PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 12, Pages 3517–3524
S 0002-9939(99)04986-2

$\mathbf{Z}_n$-GRADED POLYNOMIAL IDENTITIES OF THE FULL MATRIX ALGEBRA OF ORDER n

SERGEI YU. VASILOVSKY

(Communicated by Ken Goodearl)

Abstract. The algebra $M_n(F)$ of all $n \times n$ matrices over a field F has a natural $\mathbf{Z}_n$-grading $M_n(F) = \sum_{\alpha \in \mathbf{Z}_n} \bigoplus \mathcal{M}_n^{(\alpha)}$. In this paper graded identities of the $\mathbf{Z}_n$-graded algebra $M_n(F)$ over a field of characteristic zero are studied. It is shown that all the $\mathbf{Z}_n$-graded polynomial identities of $M_n(F)$ follow from the following:

$$x_1x_2 - x_2x_1 = 0,\ \alpha(x_1) = \alpha(x_2) = \bar{0};$$

$$x_1xx_2 - x_2xx_1 = 0,\ \alpha(x_1) = \alpha(x_2) = -\alpha(x).$$

Introduction

Let F be a field of characteristic zero, and let $M_n(F)$ be the algebra of all square matrices of order n over F. The ideal $T(M_n(F))$ of all polynomial identities of the algebra $M_n(F)$ plays an important role in the theory of varieties of associative algebras. As follows from Kemer's result [7] about the finite basis property of identities of associative algebras over a field of characteristic zero, all polynomial identities of $M_n(F)$ follow from finitely many basic identities in $T(M_n(F))$ for any n. However, the problem of finding an explicit finite basis for the ideal $T(M_n(F))$ has so far been resolved only for $n = 2$ (cf. [5], [9]), and there is still no solution in sight even for $n = 3$.

One of the general methods in the modern theory of varieties of algebras which has proved to be quite effective consists in expanding the signature of the algebra under consideration with an additional structure such as a trace or a grading that is inherent to the algebra and studying generalized identities of the arising algebraic system rather than the ordinary polynomial identities of the original algebra (see, for instance, [2], [6], [8], [10], [11]). In particular, Yu.P. Razmyslov and C. Procesi have independently established that, for all n, all the trace identities of $M_n(F)$ are obtained from the Cayley-Hamilton theorem (cf. [10] and [8]).

Let $\mathbf{Z}_n$ denote the additive group of integers modulo n. In [3] O.M. Di Vincenzo has described a finite basis for the $\mathbf{Z}_2$-graded polynomial identities of $M_2(F)$. Subsequently, I.P. Shestakov has posed the problem of finding an explicit finite basis for the graded polynomial identities of the algebra $M_n(F)$ equipped with its natural $\mathbf{Z}_n$-grading for any n. In the present paper this problem is completely resolved.

Received by the editors April 10, 1997 and, in revised form, February 26, 1998.
1991 *Mathematics Subject Classification.* Primary 16R40.
Key words and phrases. Graded polynomial identities, full matrix algebra.

Theorem. *All graded polynomial identities of the* $\mathbf{Z}_n$*-graded algebra* $M_n(F)$ *follow from*

$$x_1x_2 - x_2x_1 = 0,\ \alpha(x_1) = \alpha(x_2) = \overline{0}, \tag{0.1}$$

and

$$x_1xx_2 - x_2xx_1 = 0,\ \alpha(x_1) = \alpha(x_2) = -\alpha(x). \tag{0.2}$$

1. Definitions and preliminary results

Denote by $E_{i,j}$, $1 \leq i, j \leq n$, the $n \times n$ matrix whose only nonzero entry is 1 in the ith row and jth column. The matrices $E_{i,j}$'s, known as *matrix units*, form a basis of $M_n(F)$ as a vector space. For $t \in \mathbf{Z}$, let $\overline{t}$ denote the residue class in $\mathbf{Z}_n$ that contains t. For $\alpha \in \mathbf{Z}_n$, let $\mathcal{M}_n^{(\alpha)}$ be the subspace of $M_n(F)$ spanned by all matrix units $E_{i,j}$ such that $\overline{j-i} = \alpha$, so that $\mathcal{M}_n^{(\overline{0})}$ consists of the matrices of the form

$$\begin{pmatrix} a_{1,1} & & & \\ & a_{2,2} & 0 & \\ & 0 & \ddots & \\ & & & a_{n,n} \end{pmatrix},\ a_{1,1}, a_{2,2}, \ldots, a_{n,n} \in F,$$

and, for $0 < t \leq n-1$, $\mathcal{M}_n^{(\overline{t})}$ consists of the matrices of the form

$$\begin{pmatrix} 0 & \cdots & 0 & a_{1,t+1} & \cdots & \cdots & 0 \\ \vdots & & \vdots & \vdots & a_{2,t+2} & & \vdots \\ \vdots & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & a_{n-t,n} \\ a_{n-t+1,1} & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & & \vdots \\ 0 & \cdots & a_{n,t} & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where $a_{1,t+1}, a_{2,t+2}, \ldots, a_{n-t,n}, a_{n-t+1,1}, a_{n-t+2,2}, \ldots, a_{n,t} \in F$. Then $M_n(F)$ is a direct sum of the subspaces $\mathcal{M}_n^{(\alpha)}$'s:

$$M_n(F) = \sum_{\alpha \in \mathbf{Z}_n} \bigoplus \mathcal{M}_n^{(\alpha)}. \tag{1.1}$$

Recall (cf., for instance, [4]) that a family $\{\mathcal{A}^{(\alpha)},\ \alpha \in G\}$ of subspaces of an algebra $\mathcal{A}$, where G is an additive group, defines a G*-grading* of the algebra $\mathcal{A}$ if $\mathcal{A} = \sum_{\alpha \in G} \mathcal{A}^{(\alpha)}$ and, for $\alpha, \beta \in G$, $\mathcal{A}^{(\alpha)}\mathcal{A}^{(\beta)} \subseteq \mathcal{A}^{(\alpha+\beta)}$. Since

$$E_{i,j}E_{r,s} = \begin{cases} E_{i,s}, & \text{if } j = r, \\ 0, & \text{if } j \neq r, \end{cases}$$

it follows that, for $p, q \in \{0, 1, \ldots, n-1\}$, we have $\mathcal{M}^{(\overline{p})}\mathcal{M}^{(\overline{q})} \subseteq \mathcal{M}^{(\overline{p+q})}$, so that the decomposition (1.1) defines a $\mathbf{Z}_n$-grading of the algebra $M_n(F)$. We will denote the $\mathbf{Z}_n$-graded algebra $M_n(F)$ by $\mathcal{M}_n$.

Let $\left\{X^{(\alpha)},\ \alpha \in \mathbf{Z}_n\right\}$ be a family of disjoint countable sets $X^{(\alpha)}$. Put $X = \bigcup_{\alpha \in \mathbf{Z}_n} X^{(\alpha)}$ and denote by $A[X]$ the free associative algebra freely generated by the set X. The monomials

$$\{\, x_{i_1}x_{i_2} \cdots x_{i_k} :\ k = 1, 2, \ldots;\ x_{i_1}, x_{i_2}, \ldots, x_{i_k} \in X \,\}$$

form a basis of $A[X]$ as a vector space. An indeterminate $x \in X$ is said to be *of homogeneous degree* α, written $\alpha(x) = \alpha$, if $x \in X^{(\alpha)}$. The homogeneous degree of a monomial $\mathsf{n} = x_{i_1} x_{i_2} \cdots x_{i_k}$ is defined to be $\alpha(\mathsf{n}) = \alpha(x_{i_1}) + \alpha(x_{i_2}) + \cdots + \alpha(x_{i_k})$. For $\alpha \in \mathbf{Z}_n$, denote by $A[X]^{(\alpha)}$ the subspace of $A[X]$ spanned by all the monomials having homogeneous degree α. Notice that $A[X]^{(\alpha)} A[X]^{(\beta)} \subseteq A[X]^{(\alpha+\beta)}$ for all $\alpha, \beta \in \mathbf{Z}_n$. Thus

$$A[X] = \sum_{\alpha \in \mathbf{Z}_n} \bigoplus A[X]^{(\alpha)}$$

proves to be a $\mathbf{Z}_n$-graded algebra. The elements of the $\mathbf{Z}_n$-graded algebra $A[X]$ are referred to as $\mathbf{Z}_n$*-graded polynomials* or, simply, *graded polynomials*. An ideal I of $A[X]$ is said to be a T_n*-ideal* if it is invariant under all F-endomorphisms $\gamma\colon A[X] \to A[X]$ such that $\gamma(A[X]^{(\alpha)}) \subseteq A[X]^{(\alpha)}$ for all $\alpha \in \mathbf{Z}_n$.

Let $\mathcal{A} = \sum_{\alpha \in \mathbf{Z}_n} \mathcal{A}^{(\alpha)}$ be a $\mathbf{Z}_n$-graded associative algebra. A $\mathbf{Z}_n$-graded polynomial $f(x_1, x_2, \ldots, x_k)$, or the expression $f(x_1, x_2, \ldots, x_k) = 0$, is said to be a *graded polynomial identity* of the $\mathbf{Z}_n$-graded algebra $\mathcal{A}$ if $f(a_1, a_2, \ldots, a_k) = 0$ for all $a_1, a_2, \ldots, a_k \in \bigcup_{\alpha \in \mathbf{Z}_n} \mathcal{A}^{(\alpha)}$ such that $a_s \in \mathcal{A}^{(\alpha(x_s))}$, $s = 1, 2, \ldots, k$. The set $T_n(\mathcal{A})$ of all graded identities of a $\mathbf{Z}_n$-graded algebra $\mathcal{A}$ is a T_n-ideal of $A[X]$. A $\mathbf{Z}_n$-graded polynomial identity $f = 0$ is said to follow from a family of $\mathbf{Z}_n$-graded polynomial identities $g_\lambda = 0$, $\lambda \in \Lambda$, if the graded polynomial f lies in the smallest T_n-ideal containing all the graded polynomials g_λ, $\lambda \in \Lambda$.

Lemma 1. *The graded algebra $\mathcal{M}_n$ satisfies* (0.1) *and* (0.2).

Proof. Since any two diagonal matrices commute, it follows that $\mathcal{M}_n$ satisfies graded identity (0.1). The identities (0.2) being multilinear, it is only required to show that this identity holds when

$$x_1 = E_{i_1,j_1},\ x_2 = E_{i_2,j_2},\ x = E_{r,s}$$

for $E_{i_1,j_1}, E_{i_2,j_2} \in \mathcal{M}_n^{(\bar{t})}$, $E_{r,s} \in \mathcal{M}_n^{(\overline{n-t})}$, where $0 < t \leq n-1$; so that

$$j_1 = \begin{cases} i_1 + t, & \text{if } i_1 + t \leq n, \\ i_1 + t - n, & \text{if } i_1 + t > n; \end{cases}$$

$$i_2 = \begin{cases} j_2 - t, & \text{if } j_2 - t \geq 1, \\ j_2 - t + n, & \text{if } j_2 - t < 1; \end{cases}$$

$$r = \begin{cases} s + t, & \text{if } s + t \leq n, \\ s + t - n, & \text{if } s + t > n. \end{cases}$$

Note that $E_{i_1,j_1} E_{r,s} E_{i_2,j_2} \neq 0$ only if $j_1 = r$ and $s = i_2$. We claim that in this case $i_1 = s = i_2$ and $j_1 = r = j_2$. Observe that, if $j_1 = i_1 + t$ and $r = s + t - n$, then, from $j_1 = r$, it follows that $n = s - i_1$, which is impossible. So the equalities $j_1 = i_1 + t$ and $r = s + t - n$ cannot hold simultaneously. The same applies to the equalities $s = r - t$ and $i_2 = j_2 - t + n$. Thus, when $j_1 = i_1 + t$, we have $r = s + t$ and $i_2 = j_2 - t$, so that

$$i_2 = s = r - t = j_1 - t = i_1$$

and

$$r = j_1 = i_1 + t = i_2 + t = j_2.$$

Similarly, when $j_1 = i_1 + t - n$, we have $r = s + t - n$ and $i_2 = j_2 - t + n$, whence

$$i_2 = s = r - t + n = j_1 - t + n = i_1$$

and

$$r = j_1 = i_1 + t - n = i_2 + t - n = j_2,$$

proving the above assertion. This enables us to conclude that $E_{i_1,j_1}E_{r,s}E_{i_2,j_2} \neq 0$ if and only if $i_1 = s = i_2$ and $j_1 = r = j_2$, and if and only if $E_{i_2,j_2}E_{r,s}E_{i_1,j_1} \neq 0$. In this case

$$E_{i_1,j_1}E_{r,s}E_{i_2,j_2} = E_{i_1,j_2} = E_{i_2,j_1} = E_{i_2,j_2}E_{r,s}E_{i_1,j_1}.$$

Otherwise,

$$E_{i_1,j_1}E_{r,s}E_{i_2,j_2} = 0 = E_{i_2,j_2}E_{r,s}E_{i_1,j_1},$$

proving that (0.2) does hold in $\mathcal{M}_n$. □

From now on let I_n be the T_n-ideal generated by the graded identities (0.1) and (0.2). For a positive integer k, denote by S_k the set of all permutations of the set $\{1, 2, \ldots, k\}$. For $x_1, x_2, \ldots, x_k \in X$ and $\sigma \in S_k$, let

$$\mathsf{m}_\sigma = \mathsf{m}_\sigma(x_1, x_2, \ldots, x_k) = x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(k)}.$$

The multilinear monomial in $x_1, x_2, \ldots, x_k$ corresponding to the identity permutation will be denoted by

$$\mathsf{m} = \mathsf{m}(x_1, x_2, \ldots, x_k) = x_1x_2 \cdots x_k.$$

Obviously, $\alpha(\mathsf{m}) = \alpha(\mathsf{m}_\sigma) = \alpha(x_1) + \alpha(x_2) + \cdots + \alpha(x_k)$. It is clear that every multilinear graded polynomial $f(x_1, x_2, \ldots, x_k)$ can be expressed as

$$f = \sum_{\sigma \in S_k} a_\sigma \mathsf{m}_\sigma,$$

where $a_\sigma \in F$.

By a *standard substitution* we will understand a substitution $\mathcal{S}$ of the form

$$x_1 = E_{i_1,j_1},\ x_2 = E_{i_2,j_2}, \ldots, x_k = E_{i_k,j_k}, \tag{1.2}$$

where

$$\overline{j_s - i_s} = \alpha(x_s) \tag{1.3}$$

so that $E_{i_s,j_s} \in \mathcal{M}_n^{(\alpha(x_s))}$, $s = 1, 2, \ldots, k$. For a graded polynomial $f(x_1, x_2, \ldots, x_k)$ and a substitution $\mathcal{S}$, we denote by $f|_\mathcal{S}$ the value of f corresponding to the substitution $\mathcal{S}$. It is easy to see that, if a multilinear graded polynomial $f(x_1, x_2, \ldots, x_k)$ is such that $f|_\mathcal{S} = 0$ for every standard substitution $\mathcal{S}$, then $f = 0$ is a graded identity of $\mathcal{M}_n$. Observe that, when a substitution (1.2) is made, the value of a monomial $\mathsf{m}_\sigma(x_1, x_2, \ldots, x_k)$ differs from zero only if

$$j_{\sigma(1)} = i_{\sigma(2)},\ j_{\sigma(2)} = i_{\sigma(3)},\ \ldots\ ,\ j_{\sigma(k-1)} = i_{\sigma(k)}, \tag{1.4}$$

in which case $\mathsf{m}_\sigma|_{(1.2)} = E_{i_{\sigma(1)},j_{\sigma(k)}}$.

For a monomial $\mathsf{m}_\sigma(x_1, x_2, \ldots, x_k)$, $\sigma \in S_k$, and any two integers $1 \le p \le q \le k$, denote by $\mathsf{m}_\sigma^{[p,q]}$ the subword obtained from m_σ by discarding the first $p - 1$ and the last $k - q$ factors:

$$\mathsf{m}_\sigma^{[p,q]} = x_{\sigma(p)}x_{\sigma(p+1)} \cdots x_{\sigma(q)}.$$

Lemma 2. *For any $\sigma \in S_k$, the exists a standard substitution $\mathcal{S}$ such that*

$$\mathsf{m}_\sigma(x_1, x_2, \dots, x_k)|_{\mathcal{S}} \neq 0.$$

Proof. We apply induction on k. The case $k = 1$ is trivial. Let $k > 1$ and let $\alpha(x_k) = \overline{t}$ for some integer $0 \leq t < n$. If, when some standard substitution

$$x_{\sigma(1)} = E_{i_1,j_1},\ x_{\sigma(2)} = E_{i_2,j_2}, \dots, x_{\sigma(k-1)} = E_{i_{k-1},j_{k-1}} \tag{1.5}$$

is made, $\mathsf{m}_\sigma^{[k-1]} = E_{i_1,j_{k-1}} \neq 0$, then, setting, in addition to (1.5), $x_{\sigma(k)} = E_{j_{k-1},j_k}$, where

$$j_k = \begin{cases} j_{k-1} + t, & \text{if } j_{k-1} + t \leq n, \\ j_{k-1} + t - n, & \text{if } j_{k-1} + t > n, \end{cases}$$

we get

$$\mathsf{m}_\sigma(x_1, x_2, \dots, x_k) = E_{i_1,j_{k-1}} E_{j_{k-1},j_k} = E_{i_1,j_k} \neq 0,$$

completing the proof. $\square$

Lemma 3. *If $\mathsf{m}_\sigma|_{(1.2)} \neq 0$, then, for any $1 \leq p \leq q \leq k$,*

$$\alpha(\mathsf{m}_\sigma^{[p,q]}) = \overline{j_{\sigma(q)} - i_{\sigma(p)}}.$$

Proof. In view of (1.3) and (1.4), we have

$$\alpha(\mathsf{m}_\sigma^{[p,q]}) = \alpha(x_{\sigma(q)}) + \alpha(x_{\sigma(q-1)}) + \dots + \alpha(x_{\sigma(p)})$$

$$= \overline{(j_{\sigma(q)} - i_{\sigma(q)})} + \overline{(j_{\sigma(q-1)} - i_{\sigma(q-1)})} + \dots + \overline{(j_{\sigma(p)} - i_{\sigma(p)})} = \overline{j_{\sigma(q)} - i_{\sigma(p)}},$$

completing the proof. $\square$

Lemma 4. *If, for a permutation $\sigma \in S_k$, there exists a standard substitution* (1.2) *such that*

$$\mathsf{m}_\sigma(x_1, x_2, \dots, x_k)|_{(1.2)} = \mathsf{m}(x_1, x_2, \dots, x_k)|_{(1.2)} \neq 0,$$

then

$$\mathsf{m}_\sigma(x_1, x_2, \dots, x_k) \equiv x_1 \cdot \mathsf{n}\,(x_2, x_3, \dots, x_k)\ (mod\ I_n)$$

for some monomial $\mathsf{n}\,(x_2, x_3, \dots, x_k) = x_{l_2} x_{l_3} \cdots x_{l_k}$.

Proof. Suppose $\sigma(1) \neq 1$. Then $1 = \sigma^{-1}(\sigma(1)) < \sigma^{-1}(1)$. Let t be the least positive integer such that $\sigma^{-1}(t+1) < \sigma^{-1}(1)$. Obviously,

$$1 \leq \sigma^{-1}(t+1) < \sigma^{-1}(1) \leq \sigma^{-1}(t).$$

Since

$$E_{i_1,j_1} E_{i_2,j_2} \cdots E_{i_k,j_k} = E_{i_{\sigma(1)},j_{\sigma(1)}} E_{i_{\sigma(2)},j_{\sigma(2)}} \cdots E_{i_{\sigma(k)},j_{\sigma(k)}} \neq 0,$$

it follows that $i_1 = i_{\sigma(1)}$, $j_t = i_{t+1}$ and, for $s > 1$, $j_{\sigma(s-1)} = i_{\sigma(s)}$. So, setting $p = \sigma^{-1}(t+1)$, $q = \sigma^{-1}(1)$ and $r = \sigma^{-1}(t)$, we have $1 \leq p < q \leq r$ with $j_{\sigma(q-1)} = i_{\sigma(q)} = i_{\sigma(1)}$, $j_{\sigma(r)} = i_{\sigma(p)}$ and, when $p > 1$, $j_{\sigma(p-1)} = i_{\sigma(p)}$. First consider the case $p > 1$. From the equalities

$$j_{\sigma(r)} = i_{\sigma(p)} = j_{\sigma(p-1)} \quad \text{and} \quad j_{\sigma(q-1)} = i_{\sigma(q)} = i_{\sigma(1)},$$

we infer that

$$j_{\sigma(p-1)} - i_{\sigma(1)} = i_{\sigma(p)} - j_{\sigma(q-1)} = j_{\sigma(r)} - i_{\sigma(q)} = t_0$$

for some $t_0 \in \mathbf{Z}$. By Lemma 3, we have

$$\begin{gathered}\alpha(\mathsf{m}_\sigma^{[1,p-1]}) = \overline{j_{\sigma(p-1)} - i_{\sigma(1)}} = \overline{t_0};\\ \alpha(\mathsf{m}_\sigma^{[p,q-1]}) = \overline{j_{\sigma(q-1)} - i_{\sigma(p)}} = -\overline{t_0};\\ \alpha(\mathsf{m}_\sigma^{[q,r]}) = \overline{j_{\sigma(r)} - i_{\sigma(q)}} = \overline{t_0}.\end{gathered}$$

Hence, using (0.2), we get

$$\mathsf{m}_\sigma = \mathsf{m}_\sigma^{[1,p-1]} \cdot \mathsf{m}_\sigma^{[p,q-1]} \cdot \mathsf{m}_\sigma^{[q,r]} \cdot \mathsf{m}_\sigma^{[r+1,k]}$$

$$\equiv \mathsf{m}_\sigma^{[q,r]} \cdot \mathsf{m}_\sigma^{[p,q-1]} \cdot \mathsf{m}_\sigma^{[1,p-1]} \cdot \mathsf{m}_\sigma^{[r+1,k]} = x_{\sigma(q)} x_{l_2} \cdots x_{l_k} = x_1 x_{l_2} \cdots x_{l_k} \pmod{I_n}.$$

Now let $p = 1$. In this case $j_{\sigma(q-1)} = i_{\sigma(q)} = i_{\sigma(1)} = j_{\sigma(r)}$, which, in view of Lemma 3, gives

$$\begin{gathered}\alpha(\mathsf{m}_\sigma^{[1,q-1]}) = \overline{j_{\sigma(q-1)} - i_{\sigma(1)}} = \overline{0};\\ \alpha(\mathsf{m}_\sigma^{[q,r]}) = \overline{j_{\sigma(r)} - i_{\sigma(q)}} = \overline{0}.\end{gathered}$$

Then it follows from (0.1) that

$$\mathsf{m}_\sigma = \mathsf{m}_\sigma^{[1,q-1]} \cdot \mathsf{m}_\sigma^{[q,r]} \cdot \mathsf{m}_\sigma^{[r+1,k]}$$

$$\equiv \mathsf{m}_\sigma^{[q,r]} \cdot \mathsf{m}_\sigma^{[1,q-1]} \cdot \mathsf{m}_\sigma^{[r+1,k]} = x_{\sigma(q)} x_{l_2} \cdots x_{l_k} = x_1 x_{l_2} \cdots x_{l_k} \pmod{I_n}.$$

This completes the proof. □

Lemma 5. *If, for a permutation $\sigma \in S_k$, there exists a standard substitution* (1.2) *such that*

$$\mathsf{m}_\sigma(x_1, x_2, \ldots, x_k)|_{(1.2)} = \mathsf{m}(x_1, x_2, \ldots, x_k)|_{(1.2)} \neq 0,$$

then

$$\mathsf{m}_\sigma(x_1, x_2, \ldots, x_k) \equiv \mathsf{m}(x_1, x_2, \ldots, x_k) \ (\textit{mod } I_n).$$

Proof. Let r be the greatest positive integer such that

$$\mathsf{m}_\sigma(x_1, x_2, \ldots, x_k) \equiv x_1 x_2 \cdots x_r \cdot \mathsf{n}(x_{r+1}, \ldots, x_k) \pmod{I_n}$$

for some multilinear monomial $\mathsf{n} = \mathsf{n}(x_{r+1}, \ldots, x_k)$. We need only show that $r = k$. Suppose, on the contrary, $r < k$. Then, obviously, $r \leq k - 2$. We have

$$x_1 x_2 \cdots x_r \cdot \mathsf{n}|_{(1.2)} = \mathsf{m}_\sigma|_{(1.2)} = \mathsf{m}|_{(1.2)} \neq 0.$$

Combining this with

$$x_1 x_2 \cdots x_r \cdot \mathsf{n}|_{(1.2)} = E_{i_1,j_1} \cdots E_{i_r,j_r} \cdot \mathsf{n}|_{(1.2)} = E_{i_1,j_r} \cdot \mathsf{n}|_{(1.2)}$$

and

$$\mathsf{m}|_{(1.2)} = E_{i_1,j_1} \cdots E_{i_r,j_r} \cdot \{x_{r+1} x_{r+2} \cdots x_k\}|_{(1.2)} = E_{i_1,j_r} \cdot \{x_{r+1} x_{r+2} \cdots x_k\}|_{(1.2)},$$

we get

$$\mathsf{n}(x_{r+1}, x_{r+2}, \ldots, x_k)|_{(1.2)} = x_{r+1} x_{r+2} \cdots x_k|_{(1.2)} \neq 0.$$

By Lemma 4, there exists a multilinear monomial $\mathsf{n}'(x_{r+2}, x_{r+3}, \ldots, x_k)$ such that

$$\mathsf{n}(x_{r+1}, x_{r+2}, \ldots, x_k) \equiv x_{r+1} \cdot \mathsf{n}'(x_{r+2}, \ldots, x_k) \pmod{I_n}$$

so that

$$\mathsf{m}_\sigma \equiv x_1 \cdots x_r \cdot \mathsf{n}(x_{r+1}, x_{r+2}, \ldots, x_k) \equiv x_1 \cdots x_r x_{r+1} \cdot \mathsf{n}'(x_{r+2}, \ldots, x_k) \pmod{I_n},$$

which contradicts our choice of the number r. This completes the proof. □

From Lemma 5, we immediately obtain the following

Corollary 6. *If, for two permutations* $\sigma, \tau \in S_k$, *there exists a standard substitution* $\mathcal{S}$ *such that*

$$\mathsf{m}_\sigma(x_1, x_2, \dots, x_k)|_{\mathcal{S}} = \mathsf{m}_\tau(x_1, x_2, \dots, x_k)|_{\mathcal{S}} \neq 0,$$

then

$$\mathsf{m}_\sigma(x_1, x_2, \dots, x_k) \equiv \mathsf{m}_\tau(x_1, x_2 \dots, x_k) \ (mod\ I_n).$$

2. Proof of the theorem

Since the characteristic of the basic field is zero, we need only prove that an arbitrary multilinear graded polynomial identity $f(x_1, x_2, \dots, x_k) = 0$ of $\mathcal{M}_n$ lies in I_n (cf., for instance, [1, Theorem 4.2.3]). Let r be the least non-negative integer such that the polynomial f can be expressed, modulo I_n, as a linear combination of r multilinear monomials:

$$f \equiv \sum_{q=1}^{r} a_{\sigma_q} \mathsf{m}_{\sigma_q} \ (\mathrm{mod}\ I_n),$$

where $0 \neq a_{\sigma_q} \in F$, $\sigma_1, \sigma_2, \dots, \sigma_r \in S_k$. Show that $r = 0$. Suppose, on the contrary, $r > 0$. In view of Lemma 2, we can find a standard substitution $\mathcal{S}$ such that $\mathsf{m}_{\sigma_1}|_{\mathcal{S}} \neq 0$. Since

$$\mathsf{m}_{\sigma_q}|_{\mathcal{S}} \in \{\, E_{i,j} : i, j = 1, 2, \dots, n\} \cup \{0\}, \ q = 1, 2, \dots, r,$$

and

$$a_{\sigma_1} \mathsf{m}_{\sigma_1}|_{\mathcal{S}} = \sum_{q=2}^{r} (-a_{\sigma_q}) \mathsf{m}_{\sigma_q}|_{\mathcal{S}},$$

it follows that there is at least one integer $p \in \{2, 3, \dots, r\}$ such that $\mathsf{m}_{\sigma_p}|_{\mathcal{S}} = \mathsf{m}_{\sigma_1}|_{\mathcal{S}}$. Then, by Corollary 6, $\mathsf{m}_{\sigma_p} \equiv \mathsf{m}_{\sigma_1}$ (mod I_n), so that

$$f \equiv \sum_{q=1}^{r} a_{\sigma_q} \mathsf{m}_{\sigma_q} \equiv (a_{\sigma_1} + a_{\sigma_p}) \mathsf{m}_{\sigma_1} + \sum_{q=2}^{p-1} a_{\sigma_q} \mathsf{m}_{\sigma_q} + \sum_{q=p+1}^{r} a_{\sigma_q} \mathsf{m}_{\sigma_q} \ (\mathrm{mod}\ I_n),$$

that is, f can be expressed, modulo I_n, as a linear combination of no more than $r - 1$ multilinear monomials, which contradicts our choice of the number r. Thus

$$f \equiv 0 \ (\mathrm{mod}\ I_n).$$

This completes the proof of the theorem.

Acknowledgment

I want to thank Professor I.P. Shestakov for introducing me to this problem.

References

1. Yu.A. Bahturin, *Identities in Lie algebras*, Nauka, Moskva, 1985 [Russian]. Translation: VNU Sci. Press, Utrecht, 1987. MR **86k:**17015
2. A. Berele, *Trace identities and* $\mathbf{Z}/2\mathbf{Z}$*-graded invariants*, Trans. AMS, **309(2)** (1988), 581–589. MR **89m:**17043
3. O.M. Di Vincenzo, *On the graded identities of* $M_{1,1}(E)$, Israel J. Math. **80(3)** (1992), 323–335. MR **94f:**16041
4. O.M. Di Vincenzo, *Cocharacters of G-graded algebras*, Comm. Algebra **24(10)** (1996), 3293–3310. MR **97f:**16041

5. V.S. Drenski, *A minimal basis for the identities of a second order matrix algebra over a field of characteristic* 0, Algebra i Logika, **20** (1981), 282-290 [Russian]. Translation: Algebra and Logic, **20** (1981), 188-194. MR **83g:**16031
6. A.R. Kemer, *Varieties and Z_2-graded algebras*, Izv. Akad. Nauk SSSR Ser. Mat., **48(5)** (1984), 1042–1059 [Russian]. Translation: Math USSR Izv., **25** (1985), 359–374. MR **86f:**16015
7. A.R. Kemer, *Finite basis property of identities of associative algebras*, Algebra i Logika, **26(5)** (1987), 597-641 [Russian]. Translation: Algebra and Logic, **26(5)** (1987), 362–397. MR **90b:**08008
8. C. Procesi, *The invariant theory of $n \times n$ matrices*, Adv. in Math., **19** (1976), 306–381. MR **54:**7512
9. Yu.P. Razmyslov, *Finite basing of the identities of a matrix algebra of second order over a field of characteristic zero*, Algebra i Logika, **12** (1973), 83–113. [Russian] Translation: Algebra and Logic, **12** (1974), 47–63. MR **49:**5103
10. Yu.P. Razmyslov, *Trace identities of full matrix algebra over a field of characteristic zero*, Izv. Akad. Nauk SSSR Ser. Mat. **38(4)** (1974), 723–756 [Russian]. Translation: Math. USSR Izv., **8(4)** (1974), 727–760. MR **58:**22158
11. Yu.P. Razmyslov, *Trace identities and central polynomials in the matrix superalgebras $M_{n,k}$*, Mat. Sb., **128(2)** (1985) [Russian]. Translation: Math. USSR Sb., **56(1)** (1987), 187–206. MR **87f:**16014

Department of Mathematics, University of Swaziland, Private Bag 4, Kwaluseni, Swaziland, Southern Africa

Institute of Mathematics, Novosibirsk 630090, Russia
E-mail address: `vasilovs@realnet.co.sz`