

A MORPHISM OF INTERSECTION HOMOLOGY INDUCED BY AN ALGEBRAIC MAP

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ABSTRACT. Let $f : X \rightarrow Y$ be a map of algebraic varieties. Barthel, Brasselet, Fieseler, Gabber and Kaup have shown that there exists a homomorphism of intersection homology groups $f^* : IH^*(Y) \rightarrow IH^*(X)$ compatible with the induced homomorphism on cohomology. The crucial point in the argument is reduction to the finite characteristic. We give an alternative and short proof of the existence of a homomorphism f^* . Our construction is an easy application of the Decomposition Theorem.

Let X be an algebraic variety, $IH^*(X) = H^*(X; IC_X)$ its rational intersection homology group with respect to the middle perversity and IC_X the intersection homology sheaf which is an object of derived category of sheaves over X [GM1]. We have the homomorphism $\omega_X : H^*(X; \mathbb{Q}) \rightarrow IH^*(X)$ induced by the canonical morphism of the sheaves $\omega_X : \mathbb{Q}_X \rightarrow IC_X$.

Let $f : X \rightarrow Y$ be a map of algebraic varieties. It induces a homomorphism of the cohomology groups. The natural question arises: Does there exist an induced homomorphism for intersection homology compatible with f^* ?

$$\begin{array}{ccc} IH^*(Y) & \xrightarrow{?} & IH^*(X) \\ \uparrow \omega_Y & & \uparrow \omega_X \\ H^*(Y; \mathbb{Q}) & \xrightarrow{f^*} & H^*(X; \mathbb{Q}). \end{array}$$

The answer is positive. For topological reasons the map in question exists for normally nonsingular maps [GM1, §5.4.3] and for placid maps [GM3, §4]. The authors of [BBFGK] proved the following:

Theorem 1. *Let $f : X \rightarrow Y$ be an algebraic map of algebraic varieties. Then there exists a morphism $\lambda_f : IC_Y \rightarrow Rf_*IC_X$ such that the following diagram with the canonical morphisms commutes:*

$$\begin{array}{ccc} IC_Y & \xrightarrow{\lambda_f} & Rf_*IC_X \\ \uparrow \omega_Y & & \uparrow Rf_*(\omega_X) \\ \mathbb{Q}_Y & \xrightarrow{\alpha_f} & Rf_*\mathbb{Q}_X. \end{array}$$

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In fact, [BBFGK] proves the existence of a morphism $\mu_f : f^*IC_Y \rightarrow IC_X$, which is adjoint to λ_f .

The sheaf language can be translated to the following: an induced homomorphism of intersection homology exists in a functorial way with respect to the open subsets of Y . This means that there exists a compatible family of induced homomorphisms

$$f_{\lambda,U}^* : IH^*(f^{-1}U) \longrightarrow IH^*(U),$$

which is also compatible with the family

$$f_{|U}^* : H^*(f^{-1}U; \mathbb{Q}) \longrightarrow H^*(U; \mathbb{Q}).$$

As shown in [BBFGK] the morphism λ_f (and μ_f) is not unique. It is not possible to choose the morphisms λ_f (or μ_f) in a functorial way with respect to all algebraic maps (p.160). The simplest counterexample is the inclusion $\{(0, 0)\} \hookrightarrow \{(x_1, x_2) : x_1x_2 = 0\}$, which can be factored through the inclusions $\{(0, 0)\} \hookrightarrow \{(x_1, x_2) : x_i = 0\}$ for $i = 1$ or 2 .

We will give a short proof of the main theorem from [BBFGK]. We will derive it from the Decomposition Theorem. The reference to the Decomposition Theorem is [BBD, 6.2.8] (see also [GM2]) and in a slightly different context [Sa].

We will use only the following corollary from the Decomposition Theorem:

Corollary from the Decomposition Theorem. *Let $\pi : X \rightarrow Y$ be a proper surjective map of algebraic varieties. Then IC_Y is a direct summand in $R\pi_*IC_X$.*

The idea of the proof of our theorem is simple; the essence is the argument similar to [BBFGK, Remarque pp.172–174]. We take a resolution $\pi_Y : \tilde{Y} \rightarrow Y$ and enlarge the space X to obtain a map $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$. There exists the induced morphism of intersection homology $\lambda_{\tilde{f}}$ for \tilde{f} . By the Decomposition Theorem the intersection homology of X (and Y) is a direct summand of intersection homology of \tilde{X} (resp. \tilde{Y}). We compose $\lambda_{\tilde{f}}$ with the projection and inclusion in the direct sums to obtain the desired morphism λ_f .

Remark. If we insisted, then \tilde{X} might be even smooth of the same dimension as X with the map $\pi_X : \tilde{X} \rightarrow X$ generically finite; compare [BBFGK, p.173].

Proof of Theorem 1. We may assume that X and Y are irreducible. Let $\pi_Y : \tilde{Y} \rightarrow Y$ be a resolution of Y . Denote by \tilde{X} the fiber product (pull-back) $X \times_Y \tilde{Y}$. Note that it is a variety, which may be singular and not equidimensional. We have a commutative diagram of algebraic maps (π_X and π_Y proper):

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \pi_X \downarrow & & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y \end{array}$$

and the associated diagram of sheaves over Y :

$$\begin{array}{ccccccc} R\pi_{Y*}IC_{\tilde{Y}} & = & R\pi_{Y*}\mathbb{Q}_{\tilde{Y}} & \xrightarrow{R\pi_{Y*}(\alpha_{\tilde{f}})} & Rf_*R\pi_{X*}\mathbb{Q}_{\tilde{X}} & \xrightarrow{Rf_*R\pi_{X*}(\omega_{\tilde{X}})} & Rf_*R\pi_{X*}IC_{\tilde{X}} \\ ?\uparrow & & \uparrow \alpha_{\pi_Y} & & \uparrow Rf_*(\alpha_{\pi_X}) & & \downarrow ? \\ IC_Y & \xleftarrow{\omega_Y} & \mathbb{Q}_Y & \xrightarrow{\alpha_f} & Rf_*\mathbb{Q}_X & \xrightarrow{Rf_*(\omega_X)} & Rf_*IC_X. \end{array}$$

To prove the existence of a morphism $\lambda_f : IC_Y \rightarrow Rf_*IC_X$, we will show that the arrows with question marks exist in a way that the diagram remains commutative. The existence of such morphisms follows from the Decomposition Theorem for π_Y and π_X (see the corollary). The sheaf IC_Y is a direct summand in $R\pi_{Y*}IC_{\tilde{Y}}$:

$$i : IC_Y \hookrightarrow R\pi_{Y*}IC_{\tilde{Y}}.$$

We also have a projection:

$$p : R\pi_{X*}IC_{\tilde{X}} \twoheadrightarrow IC_X,$$

which induces

$$Rf_*(p) : Rf_*R\pi_{X*}IC_{\tilde{X}} \twoheadrightarrow Rf_*IC_X.$$

It remains to prove the commutativity of the diagram. We compare the morphisms over Y :

$$\mathbb{Q}_Y \xrightarrow{\omega_Y} IC_Y \xrightarrow{i} R\pi_{Y*}IC_{\tilde{Y}} = R\pi_{Y*}\mathbb{Q}_{\tilde{Y}}$$

and the natural one

$$\mathbb{Q}_Y \xrightarrow{\alpha_{\pi_Y}} R\pi_{Y*}\mathbb{Q}_{\tilde{Y}}.$$

Respectively over X we compare the morphisms:

$$\mathbb{Q}_X \xrightarrow{\alpha_{\pi_X}} R\pi_{X*}\mathbb{Q}_{\tilde{X}} \xrightarrow{R\pi_{X*}(\omega_{\tilde{X}})} R\pi_{X*}IC_{\tilde{X}} \xrightarrow{p} IC_X$$

and the canonical one

$$\mathbb{Q}_X \xrightarrow{\omega_X} IC_X.$$

Let U (resp. V) be the regular part of Y (resp. X). After multiplication by a constant if necessary, these morphisms are equal on U (resp. on V). We will show that an equality of morphisms over an open set implies the equality over the whole space. We have the restriction morphism

$$\begin{array}{ccc} Hom(\mathbb{Q}_Y, R\pi_{Y*}\mathbb{Q}_{\tilde{Y}}) & \xrightarrow{\rho_U} & Hom((\mathbb{Q}_Y)|_U, (R\pi_{Y*}\mathbb{Q}_{\tilde{Y}})|_U) \\ \parallel & & \parallel \\ H^0(\tilde{Y}) & & H^0(\pi_Y^{-1}(U)). \end{array}$$

The kernel of ρ_U is $H^0(\tilde{Y}, \pi_Y^{-1}(U))$, which is trivial. We have the same for the morphisms over X :

$$\begin{array}{ccc} Hom(\mathbb{Q}_X, IC_X) & \xrightarrow{\rho_V} & Hom((\mathbb{Q}_X)|_V, (IC_X)|_V) \\ \parallel & & \parallel \\ IH^0(X) & & IH^0(V). \end{array}$$

The kernel is $IH^0(X, V) = 0$. □

Remark. The restriction morphisms ρ_U and ρ_V are in fact isomorphisms. The cokernel of ρ_U is contained in $H^1(\tilde{Y}, \pi_Y^{-1}(U)) = H_{2(\dim \tilde{Y})-1}^{cl}(\tilde{Y} \setminus \pi_Y^{-1}(U))$ which is trivial for dimensional reasons. The second follows from [Bo, V.9.2 p.144] as noticed in [BBFGK, p.178].

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REFERENCES

- [Bo] A. Borel (ed.), *Intersection cohomology*, Progress in Mathematics Vol. 50 (A. Borel, ed.), Birkhäuser, 1984. MR **88d**:32024
- [BBD] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux Pervers*, Astérisque **100** (1983). MR **86g**:32015
- [BBFGK] G. Barthel, J.-P. Brasselet, K.-H. Fieseler, O. Gabber, L. Kaup, *Relèvement de cycles algébriques et homomorphismes associés en homologie d'intersection*, Ann. Math **141** (1995), 147-179. MR **96a**:14027
- [GM1] M. Goresky, R. MacPherson, *Intersection homology II*, Invent. Math. **72** (1983), 77-130. MR **84i**:57012
- [GM2] M. Goresky, R. MacPherson, *On the topology of complex algebraic maps*, Geometry La Rabida, Lecture Notes in Mathematics, vol. 961, Springer Verlag, N. Y., 1982, pp. 119-129. MR **85f**:32019
- [GM3] M. Goresky, R. MacPherson, *Lefschetz fixed point theorem for intersection homology*, Comm. Math. Helv. **60** (1985), 366-391. MR **87f**:32030
- [Sa] M. Saito, *Decomposition theorem for proper Kähler morphisms*, Tôhoku Math. J. **42** (1990), 127-148. MR **91j**:32042

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