$  is a  $\Lambda(p_1, q_1)$  set and from (7) that  $E$  is not a  $\Lambda(p, q)$  set if  $p > p_1$ .  $\square$

## 6. REMARKS

A pervasive theme in this paper has been to generalize results for  $\Lambda(p)$  sets by expressing them as theorems about  $\Lambda(p, q)$  sets. This theme has been developed by capitalizing on the structural similarities existing between  $L^p$  spaces and  $L(p, q)$  spaces. A similar idea is exploited in [5] and [6] where the theory of  $L^p$ -improving measures is extensively generalized and subsequently leads to the theory of Lorentz-improving measures.

Theorem 5.7 and the inclusions found in (1) suggest the following question. Suppose  $E$  is a  $\Lambda(2, q_1)$  set where  $q_1 > 2$ . Is it true that  $E$  is also a  $\Lambda(2, q)$  set for some  $q \in [1, q_1)$ ? As mentioned just before Theorem 5.7, the analogous question for  $\Lambda(p)$  sets has been answered affirmatively: if  $p \in (1, 2)$ , then each  $\Lambda(p)$  set is also a  $\Lambda(p + \epsilon)$  set for some  $\epsilon > 0$  (see [1, Main Theorem] and [7, Main Theorem]). The techniques used in these two papers for the analysis of  $\Lambda(p)$  sets do not yet appear to be modifiable so as to yield a definite answer to the above question for  $\Lambda(p, q)$  sets. Theorem 5.7 is, however, a natural result in this direction. Note that if  $E$  is a  $\Lambda(2, q_1)$  set for some  $q_1 > 2$ , then  $E$  is also a  $\Lambda(p)$  set for all  $p \in (1, 2)$  since  $L(2, q_1) \subset L^p$ . Consequently, if there does exist such a set  $E$  which is not a

$\Lambda(2, q)$  set for any  $q \in [1, q_1)$ , then  $E$  also has the interesting property that it is a  $\Lambda(p)$  set for all  $p \in (1, 2)$  yet is not a  $\Lambda(2)$  set. The reader is referred to [8] for some further issues regarding open questions on the relationship between  $\Lambda(p)$  sets for  $p \in (1, 2)$ , and  $\Lambda(2)$  sets.

A major contribution to the theory of  $\Lambda(p)$  sets is [3] where a solution to the celebrated  $\Lambda(p)$  set problem is presented. The solution asserts that, for each  $p > 2$ , there is a  $\Lambda(p)$  set of integers which is not a  $\Lambda(r)$  set for any  $r > p$ . The analogous problem for  $\Lambda(p, q)$  sets is as follows. Does there exist a  $\Lambda(p_1, q_1)$  set for some  $(p_1, q_1) > (2, 2)$  which is not a  $\Lambda(p, q)$  set for any  $(p, q) > (p_1, q_1)$ ? Theorem 5.8 is a minor result concerning this question. It is seen in [3] that the solution to the  $\Lambda(p)$  set problem is obtained via intricate probabilistic methods. It is quite unclear as to whether these kinds of techniques can be modified so as to provide a solution to the problem for  $\Lambda(p, q)$  sets.

A popular and successful direction for lacunary research is to generalize results for the integers and rephrase them in terms of the dual of a compact abelian group. A further degree of extension is attained when results for the dual of a compact abelian group can be abstracted in terms of the dual object of a compact group or hypergroup. There are a number of possibilities for further research suggested by these remarks and the results for  $\Lambda(p, q)$  sets we have developed in this paper.

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