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TORIC DEGENERATIONS AND VECTOR BUNDLES

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ABSTRACT. There are many affine subalgebras of polynomial rings with highly non-trivial projective modules, whose initial algebras (toric degenerations) are still finitely generated and have all projective modules free.

Let $k[X_1, \dots, X_n]$ be a polynomial algebra (k a field, $n \in \mathbb{N}$) and A an affine k-subalgebra. Let $\prec$ denote a term order on the multiplicative semigroup of monomials in the X_i and let $\mathrm{in}_\prec(A)$ denote the monomial subalgebra of $k[X_1, \dots, X_n]$, generated by the leading monomials of elements $f \in A$ with respect to $\prec$. In case the *initial algebra* $\mathrm{in}_\prec(A)$ is finitely generated, one can obtain many properties of A by checking them for $\mathrm{in}_\prec(A)$ (called sometimes a *toric degeneration* of A) (see [CHV], [RS]). However, this is not the case for the property 'all projective modules are free' – thanks to Bernd Sturmfels for asking me this question.

Theorem 1. *Let $A = k[X, Y, Z^2, Z^3 - XYZ]$ and $\prec$ be the lexicographic term order corresponding to $Z \prec Y \prec X$. Then $SK_0(A) = \mathrm{Ker}(\widetilde{K}_0(A) \overset{\det}{\to} \mathrm{Pic}(A))$ is not trivial (equivalently, there are projective A-modules which are not even stably of type free$\oplus$rank 1), while $\mathrm{in}_\prec(A)$ is finitely generated and all projective $\mathrm{in}_\prec(A)$-modules are free.*

(Here 'projective' includes 'finitely generated'.)

Proof. Since $(Z^2 - XY)k[X, Y, Z] \subset A$, the following diagram with the upper horizontal identity embedding is a pull-back diagram (all the letters refer to variables)

$$\begin{array}{ccc} A & \longrightarrow & k[X,Y,Z] \\ \Big\downarrow & & \Big\downarrow \\ k[U,V] & \longrightarrow & k[S^2, ST, T^2]\ , \end{array}$$

where $X \mapsto S^2$, $Y \mapsto T^2$, $Z \mapsto ST$, $U \mapsto S^2$, $V \mapsto T^2$. It is similarly easy to show that $\mathrm{in}_\prec(A) = k[X, Y, Z^2, XYZ]$ – a seminormal monomial algebra (as a k-vector space it is generated by the normal monomial subalgebras $k[X, Y]$, $k[X, Z^2]$, $k[Y, Z^2]$ and $k[\{X^aY^bZ^c | a > 0, b > 0, c > 0\}]$). So by [Gu1] projective modules over $\mathrm{in}_\prec(A)$ are free, while the Mayer-Vietoris sequence (see [Bass], p.490), applied to the diagram above, implies $SK_0(A) = SK_1(k[S^2, ST, T^2])$. Hence, by [Gu2] $SK_0(A) \neq 0$. □

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Remark 2. Actually, by the $(\mathrm{U}-\mathrm{Pic})$–Mayer-Vietoris sequence $\mathrm{Pic}(A)=0$ for the ring A in Theorem 1.

By [Gu2] essentially all (and conjecturely all) monomial algebras, not isomorphic to polynomial rings, have non-trivial SK_1-groups. So changing the lower right algebra in the diagram above by other monomial algebras and changing correspondingly the 'side' polynomial algebras, one can produce a big class of rings having this property. One uses the general observation that if some powers of all variables belong to A, both A itself and its initial algebra $\mathrm{in}_{\prec}(A)$ are finitely generated because they are intermediate algebras of the module finite extensions

$$k[X_1^{a_1},\dots,X_n^{a_n}]\subset k[X_1,\dots,X_n].$$

Nontriviality of SK_1 for the special monomial ring $k[S^2,ST,T^2]$ (over a characteristic zero field k) was also shown in [Sri].

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