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CHAOTIC POLYNOMIALS ON FRÉCHET SPACES

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ABSTRACT. Contrary to the case of polynomials on Banach spaces, in which it is known that no hypercyclic homogeneous polynomial of degree $m \geq 2$ exists on any Banach space, we construct for each $m \geq 2$ a chaotic m-homogeneous polynomial P on the Fréchet space $\mathcal{H}(\mathbb{C})$.

A map $T : X \to X$ on a metric space X is chaotic (see [3]) if (a) T is transitive (which for complete separable X is equivalent to the existence of $x \in X$ whose orbit $\mathrm{Orb}(T, x) := \{x, Tx, T^2x, \dots\}$ is dense in X), (b) the periodic points of T are dense in X, and (c) T has sensitive dependence on initial conditions. Banks et al. [1] showed that (c) is redundant in the definition of Devaney.

We present an example of a chaotic m-homogeneous continuous polynomial $P : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$. This must be compared with a result of Bernardes [2], who showed that for $m > 1$ there are no continuous m-homogeneous polynomials admitting a vector with dense orbit (hypercyclic in the usual terminology) on any Banach space. The result of Bernardes is a consequence of the inequality $\| P^n x \| \leq \| P \|^{1+m+\dots+m^{n-1}} \| x \|^{m^n}$ for every $x \in X$ if P is a continuous m-homogeneous polynomial $(m > 1)$ on a Banach space X, since no hypercyclic vector for P could lie on the ball centered at 0 of radius $r := 1/ \| P \|$. This implies that P does not admit any hypercyclic vector.

On $\mathcal{H}(\mathbb{C})$ we consider the increasing sequence of norms $(\| \cdot \|_k)_k$ defined by

$$\| f \|_k := \sup_{j \geq 0} \frac{|f^{(j)}(0)|}{j!} k^j, \qquad k \in \mathbb{N}, \quad f \in \mathcal{H}(\mathbb{C}),$$

which define the natural Fréchet topology on $\mathcal{H}(\mathbb{C})$.

Our notation is standard. We refer to the monograph [6] for Fréchet spaces and to [4] for polynomials on locally convex spaces.

Theorem 1. *For each m natural ($m \geq 2$) there exists a chaotic m-homogeneous polynomial $P : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$.*

Proof. Let us define $P : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ by $(Pf)(z) := \sum_{j \geq 0} \frac{(f^{(j+1)}(0))^m}{j!} z^j$ for every $f \in \mathcal{H}(\mathbb{C})$ and for every $z \in \mathbb{C}$. P is obviously a well-defined m-homogeneous polynomial. We first prove that P is hypercyclic.

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Indeed, by the hypercyclicity of the derivative operator $D : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ (see [5]), there exists $x \in \mathcal{H}(\mathbb{C})$, a hypercyclic vector for D. We denote by $x_j := x^{(j)}(0)$, $j \in \mathbb{N}$, and pick an increasing sequence $(n_k)_k$ of natural numbers such that $(D^{n_k}x)_k$ is dense in $\mathcal{H}(\mathbb{C})$ and

(i) $\| \sum_{j \geq n_{k+1}-n_k} \frac{x_{j+n_k}}{j!} z^j \|_k < \frac{1}{k}$,

(ii) $\sup \left\{ \frac{k^j}{j!} \ / \ j \geq n_{k+1} - n_k \right\} < \frac{1}{k}$,

for each $k \in \mathbb{N}$.

Now we select $(y_j)_{j \geq 0}$ such that

$$y_j^{m^{n_k}} = x_j, \quad n_k \leq j < n_{k+1} \ \ (n_0 := 0) \ \ \forall k \in \mathbb{N}.$$

Then

$$y := y(z) := \sum_{j \geq 0} \frac{y_j}{j!} z^j \in \mathcal{H}(\mathbb{C}),$$

since $|y_j| \leq 1 + |x_j|$, $j \geq 0$. Moreover, the selection of $(y_j)_{j \geq 0}$ and the definition of P imply

(iii) $(P^{n_k}y)^{(j)}(0) = (D^{n_k}x)^{(j)}(0)$, $0 \leq j < n_{k+1} - n_k$, $\forall k \in \mathbb{N}$.

Given $n \in \mathbb{N}$, $f \in \mathcal{H}(\mathbb{C})$ and $\varepsilon > 0$, we fix $k > n$ such that $1/k < \varepsilon/4$ and $\| D^{n_k}x - f \|_n < \varepsilon/4$. Then, from (i), (ii) and (iii), we get

$$\| P^{n_k}y - f \|_n$$

$$\leq \| P^{n_k}y - \sum_{n_{k+1}-n_k > j \geq 0} \frac{x_{j+n_k}}{j!} z^j \|_n + \| \sum_{j \geq n_{k+1}-n_k} \frac{x_{j+n_k}}{j!} z^j \|_n + \| D^{n_k}x - f \|_n$$

$$< \| P^{n_k}y - \sum_{n_{k+1}-n_k > j \geq 0} \frac{(D^{n_k}x)^{(j)}(0)}{j!} z^j \|_k + \frac{1}{k} + \frac{\varepsilon}{4}$$

$$< \sup \left\{ \frac{|y_{j+n_k}|^{m^{n_k}}}{j!} k^j \ / \ j \geq n_{k+1} - n_k \right\} + \frac{\varepsilon}{2}$$

$$\leq \sup \left\{ \frac{1 + |x_{j+n_k}|}{j!} k^j \ / \ j \geq n_{k+1} - n_k \right\} + \frac{\varepsilon}{2}$$

$$< \sup \left\{ \frac{|x_{j+n_k}|}{j!} k^j \ / \ j \geq n_{k+1} - n_k \right\} + \frac{1}{k} + \frac{\varepsilon}{2} < \varepsilon.$$

This implies that P is hypercyclic. To conclude the proof we show that the periodic vectors of P are dense in $\mathcal{H}(\mathbb{C})$. We are done if, given $m \in \mathbb{N}$ and an arbitrary polynomial $p(z) = \sum_{j=1}^{n} \frac{\alpha_j}{j!} z^j$, we find $u \in \mathcal{H}(\mathbb{C})$ periodic for P such that $\| u - p \|_m < 1/m$. To do this, take $k > n$ satisfying

(iv) $\sup \left\{ \frac{M}{j!} m^j \ / \ j \geq k \right\} < \frac{1}{m}$,

where $M := \sup\{1 + |\alpha_j| \ / \ 1 \leq j \leq n\}$. Choose $(u_j)_{j \geq 0}$ such that

$$u_j := \alpha_j \ \ (\alpha_j := 0 \quad \text{if} \quad j > n), \quad u_{j+rk}^{m^k} = u_{j+(r-1)k} \quad \text{if} \quad 0 \leq j < k \quad \text{and} \quad r \geq 1.$$

We have that $u(z) := \sum_{j\geq 0} \frac{u_j}{j!} z^j \in \mathcal{H}(\mathbb{C})$, since $|u_j| \leq 1 + |\alpha_j|$, $j \geq 0$. By the selection of $(u_j)_{j\geq 0}$ and (iv) we also get that u is k-periodic for P and

$$\| u - p \|_m = \| \sum_{j\geq k} \frac{u_j}{j!} z^j \|_m \leq \sup \left\{ \frac{M}{j!} m^j \quad / \quad j \geq k \right\} < \frac{1}{m}.$$

We conclude that P is chaotic. □

References

[1] J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, *On Devaney's definition of chaos*, Amer. Math. Monthly **99** (1992), 332-334. MR **93d:**54059

[2] N. C. Bernardes, *On orbits of polynomial maps in Banach spaces*, Quaestiones Math. (to appear).

[3] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, 1989. MR **91a:**58114

[4] S. Dineen, *Complex Analysis in Locally Convex Spaces*, North-Holland, Amsterdam-New York-Oxford, 1981. MR **84b:**46050

[5] G. R. MacLane, *Sequences of derivatives on normal families*, J. Analyse Math. **2** (1952), 72-87. MR **14:**741d

[6] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Oxford Univ. Press, 1997. CMP 98:04

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