

## ON TIGHTNESS AND DEPTH IN SUPERATOMIC BOOLEAN ALGEBRAS

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ABSTRACT. We introduce a large cardinal property which is consistent with  $L$  and show that for every superatomic Boolean algebra  $B$  and every cardinal  $\lambda$  with the large cardinal property, if  $\text{tightness}^+(B) \geq \lambda^+$ , then  $\text{depth}(B) \geq \lambda$ . This improves a theorem of Dow and Monk.

In [DM, Theorem C], Dow and Monk have shown that if  $\lambda$  is a Ramsey cardinal (see [J, p.328]), then every superatomic Boolean algebra with tightness at least  $\lambda^+$  has depth at least  $\lambda$ . Recall that a Boolean algebra  $B$  is *superatomic* iff every homomorphic image of  $B$  is atomic. The *depth* of  $B$  is the supremum of all cardinals  $\lambda$  such that there is a sequence  $(b_\alpha : \alpha < \lambda)$  in  $B$  with  $b_\beta < b_\alpha$  for all  $\alpha < \beta < \lambda$  (a *well-ordered chain* of length  $\lambda$ ). Then  $\text{depth}^+$  of  $B$  is the first cardinal  $\lambda$  such that there is no well-ordered chain of length  $\lambda$  in  $B$ . The *tightness* of  $B$  is the supremum of all cardinals  $\lambda$  such that  $B$  has a *free* sequence of length  $\lambda$ , where a sequence  $(b_\alpha : \alpha < \lambda)$  is called *free* provided that if  $\Gamma$  and  $\Delta$  are finite subsets of  $\lambda$  such that  $\alpha < \beta$  for all  $\alpha \in \Gamma$  and  $\beta \in \Delta$ , then

$$\bigcap_{\alpha \in \Gamma} -b_\alpha \cap \bigcap_{\beta \in \Delta} b_\beta \neq 0.$$

By  $\text{tightness}^+(B)$  we denote the first cardinal  $\lambda$  for which there is no free sequence of length  $\lambda$  in  $B$ .

For  $b \in B$  we sometimes write  $b^0$  for  $-b$  and  $b^1$  for  $b$ .

We improve Theorem C from [DM] in two directions. We introduce a large cardinal property which is much weaker than Ramseyness and even consistent with  $L$  (the constructible universe) and show that in Theorem C from [DM] it suffices to assume that  $\lambda$  has this property. Moreover we show that it suffices to assume  $\text{tightness}^+(B) \geq \lambda^+$  instead of  $\text{tightness}(B) \geq \lambda^+$  to conclude that  $\text{depth}(B) \geq \lambda$ . In particular we get:

**Theorem 1.** *Suppose that  $0^\sharp$  exists. Let  $B$  be a superatomic Boolean algebra in the constructible universe  $L$ , and let  $\lambda$  be an uncountable cardinal in  $V$ . Then in  $L$  it is true that  $\text{tightness}^+(B) \geq \lambda^+$  implies that  $\text{depth}^+(B) \geq \lambda$ .*

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For the theory of  $0^\sharp$  see [J, §30]. Note that  $\lambda$  as in Theorem 1 is a limit cardinal in  $L$ ; hence it suffices to show that, in  $L$ ,  $\text{depth}(B) \geq \kappa$  for all cardinals  $\kappa < \lambda$ . As was the case with the proof of Theorem C of [DM], we can't show that, under the assumptions of Theorem 1,  $\text{depth}(B) = \lambda$  is attained, i.e. that there is a well-ordered chain of length  $\lambda$ .

For the proof we consider the following large cardinal property:

**Definition 2.** Let  $\lambda, \kappa, \theta$  be infinite cardinals, and let  $\gamma$  be an ordinal. The relation  $R_\gamma(\lambda, \kappa, \theta)$  is defined as follows:

For every  $c : [\lambda]^{<\omega} \rightarrow \theta$  there exists  $A \subseteq \lambda$  of order-type  $\gamma$ , such that for every  $u \in [A]^{<\omega}$  there exists  $B \subseteq \lambda$  of order-type  $\kappa$  such that  $\forall w \in [B]^{|u|} \quad c(w) = c(u)$ .

**Lemma 3.** Assume  $R_\gamma(\lambda, \kappa, \theta)$ , where  $\gamma$  is a limit ordinal. For every  $c : [\lambda]^{<\omega} \rightarrow \theta$  there exists  $A \subseteq \lambda$  as in the definition of  $R_\gamma(\lambda, \kappa, \theta)$  such that additionally  $c \upharpoonright [A]^n$  is constant for every  $n < \omega$ .

*Proof.* Define  $c'$  on  $[\lambda]^{<\omega}$  by

$$c' \{ \beta_0, \dots, \beta_{n-1} \} = \{ (v, c \{ \beta_i : i \in v \}) : v \subseteq n \}.$$

As  $\theta$  is infinite we can easily code the values of  $c'$  as ordinals in  $\theta$  and therefore apply  $R_\gamma(\lambda, \kappa, \theta)$  to it. We get  $A \subseteq \lambda$  of order-type  $\gamma$ . We shall prove that  $c \upharpoonright [A]^n$  is constant, for every  $n < \omega$ . Fix  $w_1, w_2 \in [A]^n$ . Since  $\gamma$  is a limit, without loss of generality we may assume that  $\max(w_1) < \min(w_2)$ . Let  $w = w_1 \cup w_2$ . By Definition 2 there exists  $B \subseteq \lambda$ , o.t. $B = \kappa$ , such that  $c' \upharpoonright [B]^{2n}$  is constant with value  $c'(w)$ . Let  $(\beta_\nu : \nu < \kappa)$  be the increasing enumeration of  $B$ . We have

$$c' \{ \beta_0, \dots, \beta_{2n-1} \} = c' \{ \beta_n, \dots, \beta_{3n-1} \}.$$

By the definition of  $c'$  we get

$$c \{ \beta_0, \dots, \beta_{n-1} \} = c \{ \beta_n, \dots, \beta_{2n-1} \} =: c_0.$$

This information is coded in  $c' \{ \beta_0, \dots, \beta_{2n-1} \}$ , i.e.

$$(\{0, \dots, n-1\}, c_0), (\{n, \dots, 2n-1\}, c_0) \in c' \{ \beta_0, \dots, \beta_{2n-1} \}.$$

As  $c' \{ \beta_0, \dots, \beta_{2n-1} \} = c'(w)$ , we conclude  $c(w_1) = c(w_2) = c_0$ .  $\square$

**Theorem 4.** Assume  $R_\gamma(\lambda, \kappa, \omega)$ , where  $\gamma$  is a limit ordinal. If  $B$  is a Boolean algebra and  $(a_\nu : \nu < \lambda)$  is a sequence in  $B$ , then one of the following holds:

- (a) there exists  $A \subseteq \lambda$ , o.t. $(A) = \gamma$ , such that  $(a_\nu : \nu \in A)$  is independent;
- (b) there exist  $n < \omega$  and a strictly increasing sequence  $(\beta_\nu : \nu < \kappa)$  in  $\lambda$  such that, letting

$$(*) \quad b_\nu = \bigcup_{k < n} \bigcap_{l < n} a_{\beta_{n+2\nu+k+l}},$$

we have that  $(b_\nu : \nu < \kappa)$  is constant;

- (c) there exists a strictly decreasing sequence in  $B$  of length  $\kappa$ .

**Corollary 5.** Assume  $R_\gamma(\lambda, \kappa, \omega)$ , where  $\gamma$  is a limit ordinal. If  $B$  is a superatomic Boolean algebra, then  $\text{tightness}^+(B) > \lambda$  implies  $\text{Depth}^+(B) > \kappa$ .

*Proof of Corollary 5.* Let  $(a_\nu : \nu < \lambda)$  be a free sequence in  $B$ . As a superatomic Boolean algebra does not have an infinite independent subset, (a) is impossible. Suppose (b) were true. Define  $b_\nu$  as in (\*). Clearly we have

$$-b_\nu \geq \bigcap_{k,l < n} a_{\beta_n 2^{\nu+nk+l}}^0, \text{ and}$$

$$b_\nu \geq \bigcap_{k,l < n} a_{\beta_n 2^{\nu+nk+l}}.$$

Hence if  $\nu < \mu$  and  $b_\nu = b_\mu$ , we obtain

$$0 = -b_\nu \cap b_\mu \geq \bigcap_{k,l < n} a_{\beta_n 2^{\nu+nk+l}}^0 \cap \bigcap_{k,l < n} a_{\beta_n 2^{\mu+nk+l}}.$$

This contradicts freeness of  $(a_\nu : \nu < \kappa)$ . We conclude that (c) must hold. □

*Proof of Theorem 4.* Define  $c : [\lambda]^{<\omega} \rightarrow [^{<\omega}2]^{<\omega}$  by

$$c\{\beta_0 < \dots < \beta_{n-1}\} = \{\eta \in {}^n 2 : \bigcap_{i < n} a_{\beta_i}^{\eta(i)} = 0\}.$$

Note that  $c\{\beta_0 < \dots < \beta_{n-1}\} = c\{\alpha_0 < \dots < \alpha_{n-1}\}$  implies that  $\{a_{\beta_0}, \dots, a_{\beta_{n-1}}\}$  and  $\{a_{\alpha_0}, \dots, a_{\alpha_{n-1}}\}$  have the same quantifier-free diagram, i.e. for every quantifier-free formula  $\phi(x_0, \dots, x_{n-1})$  in the language of Boolean algebra,

$$B \models \phi[a_{\beta_0}, \dots, a_{\beta_{n-1}}] \Leftrightarrow B \models \phi[a_{\alpha_0}, \dots, a_{\alpha_{n-1}}].$$

Let  $A \subseteq \lambda$  be as guaranteed for  $c$  by  $R_\gamma(\lambda, \kappa, \omega)$ . By Lemma 3 we may assume that  $c[A]^n$  is constant, for every  $n < \omega$ .

If  $(a_\alpha : \alpha \in A)$  is independent, we are done. Therefore we may assume that this is false. For  $m < \omega$  define

$$\Gamma_m = \{\eta \in {}^m 2 : \exists\{\beta_0 < \dots < \beta_{m-1}\} \subseteq A \quad \bigcap_{i < m} a_{\beta_i}^{\eta(i)} = 0\}.$$

By assumption, in the definition of  $\Gamma_m$  the existential quantifier can be replaced by a universal one to give the same set. There exists  $m < \omega$  such that  $\Gamma_m \neq \emptyset$ . Define

$$\Gamma'_m = \{\eta \in \Gamma_m : \text{no proper subsequence of } \eta \text{ belongs to } \bigcup_{k < m} \Gamma_k\}.$$

By Kruscal's Theorem [K], we have that  $\bigcup_{m < \omega} \Gamma'_m$  is finite. Let  $n^*$  be minimal such that  $\bigcup_{m < \omega} \Gamma'_m = \bigcup_{m < n^*} \Gamma'_m$ . Then clearly we have that, for every  $m < \omega$  and  $\eta \in \Gamma_m$ ,  $\eta$  has a subsequence in  $\bigcup_{k < n^*} \Gamma'_k$ . Let  $m^* = (n^*)^2$ , and let

$$\tau(x_0, \dots, x_{m^*-1}) = \bigcup_{l < n^*} \bigcap_{k < n^*} x_{n^*l+k}.$$

*Claim 1.* If  $\eta \in {}^{m^*}2$ ,  $t \in \{0, 1\}$ , and  $\tau[\eta(0), \dots, \eta(m^* - 1)] = t$  in the Boolean algebra  $\{0, 1\}$ , then  $|\{i < m^* : \eta(i) = t\}| \geq n^*$ . □

Let  $(\beta_\nu : \nu < \gamma)$  be the strictly increasing enumeration of  $A$ , and define

$$b_\nu = \tau[a_{\beta_{m^*\nu}}, a_{\beta_{m^*\nu+1}}, \dots, a_{\beta_{m^*\nu+m^*-1}}],$$

for every  $\nu < \gamma$ , where the evaluation of  $\tau$  takes place in  $B$ , of course. It is easy to see that the sequence  $(b_\nu : \nu < \gamma)$  inherits from  $(a_{\beta_\nu} : \nu < \gamma)$  the property that any two finite subsequences of same length have the same quantifier-free diagram.

*Claim 2.* If  $\eta \in \Gamma_n$ , then  $\bigcap_{i < n} b_i^{\eta(i)} = 0$ .

*Proof of Claim 2.* Otherwise there exists an ultrafilter  $D$  on  $B$  such that  $\bigcap_{i < n} b_i^{\eta(i)} \in D$ . Define  $\zeta \in {}^{nm^*}2$  by  $\zeta(i) = 1$  iff  $a_{\beta_i} \in D$ . Then  $\bigcap_{i < nm^*} a_{\beta_i}^{\zeta(i)} \in D$ , and hence  $\zeta \notin \Gamma_{nm^*}$ . Let  $h : B \rightarrow B/D = \{0, 1\}$  be the canonical homomorphism induced by  $D$ . We calculate

$$\begin{aligned} 1 &= h(\bigcap_{i < n} b_i^{\eta(i)}) = \bigcap_{i < n} h(b_i)^{\eta(i)} = \bigcap_{i < n} \tau[h(a_{\beta_{m^*i}}, \dots, h(a_{\beta_{m^*(i+1)-1}}))]^{\eta(i)} \\ &= \bigcap_{i < n} \tau[\zeta(m^*i), \dots, \zeta(m^*i + k), \dots, \zeta(m^*(i + 1) - 1)]^{\eta(i)}. \end{aligned}$$

We conclude that  $\tau[\zeta(m^*i), \dots, \zeta(m^*i + k), \dots, \zeta(m^*(i + 1) - 1)] = \eta(i)$ , for all  $i < n$ , and hence by Claim 1 we can choose  $j_i \in [m^*i, m^*(i + 1))$  such that  $\zeta(j_i) = \eta(i)$ . Clearly  $i_0 < i_1$  implies that  $j_{i_0} < j_{i_1}$ . But this implies  $\zeta \in \Gamma_{nm^*}$ , a contradiction.  $\square$

*Claim 3.* If  $t < \omega$ ,  $\eta \in \Gamma_n$ ,  $0 = k_0 < k_1 < \dots < k_t = n$ , and  $\eta \upharpoonright [k_i, k_{i+1})$  is constant for all  $i < t$ , and if  $\rho \in {}^t2$  is defined by  $\rho(i) = \eta(k_i)$ , then  $\bigcap_{i < t} b_i^{\rho(i)} = 0$ .

*Proof of Claim 3.* Wlog we may assume that  $\eta \in \Gamma'_n$  for some  $n < n^*$ . Indeed, otherwise we can find  $m < n^*$ ,  $\eta' \in \Gamma'_m$  and some increasing  $h : m \rightarrow n$  such that  $\eta'(i) = \eta(h(i))$ , for all  $i < m$ . Then  $\{h^{-1}[k_i, k_{i+1}) : i < t\}$  equals  $\{[l_i, l_{i+1}) : i < s\}$  for some  $l_0 = 0 < l_1 < \dots < l_{s-1} = m$ . Note that  $\eta' \upharpoonright [l_i, l_{i+1})$  is constant, and letting  $\rho' \in {}^s2$  be defined by  $\rho'(i) = \eta'(l_i)$ , we have  $\rho'(i) = \rho(h(i))$ . Hence  $\bigcap_{i < s} b_i^{\rho'(i)} = 0$  implies  $\bigcap_{i < t} b_i^{\rho(i)} = 0$ .

Therefore we assume  $\eta \in \Gamma'_n$ , for some  $n < n^*$ . Suppose we had  $\bigcap_{i < t} b_i^{\rho(i)} > 0$ . Let  $D$  be an ultrafilter on  $B$  containing  $\bigcap_{i < t} b_i^{\rho(i)}$ . Let  $h : B \rightarrow B/D$  be the canonical homomorphism. Define  $\zeta \in {}^{tm^*}2$  such that  $\zeta(i) = 1$  iff  $a_i \in D$ . Hence  $\zeta \notin \Gamma_{tm^*}$ . We get

$$h(\bigcap_{i < t} b_i^{\rho(i)}) = \bigcap_{i < t} \tau[\zeta(im^*), \dots, \zeta((i + 1)m^* - 1)]^{\rho(i)} = 1.$$

Hence by Claim 1,

$$\forall i < t \exists a_i \in [\{im^*, \dots, (i + 1)m^* - 1\}]^{n^*} \forall j \in a_i \quad \zeta(j) = \rho(i).$$

Define  $\mu \in {}^{tn^*}2$  by  $\mu(j) = \rho(i)$  iff  $j \in [in^*, (i + 1)n^*)$ . Then  $\mu$  is a subsequence of  $\zeta$  and therefore  $\mu \notin \Gamma_{tn^*}$ . But also  $\eta$  is a subsequence of  $\mu$ , and hence  $\eta \notin \Gamma_n$ , a contradiction.  $\square$

*Claim 4.* Suppose  $\rho \in {}^t2$  and  $\bigcap_{i < t} b_i^{\rho(i)} = 0$ . Let  $\zeta \in {}^{m^*t}2$  be defined such that  $\zeta(m^*i) = \rho(i)$  and  $\zeta \upharpoonright [m^*i, m^*(i + 1))$  is constant for every  $i < t$ . Then  $\zeta \in \Gamma_{m^*t}$ .

*Proof of Claim 4.* Otherwise,  $\bigcap_{i < m^*t} a_i^{\zeta(i)} > 0$ . Let  $D$  be an ultrafilter containing  $\bigcap_{i < m^*t} a_i^{\zeta(i)}$ . Let  $h : B \rightarrow B/D$  be the canonical homomorphism. We have

$$h(\bigcap_{i < t} b_i^{\rho(i)}) = \bigcap_{i < t} \tau[\zeta(m^*i), \dots, \zeta(m^*(i + 1) - 1)]^{\rho(i)} = \bigcap_{i < t} \tau[\rho(i), \dots, \rho(i)]^{\rho(i)} = 1.$$

This is a contradiction.  $\square$

Since we assume that  $(a_\alpha : \alpha \in A)$  is not independent, by Claim 2 we can find  $k^* < \omega$  minimal such that for some  $\rho^* \in {}^{k^*}2$ ,  $\bigcap_{i < k^*} b_i^{\rho^*(i)} = 0$ . Note that  $\rho^*(i + 1) \neq \rho^*(i)$  for every  $i < k^* - 1$ . Indeed, otherwise let  $\zeta \in {}^{m^*k^*}2$  be defined

as in Claim 4. So  $\zeta \in \Gamma_{m^*k^*}$ . By Claim 3 we can find  $\rho'$  of shorter length than  $\rho^*$  such that  $\bigcap_{i < |\rho'|} b_i^{\rho'(i)} = 0$ , contradicting the minimal choice of  $k^*$ .

Suppose first that  $k^* = 1$ . We conclude that  $(b_\nu : \nu < \gamma)$  either is constantly 1 or 0. The main part of the definition of  $R_\gamma(\lambda, \kappa, \omega)$  then gives a sequence of length  $\kappa$  as desired in (b) of Theorem 4.

Second suppose  $k^* > 1$ . If  $\bigcap_{i < k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-2} \cap b_{k^*-1}^0 = 0$  and  $\bigcap_{i < k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-2}^0 \cap b_{k^*-1} = 0$ , then  $\bigcap_{i < k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-2} = \bigcap_{i < k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-1}$ , and an application of the main part of the definition of  $R_\gamma(\lambda, \kappa, \omega)$  gives a sequence as desired in (b).

Otherwise, if  $\rho^*(k^* - 2) = 1$  and  $\rho^*(k^* - 1) = 0$ , then

$$\bigcap_{i < k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-2} < \bigcap_{i < k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-1},$$

and applying the definition gives (c). A similar argument applies if  $\rho^*(k^* - 2) = 0$  and  $\rho^*(k^* - 1) = 1$ .

**Theorem 6.** *Assume the following:*

- (1)  $0^\sharp$  exists,
- (2)  $V \models \lambda$  is an uncountable cardinal,
- (3)  $\kappa, \theta < \lambda$ , and  $L \models \kappa$  is a regular cardinal.

*Then  $L \models R_\omega(\lambda, \kappa, \theta)$ .*

*Proof.* Let  $c : [\lambda]^{<\omega} \rightarrow \theta$ ,  $c \in L$ , be arbitrary.

Let  $Y$  be the set of all  $w \in [\lambda]^{<\omega}$  such that for every  $n \leq |w|$  and  $u \in [w]^n$  there exists  $B \subseteq \lambda$  of order-type  $\kappa$  in  $L$  such that  $\forall v \in [B]^n \ c(u) = c(v)$ . Clearly  $Y \in L$ . □

*Claim 1.* If in  $V$  there exists  $A \in [\lambda]^\omega$  with  $[A]^{<\omega} \subseteq Y$ , then  $L \models R_\omega(\lambda, \kappa, \theta)$ .

*Proof of Claim 1.* Let  $T$  be the set of all one-to-one sequences  $\rho \in {}^{<\omega}\lambda$  with  $\text{ran}(\rho) \in Y$ , ordered by extension. Then  $T$  is a tree and by assumption,  $T$  has an  $\omega$ -branch in  $V$ . By absoluteness,  $T$  has an  $\omega$ -branch  $b$  in  $L$ . Then  $\text{ran}(b)$  (or some subset) witnesses  $L \models R_\omega(\lambda, \kappa, \theta)$ . □

Let  $(i_\nu : \nu < \lambda^+)$  be the increasing enumeration of the club of indiscernibles of  $L_{\lambda^+}$ . Then  $(i_\nu : \nu < \lambda)$  is the club of indiscernibles of  $L_\lambda$ . As  $c \in L_{\lambda^+}$  there exist ordinals  $\xi_0 < \dots < \xi_{p-1} < \lambda \leq \xi_p < \dots < \xi_{q-1} < \lambda^+$  and a Skolem term  $t_c$  such that

$$L_{\lambda^+} \models c = t_c[i_{\xi_0}, \dots, i_{\xi_{q-1}}].$$

By indiscernibility and remarkability (see [J, p.345]) it easily follows that if  $\alpha^* = \max\{\xi_{p-1}, \theta\} + 1$ , then  $c|[\{i_\nu : \alpha^* \leq \nu < \lambda\}]^n$  is constant for every  $n < \omega$ , say with value  $c_n$ . Let  $n < \omega$  be arbitrary. Let  $\delta_0 = i_{\alpha^*+\kappa}$ ,  $\delta_1 = i_{\alpha^*+\kappa+1}, \dots, \delta_{n-1} = i_{\alpha^*+\kappa+n-1}$ .

*Claim 2.* For every  $\alpha < \delta_0$  there exists a limit  $\delta$ ,  $\alpha < \delta < \delta_0$ , such that for all  $\beta_0 < \dots < \beta_{n-2} < \delta$  the following hold:

- (\*)<sub>0</sub>  $c\{\delta, \delta_1, \dots, \delta_{n-1}\} = c\{\delta_0, \dots, \delta_{n-1}\} (= c_n)$ ,
- (\*)<sub>1</sub>  $c\{\beta_0, \delta, \delta_2, \dots, \delta_{n-1}\} = c\{\beta_0, \delta_1, \dots, \delta_{n-1}\}$ ,
- (\*)<sub>2</sub>  $c\{\beta_0, \beta_1, \delta, \delta_3, \dots, \delta_{n-1}\} = c\{\beta_0, \beta_1, \delta_2, \dots, \delta_{n-1}\}$ ,
- ...
- (\*)<sub>n-1</sub>  $c\{\beta_0, \dots, \beta_{n-2}, \delta\} = c\{\beta_0, \dots, \beta_{n-2}, \delta_{n-1}\}$ .

*Proof of Claim 2.* Let  $\alpha < \delta_0$  be arbitrary. Choose  $\gamma < \kappa$  such that  $\gamma$  is a limit and  $i_{\alpha^*+\gamma} > \alpha$ , and let  $\delta = i_{\alpha^*+\gamma}$ .

Then clearly  $(*)_0$  holds.

In order to prove  $(*)_1$ , let  $\beta < \delta$  be arbitrary. There exist ordinals  $\nu_0 < \dots < \nu_{k-1} < \alpha^* + \gamma$  and a Skolem term  $t_\beta$  such that

$$t_\beta^{L_\lambda}[i_{\nu_0}, \dots, i_{\nu_{k-1}}] = \beta.$$

Moreover there exist ordinals  $\mu_0 < \dots < \mu_{l-1} < \alpha^*$  and a Skolem term  $t$  such that

$$(+)\quad L_{\lambda^+} \models t[i_{\mu_0}, \dots, i_{\mu_{l-1}}] = t_c[i_{\xi_0}, \dots, i_{\xi_{q-1}}]\{t_\beta[i_{\nu_0}, \dots, i_{\nu_{k-1}}], \delta_1, \dots, \delta_{n-1}\}.$$

Note that all indices of occurring indiscernibles, except for  $\delta_1, \dots, \delta_{n-1}$ , either are at least  $\lambda$  or else below  $\alpha^* + \gamma$ . We conclude that, in (+),  $\delta_1$  can be replaced by  $\delta$ . The resulting statement is

$$c\{\beta, \delta_1, \dots, \delta_{n-1}\} = c\{\beta, \delta, \delta_2, \dots, \delta_{n-1}\},$$

as desired.

The proof of  $(*)_2$ - $(*)_{n-1}$  is similar.  $\square$

It is clear that the statement of Claim 2 is absolute. Hence it is also true in  $L$ . Using this we shall prove that  $\{\{i_\nu : \alpha^* \leq \nu < \lambda\}\}^{<\omega} \subseteq Y$ . By Claim 1, this will suffice. We only have to prove that for every  $n < \omega$  there exists  $B \subseteq \lambda$  of order-type  $\kappa$  such that  $B \in L$  and  $\forall v \in [B]^n \quad c(v) = c_n$ . Fix  $n < \omega$ . Working in  $L$ , we construct  $B$  inductively as  $\{\gamma_\nu : \nu < \kappa\}$ .

Fix  $\delta_0 < \delta_1 < \dots < \delta_{n-2} < \lambda$  as above. Apply Claim 2 in  $L$  with  $\alpha = 0$  and obtain  $\gamma_0 \in (0, \delta_0)$ . Suppose we have gotten  $(\gamma_\nu : \nu < \mu)$  for some  $\mu < \kappa$ . Let  $\gamma^* = \sup_{\nu < \mu} \gamma_\nu + 1$ . Since  $\text{cf}^L(\delta_0) \geq \kappa$  and  $(\gamma_\nu : \nu < \mu) \in L$ , we have that  $\gamma^* < \delta_0$ . Apply Claim 2 with  $\alpha = \gamma^*$  and get  $\gamma_\mu \in (\gamma^*, \delta_0)$ .

We claim that  $(\gamma_\nu : \nu < \kappa)$  is as desired. Indeed, let  $\{\gamma_{\nu_0} < \gamma_{\nu_1} < \dots < \gamma_{\nu_{n-1}}\}$  be arbitrary. We have

$$\begin{aligned} c\{\gamma_{\nu_0}, \dots, \gamma_{\nu_{n-1}}\} &=^{(*)_{n-1}} c\{\gamma_{\nu_0}, \dots, \gamma_{\nu_{n-2}}, \delta_{n-1}\} \\ &=^{(*)_{n-2}} c\{\gamma_{\nu_0}, \dots, \gamma_{\nu_{n-3}}, \delta_{n-2}, \delta_{n-1}\} \\ &= \dots \\ &=^{(*)_1} c\{\gamma_{\nu_0}, \delta_1, \dots, \delta_{n-1}\} \\ &=^{(*)_0} c_n. \quad \square \end{aligned}$$

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