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PRODUCTS OF ORTHOGONAL PROJECTIONS

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Abstract. We give a characterization of operators on a separable Hilbert space of norm less than one that can be represented as products of orthogonal projections and give an estimate on the number of factors. We also describe the norm closure of the set of all products of orthogonal projections.

1. Introduction

In this paper, we study products of orthogonal projections on separable (finite or infinite-dimensional) Hilbert spaces. Throughout the work, the word *projection* is reserved for orthogonal projections on a Hilbert space. The word *idempotent* refers to a (not necessarily orthogonal) projection.

The problem of describing operators acting on a finite-dimensional complex Hilbert space which can be represented as products of orthogonal projections was solved by Kuo and Wu. In [KW1] they proved that $u : \ell_2^n \to \ell_2^n$ is a product of projections if and only if either u is the identity map or $u = I_m \oplus S$, where I_m is the identity on an m-dimensional subspace of ℓ_2^n and S is a singular strict contraction. In [KW2], Kuo and Wu characterized all the selfadjoint operators on a separable Hilbert space which factor into a product of projections.

In Section 2 we give estimates on the number of projections necessary to represent a given operator $u : \ell_2^n \to \ell_2^n$ (Theorem 1). We also give a new characterization of finite-dimensional Hilbert spaces: E is isometric to a Hilbert space if and only if every singular strict contraction $u : E \to E$ is a norm limit of products of contractive projections (Corollary 7).

In Section 3 we characterize all operators on a separable Hilbert space of the form $I \oplus S$ with $||S|| < 1$, which are products of projections (Theorem 10), and describe the norm closure of the set of products of projections (Corollary 11). We also disprove Conjecture 3.3 of [KW2] by constructing an operator $T : \ell_2 \to \ell_2$ which is a product of idempotents such that for every $x \in \ell_2$, $||(I-T)x||^2 \leq ||x||^2 - ||Tx||^2$, yet T cannot be represented as a product of projections (Proposition 14).

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2. Finite-dimensional results

The main result of this section is the following

Theorem 1. *If $u : \ell_2^n \to \ell_2^n$ is an operator of norm less than 1 with nontrivial kernel, then u can be factored into a product of projections. Moreover, if $M(u)$ denotes the minimal number of projections necessary to represent u, then there exists a constant C such that*

$$M(u) \leq C \frac{n}{n - \operatorname{rank} u} \frac{1}{1 - ||u||}.$$

This estimate is optimal: for all positive integers $n > m$ and $0 < a < 1$ there exists an operator $u : \ell_2^n \to \ell_2^n$ of norm a and rank m such that

$$M(u) \geq \frac{1}{2} \frac{n}{n - m} \frac{1}{1 - a}.$$

The fact that every singular strict contraction $u : \ell_2^n \to \ell_2^n$ can be represented as a product of projections was proved in [KW1] (in the complex case). Their method shows that $M(u) \leq Cn/(1 - ||u||)$, and the lower estimate Kuo and Wu get (for a certain u) is $c/(1 - ||u||)$.

To prove Theorem 1, we need four lemmas.

Lemma 2. (*a*) *Suppose x and y are vectors in ℓ_2^n that are not parallel and such that $||y|| < ||x||$ and $\langle x, y\rangle$ is real. Then there exist projections $P_1, \dots, P_k$ such that $P_1 \dots P_k x = y$. Moreover, $P_1, \dots, P_k$ can be chosen in such a way that they fix $\operatorname{span}[x, y]^\perp$ and $k \leq c_1/(1 - ||y||/||x||)$, where c_1 is an absolute constant.*

(*b*) *Suppose $u_i : \ell_2^{n_i} \to \ell_2^{n_i}$ $(i \in I)$ are linear operators such that $u_i = P_{i1} \dots P_{ik_i}$, where $P_{ij} : \ell_2^{n_i} \to \ell_2^{n_i}$ are projections and $\sup_i k_i < \infty$. Consider $u = \bigoplus_{i \in I} u_i : \ell_2^n \to \ell_2^n$, where $n = \sum_{i \in I} n_i$. Then $u = Q_1 \dots Q_k$, where the Q_i's are projections and $k = \sup_{i \in I} k_i$.*

Proof. Part (a) is essentially contained in [KW1]; we include the proof for the sake of completeness. If $||y|| = 0$, the proof is obvious. Otherwise, assume, w.l.og., that $||x|| = 1$, let $a = ||y||$, and set $z = y/||y||$. Let $H = \operatorname{span}[x, z]^\perp$. Find $k \geq 5$ such that $(\cos(\pi/k))^k > a$. Since

$$\left(\cos \frac{\pi}{k} \right)^k \geq \left(1 - \frac{\pi^2}{k^2} \right)^{k/2} \geq e^{-\pi^2/k},$$

we can take $k = \max\{5, \lceil -\pi^2/\ln a \rceil\}$. Find $t \in \ell_2^n$ of norm 1 such that $x = (\cos \phi_0)\, z + (\sin \phi_0)\, t$ with $0 \leq \phi_0 \leq \pi$ (so $t \in \operatorname{span}[x, z]$ and t is orthogonal to z). Let Q_i $(1 \leq i \leq k)$ be the orthogonal projection onto $(\cos \phi_i)\, z + (\sin \phi_i)\, t$, where $\phi_k = 0$, and set $P_i = I_H \oplus Q_i$ (I_H stands for the identity map on H). Then P_i fixes H and $P_k \dots P_1 x = \prod_{i=1}^k \big(\cos(\phi_i - \phi_{i-1}) \big) z$. By a suitable choice of $\phi_1, \dots, \phi_{k-1}$, we insure that $P_k \dots P_1 x = az = y$. To prove part (b), let $P_{ij} = I_{\ell_2^{n_i}}$ (the identity map on $\ell_2^{n_i}$) for $k_i + 1 \leq j \leq k$, and set $Q_j = \bigoplus_{i \in I} P_{ij}$ for $1 \leq j \leq k$. Then Q_j is a projection, and $u = Q_1 \dots Q_k$. □

Lemma 3. *If $u : \ell_2^n \to \ell_2^n$ is a normal operator of norm < 1 and rank $m < n$, then there exist projections $P_1, \dots, P_k$ such that $u = P_k \dots P_1$ and*

$$k \leq c_2 n/((n - m)(1 - ||u||)),$$

where c_2 is an absolute constant.

Proof. Let $E = (\ker u)^\perp = \operatorname{ran} u$. Then E is an m-dimensional subspace of ℓ_2^n. Fix orthonormal vectors $f_1, \dots, f_{n-m}$ in $E^\perp$. Let $a = \sqrt{||u||}$. Consider the complex case first. Let $a_1, \dots, a_m$ be nonzero eigenvalues of u, and $e_1, \dots, e_m$ the corresponding orthonormal eigenvectors. By Lemma 2, there exist projections $P_1, \dots, P_{\ell_1}$ which fix $e_{n-m+1}, \dots, e_m$, $P_{\ell_1} \dots P_1 e_i = a f_i$ for $1 \le i \le n-m$, and $\ell_1 \le c_1/(1-a)$. Similarly, there exist projections $P_{\ell_1+1}, \dots, P_{k_1}$ which fix $e_{n-m+1}, \dots, e_m$, $P_{k_1} \dots P_{\ell_1+1} a f_i = a_i e_i$ for $1 \le i \le n-m$, and $k_1 - \ell_1 \le c_1/(1 - a/||u||) = c_1/(1-a)$. Thus,

$$P_{k_1} \dots P_1 e_i = \begin{cases} a_i e_i, & 1 \le i \le n-m, \\ e_i, & i > n-m, \end{cases}$$

and $k_1 \le 2c_1/(1-\sqrt{||u||}) \le 4c_1/(1-||u||)$. Likewise, we can find $P_{k_2}, \dots, P_{k_1+1}$ such that

$$P_{k_2} \dots P_1 e_i = \begin{cases} a_i e_i, & 1 \le i \le 2(n-m), \\ e_i, & i > 2(n-m), \end{cases}$$

and $k_2 - k_1 \le 4c_1/(1-||u||)$; hence $k_2 \le 2 \cdot 4c_1/(1-||u||)$. Proceeding further in the same manner, we show the existence of projections $P_1, \dots, P_k$ ($k \le \lceil m/(n-m) \rceil \cdot 4c_1/(1-||u||)$) such that $P_k \dots P_1 e_i = a_i e_i$ for $1 \le i \le m$. Then $u = P_k \dots P_1 P_E$, where P_E is the orthogonal projection onto E.

In the real case the situation is a little more complicated: there exists an orthonormal basis $e_1, \dots, e_s, g_{11}, g_{12}, \dots, g_{r1}, g_{r2}$ in E ($s + 2r = m$) such that $ue_i = a_i e_i$ ($1 \le i \le s$), $ug_{j1} = b_{j1} g_{j1} + b_{j2} g_{j2}$ and $ug_{j2} = -b_{j2} g_{j1} + b_{j1} g_{j2}$ ($1 \le j \le r$); here, a_i, b_{j1} and b_{j2} are real, $0 < a_i \le ||u||$ and $0 < b_{j1}^2 + b_{j2}^2 \le ||u||^2$. As before, we can find projections $P_1, \dots, P_\ell$ ($\ell \le c_1 \lceil s/(n-m) \rceil/(1-||u||)$) which fix g_{jp} ($1 \le j \le r$, $p = 1, 2$) and such that $P_\ell \dots P_1 e_i = a_i e_i$ ($1 \le i \le s$). By Lemma 2, there exist projections $P_{\ell+1}, \dots, P_{\ell+\ell_1}$ ($\ell_1 \le c_1/(1-a)$) which fix e_i ($1 \le i \le s$), g_{j1} ($n-m+1 \le j \le r$) and g_{j2} ($1 \le j \le r$), such that $P_{\ell+\ell_1} \dots P_{\ell+1} g_{j1} = a f_j$ ($1 \le j \le n-m$). Also, there exist projections $P_{\ell+\ell_1+1}, \dots, P_{\ell+\ell_2}$ ($\ell_2 - \ell_1 \le c_1/(1-||u||)$) which fix e_i ($1 \le i \le s$), g_{jp} ($n-m+1 \le j \le r$, $p = 1, 2$) and f_j ($1 \le j \le n-m$), such that $P_{\ell+\ell_2} \dots P_{\ell+\ell_1+1} g_{j2} = -b_{j2} g_{j1} + b_{j1} g_{j2}$. Finally, find projections $P_{\ell+\ell_2+1}, \dots, P_{\ell+k_1}$ ($k_1 - \ell_2 \le c_1/(1-a)$) which fix e_i ($1 \le i \le s$), g_{jp} ($n-m+1 \le j \le r$, $p = 1, 2$) and $-b_{j2} g_{j1} + b_{j1} g_{j2}$ ($1 \le j \le n-m$) such that $P_{\ell+k_1} \dots P_{\ell+\ell_2+1} a f_j = b_{j1} g_{j1} + b_{j2} g_{j2}$.

Then, $P_{\ell+1}, \dots, P_{\ell+k_1}$ fix e_i ($1 \le i \le s$) and g_{jp} ($n-m+1 \le j \le s$, $p = 1, 2$), $k_1 \le 5c_1/(1-||u||)$, and $P_{\ell+k_1} \dots P_{\ell+1} g_{j1} = b_{j1} g_{j1} + b_{j2} g_{j2}$, $P_{\ell+k_1} \dots P_{\ell+1} g_{j2} = b_{j1} g_{j2} - b_{j2} g_{j1}$ for $1 \le j \le n-m$. Proceeding further in the same manner, we find projections $P_{\ell+1}, \dots, P_k$ ($k - \ell \le 5c_1 \lceil r/(n-m) \rceil/(1-||u||)$) which fix e_i ($1 \le i \le s$) and

$$P_k \dots P_{\ell+1} g_{j1} = b_{j1} g_{j1} + b_{j2} g_{j2}\ , \ \ P_k \dots P_{\ell+1} g_{j2} = -b_{j2} g_{j1} + b_{j1} g_{j2}.$$

Therefore, $P_k \dots P_1 P_E = u$, and $k \le c_2 n/((n-m)(1-||u||))$. This completes the proof of the lemma. □

Lemma 4. *Suppose H is a Hilbert space and $(e_i)_{i \in I}$, $(g_i)_{i \in I}$ are systems of norm 1 vectors such that $\langle e_i, g_i \rangle \ge 0$ and* span$[e_i, g_i]$ *is orthogonal to* span$[e_j, g_j]$ *if $i \ne j$. Suppose, furthermore, that there exists $0 < a < 1$ and scalars $(a_i)_{i \in I}$ such that $a_i = 1$ if $e_i = g_i$ and $0 \le a_i \le a$ if $e_i \ne g_i$. Then there exist projections $Q_1, \dots, Q_k$ ($k \le c_3/(1-a)$) such that $Q_k \dots Q_1 e_i = a_i g_i$ for every $i \in I$.*

Proof. Let $I_1 \stackrel{\text{def}}{=} \{i \in I \,|\, e_i \neq g_i\}$. Let $E_i = \operatorname{span}[e_i, g_i]$. By Lemma 2(b), it suffices to show that, for every $i \in I_1$, there exist projections $P_1, \ldots, P_k \in B(E_i)$ ($k \leq c_3/(1-a)$) such that $P_k \ldots P_1 e_i = a_i g_i$. However, the existence of such projections follows from Lemma 2(a). □

Lemma 5. *Suppose $0 < a < 1$ and $m < n$, and let E be an m-dimensional subspace of ℓ_2^n. Consider $u = -aP_E$, where P_E is the orthogonal projection onto E. Then, if $P_1, \ldots, P_k \in B(\ell_2^n)$ are projections such that $u = P_k \ldots P_1$, it follows that $k \geq n/(2(n-m)(1-a))$.*

Proof. Below, I and I_E will stand for the identities on ℓ_2^n and E, respectively. For $1 \leq j \leq k$ set

$$u_j \stackrel{\text{def}}{=} \Big(P_E P_j P_{j-1} \ldots P_1 - P_E P_{j-1} \ldots P_1\Big)\Big|_E = P_E\big(P_j - I\big)P_{j-1} \ldots P_1\Big|_E \in B(E).$$

Then, $u|_E - I_E = \sum_{j=1}^k u_j$. Hence $|| \sum_j u_j||_1 = ||u|_E - I_E||_1 = (1+a)m$, where $|| \cdot ||_p$ stands for the norm in Schatten class S_p. Note also that $\operatorname{rank} P_j \geq m$ for $1 \leq j \leq k$; hence $\operatorname{rank} u_j \leq s = \min\{m, n-m\}$.

Consider the polar decomposition $u_j = \sum_{\ell=1}^s e_{j\ell} \otimes f_{j\ell}$; in other words, for every $x \in E$, $u_j x = \sum_{\ell=1}^s f_{j\ell}(x) e_{j\ell}$. We may assume that $(f_{j\ell})_{\ell=1}^s$ are mutually orthogonal, and $(e_{j\ell})_{\ell=1}^s$ are orthonormal for every j. Note also that $P_j \ldots P_1 x$ is orthogonal to $P_j \ldots P_1 x - P_{j-1} \ldots P_1 x = (P_j - I)P_{j-1} \ldots P_1 x$. Therefore, $||P_{j-1} \ldots P_1 x||^2 - ||P_j \ldots P_1 x||^2 = ||(P_j - I)P_{j-1} \ldots P_1 x||^2 \geq ||u_j x||^2$. Hence,

$$\begin{aligned}(1-a^2)||x||^2 = ||x||^2 - ||P_k \ldots P_1 x||^2 &= \sum_{j=1}^k \Big(||P_{j-1} \ldots P_1 x||^2 - ||P_j \ldots P_1 x||^2\Big) \\ = \sum_{j=1}^k ||\big(I - P_j\big)P_{j-1} \ldots P_1 x||^2 &\geq \sum_{j=1}^k ||u_j x||^2 = \sum_{j,\ell} |f_{j\ell}(x)|^2.\end{aligned}$$

Now, let $N = ks$ and define operators $v : E \to \ell_2^N$ and $w : \ell_2^N \to E$ as follows:

$$vx \stackrel{\text{def}}{=} \Big(f_{j\ell}(x)\Big)_{j,\ell} \ , \ \ w\big((\delta_{j\ell})\big) \stackrel{\text{def}}{=} e_{j\ell}$$

(here, $(\delta_{j\ell})_{j,\ell}$ stands for the canonical basis in $\ell_2^N = \ell_2^k \otimes_2 \ell_2^s$). Then,

$$||v|| = \sup_{||x|| \leq 1} \Big(\sum_{j,\ell} |f_{j\ell}(x)|^2\Big)^{1/2} \leq \sqrt{1-a^2}.$$

Therefore, $||v||_2 \leq \sqrt{m}\sqrt{1-a^2}$. Moreover,

$$||w||_2 = \big(\sum_{j,\ell} ||w\delta_{j\ell}||^2\big)^{1/2} = \big(\sum_{j,\ell} ||e_{j\ell}||^2\big)^{1/2} = \sqrt{ks}.$$

Note that $u = wv$; hence $||u||_1 \leq ||w||_2 ||v||_2$. Thus, $(1+a)m \leq \sqrt{m}\sqrt{k}\sqrt{s}\sqrt{1-a^2}$. Therefore,

$$k \geq \frac{1+a}{1-a}\frac{m}{s} = \frac{1+a}{1-a}\frac{m}{\min\{m, n-m\}} \geq \frac{n}{2(n-m)(1-a)}.$$

The proof is now complete. □

Proof of Theorem 1. Consider an operator $u : \ell_2^n \to \ell_2^n$ of rank $m < n$. Set $a = ||u||^{1/3} < 1$; let $E = (\ker u)^{\perp}$, $F = \operatorname{ran} u$. If G is a subspace of ℓ_2^n, P_G will denote the projection onto G. First, we will introduce an operator $w_1 : E \to F$ such that $||w_1|| \le 1$, $||w_1^{-1}|| \le 1/a$ and $w_1 P_E$ is a product of $k \le c_3/(1-a) \le c_4/(1-||u||)$ projections.

To produce such a w_1, consider the restriction of the projection P_F onto E ($P_F|_E$) and the polar decomposition of this operator: there exist orthonormal bases $(e_i)_{i=1}^m$ and $(g_i)_{i=1}^m$ in E and F, respectively, such that $P_F e_i = b_i g_i$ ($0 \le b_i \le 1$). Since $P_F e_i = \sum_j \langle e_i, g_j\rangle g_j = b_i g_i$, $\operatorname{span}[e_i, g_i]$ is orthogonal to $\operatorname{span}[e_j, g_j]$ if $i \ne j$. Define a_i ($1 \le i \le m$) as follows:

$$a_i \stackrel{\text{def}}{=} \begin{cases} 1, & e_i = g_i, \\ a, & \text{otherwise}. \end{cases}$$

Define $w_1 : E \to F$ by setting $w_1 e_i = a_i g_i$. Then $||w_1|| \le 1$, $||w_1^{-1}|| \le 1/a$ and, by Lemma 4, $w_1 P_E$ can be represented as a product of at most $c_4/(1-||u||)$ projections.

Note that $\operatorname{ran}(w_1^{-1} u) = E$. Hence we can write $w_1^{-1} u|_E = w_2 w_3$, where $w_2 : E \to E$ is a unitary, $w_3 : E \to E$ is selfadjoint and $||w_3|| = ||w_1^{-1} u|_E|| \le ||w_1^{-1}||||u|| = a^2 < 1$. Then, $a w_2 P_E$ and $a^{-1} w_3 P_E$ are normal operators of rank m and norm $\le a$. By Lemma 3, each of them can be represented as a product of $\le c_2 n/((n-m)(1-a)) \le c_5 n/((n-m)(1-||u||))$ projections. However, u can be written as $u = (w_1 P_E)(a w_2 P_E)(a^{-1} w_3 P_E)$. Therefore, u can be factored into a product of

$$k \le \frac{c_4}{1-||u||} + 2\frac{c_5 n}{(n-m)(1-||u||)} \le C\frac{n}{n-m}\frac{1}{1-||u||}$$

projections. This establishes the upper estimate for $M(u)$. The optimality of this estimate follows from Lemma 5. □

Corollary 6. *Suppose $u : \ell_2^n \to \ell_2^n$ is a contraction with nontrivial kernel. Then u is a norm limit of products of projections.*

Corollary 7. *Suppose X is a finite-dimensional Banach space of dimension more than 2. Then the following are equivalent*:

(1) *X is isometric to a Hilbert space.*
(2) *There exists $0 < a \le 1$ such that for every operator $u : X \to X$ with 1-dimensional kernel and $||u|| \le a$ and for every $\epsilon > 0$, there exist contractive idempotents $P_1, \dots, P_k \in B(X)$ for which $||u - P_1 \dots P_k|| < \epsilon$.*

Proof. (1) ⇒ (2) follows from Corollary 6. We need to prove (2) ⇒ (1). Suppose, for the sake of contradiction, that X is not isometric to a Hilbert space. Then, by the result of Papini (see [P] or Theorem 12.8 of [A]), there exists a subspace $E \hookrightarrow X$ of codimension 1 which is not 1-complemented in X. If E' is another subspace of X, define a "quasidistance"

$$\operatorname{dist}(E, E') \stackrel{\text{def}}{=} \sup_{x \in E, ||x|| \le 1} \inf_{y \in E'} ||x - y||.$$

By Lemma 2.c.8 of [LT], if $\operatorname{codim} E' \ge 2$, $\operatorname{dist}(E, E') \ge 1$. Moreover, $\operatorname{dist}(E, E')$ is equivalent to the Hausdorff distance between $E \cap B_X$ and $E' \cap B_X$, where B_X stands for the unit ball of X. By compactness, there exists $\delta > 0$ such that, whenever $\operatorname{dist}(E, E') \le \delta$, E' is not 1-complemented.

We also know there exists a projection $P : X \to E$ with $||P|| \le 2$. Let $u = aP/2$. We will show that if $P_1, \dots, P_k \in B(X)$ are contractive idempotents, then

$||u - P_1 \dots P_k|| \geq a\delta/2$. Indeed, let $F = \operatorname{ran} P_1$. Then $\operatorname{dist}(E, F) \geq \delta$. On the other hand, $(a/2)B_X \cap E \subset uB_X$. Since, for every $x \in B_X$, $P_1 \dots P_k x \in F$,

$$\operatorname{dist}(E,F) = \max_{e \in E, ||e|| \leq 1} \min_{f \in F} ||e - f|| = \frac{2}{a} \max_{e \in E, ||e|| \leq a/2} \min_{f \in F} ||e - f||$$
$$\leq \frac{2}{a} \max_{x \in B_X} \min_{f \in F} ||e - f|| \leq \frac{2}{a} \max_{x \in B_X} ||ux - P_1 \dots P_k x|| \leq \frac{2}{a} ||u - P_1 \dots P_k||.$$

Therefore, $||u - P_1 \dots P_k|| \geq a\delta/2$ for every choice of contractive idempotents $P_1, \dots, P_k$. This contradicts (2). Thus, (2) implies (1). □

Remark. Products of idempotents on a Hilbert space have also been studied (see [D] or [E]). In particular, J. A. Erdos proved in [E] that every operator $u : \ell_2^m \to \ell_2^m$ with nontrivial kernel is a product of idempotents. However, the products of norms of these idempotents might be very large. More precisely, let $F(u) = \inf\{||P_1|| \dots ||P_k|| \mid P_k \dots P_1 = u\}$ (here, the infimum runs over all positive integers k and over all k-tuples of idempotents $P_1, \dots, P_k$). Let E be an n-dimensional subspace of ℓ_2^{n+1}, and denote the orthogonal projection onto it by P_E; let $G(b, n) = F(bP_E)$. If $||u : \ell_2^m \to \ell_2^m|| \leq b$, $\operatorname{rank} u = n < m$ and $b > 1$, then, by Theorem 1, $G(b, n) \geq F(u)$.

Proposition 8. *In the notation of the Remark above, if $b > 1$ and n is a positive integer, then $G(b, n) = b^n$.*

Proof. Suppose E is an n-dimensional subspace of ℓ_2^{n+1}, I an identity map on E, and b is as in the formulation of the proposition. Suppose, furthermore, that $P_1, \dots, P_k : \ell_2^{n+1} \to \ell_2^{n+1}$ are idempotents, for which $P_k \dots P_1|_E = bI$. Clearly, $P_1, \dots, P_k$ have rank n. Set $E_0 = E$, $E_i = \operatorname{ran} P_i$, and $Q_i = P_i|_{E_{i-1}}$ $(1 \leq i \leq k)$. We will view $E_0, \dots, E_k$ as copies of ℓ_2^n, and Q_i $(1 \leq i \leq k)$ as linear maps between E_{i-1} and E_i. Note that Q_i coincides with the identity map on a subspace H_i of E_{i-1} of dimension $n - 1$. Moreover, one can show that, if $x \in E_{i-1}$ is orthogonal to H_i, then $Q_i x$ is orthogonal to $Q_i H_i$. Hence, we can identify E_i and E_{i+1} in such a way that $Q_i = \operatorname{diag}\{1, \dots, 1, y_i, 1, \dots, 1\}$ (y_i stands on k_i-th place). Thus, $|\det Q_i| = |y_i| \leq ||Q_i|| \leq ||P_i||$. If $Q_k \dots Q_1 = bI$,

$$b^n \leq \left|\det(Q_k \dots Q_1)\right| \leq \prod_i \left|\det Q_i\right| \leq \prod_i \left|\left|P_i\right|\right|.$$

On the other hand, fix $\delta > 0$, and let $Q_i = \operatorname{diag}\{1, \dots, 1, b + \delta, 1, \dots, 1\}$ ($b + \delta$ stands on i-th place, $1 \leq i \leq n$). Let $P_1, \dots, P_k$ be idempotents such that $\operatorname{ran} P_i = E_i = \left(\ker P_{i+1}\right)^\perp$ and $Q_i = P_i\big|_{E_{i-1}}$. In this case, $||P_i|| = ||Q_i|| = b + \delta$ and, for any $x \in E$, $||P_k \dots P_1 x|| \geq (b + \delta)||x||$. By Theorem 1, we can find orthogonal projections $P_{k+1}, \dots, P_\ell$ such that $P_\ell \dots P_{k+1} P_k \dots P_1\big|_E = bI$. Thus, $P_\ell \dots P_{k+1} P_k \dots P_1 P_E = bP_E$. This completes the proof, since δ can be chosen to be arbitrarily small. □

3. Infinite-dimensional results

In this section we consider products of projections on a separable Hilbert space. We start by quoting Proposition 2.4 of [KW2]:

Proposition 9. *If T is a product of projections, then $T = I \oplus S$ with respect to $K^\perp \oplus K$, where I is the identity on $K^\perp$ and S is a completely nonunitary contraction* *and a product of projections. In this case, either* $\dim\ker S = \dim\ker S^* = \infty$ *or* $\dim K < \infty$ *and* S *is noninvertible with* $||S|| < 1$.

We have not been able to formulate necessary and sufficient conditions for an operator on a Hilbert space to be representable as a product of projections. It turns out that this problem may be quite complicated (see Proposition 14 and the Remark preceding it). We were, however, able to characterize all the operators of the form $T = I \oplus S$ with $||S|| < 1$ that are products of projections, and to estimate the minimal number of projections $M(T)$ necessary to represent such an operator.

Theorem 10. *Suppose* $T : \ell_2 \to \ell_2$ *is a linear operator and* $T = I \oplus S$ *with respect to* $K^\perp \oplus K$ *with* $||S|| < 1$. *Then* T *is a product of projections if and only if either* $\dim K < \infty$ *and* S *has nontrivial kernel, or* $\dim\ker S = \operatorname{codim}\operatorname{ran} S = \infty$. *Moreover, there exists a constant* C *such that*

$$M(T) \leq C \dim K/((\dim(\ker S))(1 - ||S||))$$

if $\dim K < \infty$, *and* $M(T) \leq C/(1 - ||S||)$ *if* $\dim K = \infty$. *These estimates are optimal*: *for every* a, $0 < a < 1$, *and for every subspace* $K \hookrightarrow \ell_2$ *of infinite codimension, there exist operators* $T = I \oplus S$ *with* $||S|| = a$ *such that* $M(T) \geq \dim K/(2(\dim(\ker S))(1 - a))$ (*K finite-dimensional*) *or* $M(T) \geq 1/(2(1 - a))$ (*K infinite-dimensional*).

Here and below, $\overline{\mathcal{A}}$ stands for the norm closure of $\mathcal{A} \subset B(\ell_2)$.

Corollary 11. *Let* $\mathcal{P}$ *be the set of all products of projections in* $B(\ell_2)$. *Then*

$$\overline{\mathcal{P}} = \{I\} \cup \big((I + \mathcal{K}_1) \cap \overline{B_1}\big) \cup \big((\mathcal{S} + \mathcal{K}) \cap \overline{B_1}\big).$$

Here, I *stands for the identity on* ℓ_2, $\mathcal{K}$ *is the set of compact operators,* $\mathcal{K}_1$ *is the set of compact operators* u *for which there exists* $x \in \ell_2$ *such that* $ux = -x$, $\mathcal{S}$ *is the set of operators with infinite-dimensional kernel and infinite codimensional range, and* B_1 *is the open unit ball of* $B(\ell_2)$.

To prove Theorem 10, we need the following

Lemma 12. *Suppose* $u : \ell_2 \to \ell_2$ *is a linear map and* $u = 0 \oplus aU$ *with respect to* $K^\perp \oplus K$, *where* $\dim K = \dim K^\perp = \infty$, U *is a unitary and* $0 < |a| < 1$. *Then* u *is a product of projections and* $M(u) \leq c_6/(1 - |a|)$, *where* c_6 *is a constant. Moreover, this estimate is optimal*: *if* $0 < a < 1$, I_K *is the identity on* K *and* $u = 0 \oplus (-aI_K)$, *then* $M(u) \geq 1/(2(1 - a))$.

It follows from Theorem 2.5 of [KW2] that u is a product of projections. However, Kuo and Wu did not estimate the number of projections necessary to represent u.

Proof. Fix orthonormal bases $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ in K and $K^\perp$, respectively. By Lemma 2, there exist projections $P_1, \ldots, P_k$ ($k \leq c_1/(1 - \sqrt{|a|})$) such that $P_k \ldots P_1 e_i = \sqrt{|a|} f_i$ for every i. Applying Lemma 2 again, we find projections $P_{k+1}, \ldots, P_\ell$ ($\ell - k \leq c_1/(1 - \sqrt{|a|})$) such that $P_\ell \ldots P_{k+1} \sqrt{|a|} f_i = aUe_i$. Thus, $u = P_K P_\ell \ldots P_1 P_K$, where P_K is the projection onto K. This proves that $M(u) \leq c_6/(1 - |a|)$.

Now fix a, $0 < a < 1$, and set $u = 0 \oplus (-aI_K) = -aP_K$. Fix a nonzero $x \in K$. Then $ux = -ax$. Define $\tilde{u} : \ell_2 \to \ell_2$ by setting $\tilde{u}x = -ax$, $\tilde{u}|_{\operatorname{span}[x]^\perp} = 0$. Then $M(u) \geq M(\tilde{u})$ and, by Lemma 5, $M(\tilde{u}) \geq 1/(2(1 - a))$. This establishes the lower estimate for $M(u)$. □

As a direct corollary of Lemma 12 and the proof of Theorem 2.5 of [KW2] we obtain

Theorem 13. *If $T = S \oplus 0 \in B(\ell_2)$, where $||S|| < 1$ and 0 acts on an infinite-dimensional space, then T is a product of projections and $M(T) \le c_7/(1 - ||S||)$, where c_7 is a constant.*

Proof of Theorem 10. We only need to prove the "only if" part and estimate $M(T)$. Lemma 2(b) implies that $M(T) = M(S)$, where S is viewed as an element of $B(K)$. If K is finite-dimensional, $||S|| < 1$ and S has nontrivial kernel, S is a product of projections by Theorem 1; the same theorem gives us the estimates for $M(S)$. Consider now the case $\dim K = \dim \ker S = \operatorname{codim} \operatorname{ran} S = \infty$. Let $E_1 = (\ker S)^\perp$, $E_2 = \overline{\operatorname{ran} S}$. If E_1 and E_2 are finite-dimensional, S is a product of at most $C/(1 - ||S||)$ projections by Theorem 1.

Otherwise, both E_1 and E_2 are infinite-dimensional. It suffices to show that there exist infinite-dimensional subspaces $F_1 \hookrightarrow E_1$ and $F_2 \hookrightarrow E_2$ and an isomorphism $U : F_1 \to F_2$ such that UP_{F_1} (here and below, P_G denotes the projection mapping K onto its subspace G) is a product of at most $c_1/(1 - ||S||^{1/3})$ projections acting on K and $||U|| \le 1$, $||U^{-1}|| \le ||S||^{1/3}$. Indeed, then there exist operators $V_1 : E_1 \to F_1$ and $V_2 : F_2 \to E_2$ such that $||V_1|| \le ||S||^{1/3}$, $||V_2|| \le ||S||^{1/3}$ and $S = V_2UV_1P_{E_1} = V_2P_{F_2}UP_{F_1}V_1P_{E_1}$. By Theorem 13, $V_1P_{E_1}$ and $V_2P_{F_2}$ are products of at most $c_7/(1 - ||S||^{1/3})$ projections each. Hence S is a product of projections, and $M(S) \le M(V_1P_{E_1}) + M(UP_{F_1}) + M(V_2P_{F_2}) \le C/(1 - ||S||)$.

To produce F_1, F_2 and U as above, we begin by selecting orthonormal systems $e_1, e_2, \dots$ and $g_1, g_2, \dots$ in E_1 and E_2, respectively, in such a way that e_i is orthogonal to g_j if $i \ne j$. First pick a unit vector $e_1 \in E_1$. If $e_1 \in E_2$, set $g_1 = e_1$. Otherwise, choose a unit vector $g_1 \in E_2$ arbitrarily. Now consider $E_1^{(1)} = E_1 \cap \big(\operatorname{span}[e_1, f_1]\big)^\perp$ and $E_2^{(1)} = E_2 \cap \big(\operatorname{span}[e_1, f_1]\big)^\perp$. In the same fashion, find unit vectors $e_2 \in E_1^{(1)}$ and $g_2 \in E_2^{(1)}$. Proceeding further in this manner, we obtain two systems of orthonormal vectors – $e_1, e_2, \dots \in E_1$ and $g_1, g_2, \dots \in E_2$ such that $\operatorname{span}[e_i, g_i]$ is orthogonal to $\operatorname{span}[e_j, g_j]$ if $i \ne j$. We can assume w.l.o.g. that $\langle e_i, g_i\rangle \ge 0$. Set $F_1 = \operatorname{span}[e_1, e_2, \dots] \hookrightarrow E_1$ and $F_2 = \operatorname{span}[g_1, g_2, \dots] \hookrightarrow E_2$. Let $a_i = 1$ if $e_i = g_i$, and $a_i = ||S||^{1/3}$ otherwise. Define $U : F_1 \to F_2$ by setting $Ue_i = a_ig_i$. By Lemma 4, UP_{F_1} can be represented as a product of no more than $c_1/(1 - ||S||^{1/3})$ projections. Thus, U is the isomorphism we need.

It follows from Lemma 12 that our estimate on $M(T)$ is optimal. □

Proof of Corollary 11. Let $\mathcal{P}_1$ be the set of products of projections with nontrivial finite-dimensional kernel, and let $\mathcal{P}_2$ stand for the set of products in which at least one projection has infinite-dimensional kernel. Note that if $u_1 \in \mathcal{P}_1$ and $u_2 \in \mathcal{P}_2$, then $||u_1 - u_2|| \ge 1$. Therefore, $\overline{\mathcal{P}} = I \cup \overline{\mathcal{P}_1} \cup \overline{\mathcal{P}_2}$.

We start by showing that $\overline{\mathcal{P}_1} = (I + \mathcal{K}_1) \cap \overline{B_1}$. By Corollary 6, $\overline{\mathcal{P}_1} \subset (I + \mathcal{K}_1) \cap \overline{B_1}$. To show the converse inclusion, consider $u \in \overline{\mathcal{P}_1}$; then there exists a sequence of operators $u_i = I + v_i$ such that $||u_i|| \le 1$, v_i is finite-dimensional, u_i has nontrivial kernel (in other words, $-1 \in \sigma(v_i)$) and $u = \lim_{i\to\infty} u_i$. Then $u = I + v$, where $v = \lim_{i\to\infty} v_i$; $||u|| \le 1$, and v is compact. Moreover, since u is a limit of operators with nontrivial kernels and the set of noninvertible operators is closed, u itself is not invertible. In other words, $-1 \in \sigma(v)$, and, by F. Riesz's theorem (see, e.g., Theorem VII.7.1 of [C]), there exists $x \in \ell_2$ such that $vx = -x$.

Now we are going to prove that $\overline{\mathcal{P}_2} = \big((\mathcal{S}+\mathcal{K})\cap\overline{B_1}\big)$. Clearly, $\mathcal{P}_2 \subset \mathcal{S}\cap\overline{B_1}$. On the other hand, by Theorem 10, $\mathcal{S}\cap B_1 \subset \mathcal{P}_2$. Therefore, $\overline{\mathcal{P}_2} = \overline{\mathcal{S}}\cap\overline{B_1}$. To complete the proof, we will show that $\overline{\mathcal{S}} = \mathcal{S}+\mathcal{K}$. Since, if $u \in \mathcal{S}$ and w is a finite rank operator, $u+w \in \mathcal{S}$, it is clear that $\mathcal{S}+\mathcal{K} \subset \overline{\mathcal{S}}$. We will prove the opposite inclusion. Consider $u \in \overline{\mathcal{S}}$. Note first that, for any $\epsilon > 0$, there are projections P and Q with infinite-dimensional kernels such that $||u-QuP|| < \epsilon/2$ and $||u-Qu|| < \epsilon$. Indeed, there exists $v \in \mathcal{S}$ for which $||u-v|| < \epsilon/4$. Let P and Q be projections onto $(\ker v)^\perp$ and $\overline{\operatorname{ran} v}$, respectively. Then $u-QuP = u-v+QvP-QuP = (u-v)+Q(u-v)P$. Hence $||u-QuP|| \le 2||u-v|| < \epsilon/2$. Moreover, $||Qu-QuP|| \le ||u-QuP||$. Therefore $||u-Qu|| \le ||u-QuP|| + ||QuP-Qu|| < \epsilon$.

Fix $u \in \overline{\mathcal{S}}$. By the reasoning above, we can find projections P_1 and Q_1 with infinite-dimensional kernels such that $||u-Q_1uP_1|| \le 1/6$ and $||Q_1u-Q_1uP_1|| \le 1/3$. Pick rank 1 projections p_1, q_1 such that $\ker(I-p_1) \subset \ker P_1$, $\ker(I-q_1) \subset \ker Q_1$. Let $u_1 = (1-q_1)u(1-p_1) = u-w_1$, where $w_1 = u-(1-q_1)u(1-p_1) = q_1u+up_1-q_1up_1$ is a finite rank operator. Thus, $u_1 \in \overline{\mathcal{S}}$. We will show that $||w_1|| \le 1/2$. Indeed,

$$w_1 = up_1 + q_1u(I-p_1) = (u-Q_1uP_1)p_1 + q_1(I-Q_1)u(I-p_1).$$

Therefore, $||w_1|| \le ||u-Q_1uP_1|| + ||(I-Q_1)u|| \le 1/2$.

Similarly, there exist projections P_2 and Q_2 with infinite-dimensional kernels such that $||u_1-Q_2u_1P_2|| \le 1/(2\cdot 6)$. We can assume w.l.o.g. that $\ker(I-p_1) \subset \ker P_2$ and $\ker(I-q_1) \subset \ker Q_2$. As before, we find rank 1 orthogonal projections p_2 and q_2 orthogonal to p_1 and q_1. Set $u_2 = (1-q_2)u_1(1-p_2)$ and $w_2 = u_1-u_2$. As above, we can show that w_2 is a finite rank operator and $||w_2|| \le 1/2^2$.

Proceed further in the same manner. After n steps we obtain two sets of mutually orthogonal projections $p_1,\dots,p_n$ and $q_1,\dots,q_n$ and finite rank operators $w_1,\dots,w_n$ such that $||w_k|| \le 1/2^k$, and $u_n = u-w_1-\dots-w_n$ satisfies the conditions $\ker u_n \subset \ker\big((I-p_n)\dots(I-p_1)\big)$ and $\operatorname{ran} u_n \subset \ker\big((I-q_n)\dots(I-q_1)\big)$, i.e. $\dim(\ker u_n) \ge n$ and $\operatorname{codim}(\operatorname{ran} u_n) \ge n$. Clearly, the sequence $\sum_k w_k$ converges to a compact operator w. Moreover, $\dim(\ker(u-w)) = \operatorname{codim}(\operatorname{ran}(u-w)) = \infty$. This shows that $\overline{\mathcal{S}} \subset \mathcal{K}+\mathcal{S}$. This completes the proof. □

Remark. In [KW2] it was noted that if $T : H \to H$ is a product of k nonnegative contractions, then, for every $x \in H$,

$$||(I-T)x||^2 \le k^2(||x||^2 - ||Tx||^2).$$

Kuo and Wu conjectured that if T is a product of idempotents and the above inequality holds for some k, then T is a product of nonnegative contractions (Conjecture 3.3). According to [D], T is a product of idempotents if and only if either of the following holds:

(1) $T = I$;

(2) $\dim\ker T = \dim\ker T^* = \infty$;

(3) $0 < \dim\ker T = \dim\ker T^*$ and $\dim\big(\{x : Tx = x\}^\perp\big) < \infty$.

Below, we present a counterexample to this conjecture of Kuo and Wu.

Proposition 14. *There exists an operator $T = u\oplus 0 : \ell_2\oplus_2\ell_2 \to \ell_2\oplus_2\ell_2$ such that $||(I-T)x||^2 \le ||x||^2 - ||Tx||^2$ for every $x \in \ell_2$, but T cannot be represented as a product of finitely many nonnegative contractions.*

Proof. Consider a block-diagonal operator

$$u = \operatorname{diag}\Big\{ \begin{pmatrix} (n^2-1)/n^2 & -\sqrt{n^2-1}/n^2 \\ \sqrt{n^2-1}/n^2 & (n^2-1)/n^2 \end{pmatrix} \Big\}_{n=1}^{\infty} : \ell_2 \to \ell_2,$$

and set $T = u \oplus 0$. Then u is normal and has eigenvalues a_n and $\overline{a_n}$ ($1 \leq n < \infty$), where

$$a_n = \frac{n^2-1}{n^2}\Big(1 + \frac{i}{\sqrt{n^2-1}}\Big).$$

Note that $|a_n|^2 = 1 - 1/n^2$ and $|1-a_n| = 1/n$. Therefore, for every $x \in \ell_2 \oplus_2 \ell_2$, $||(I-T)x||^2 \leq ||x||^2 - ||Tx||^2$. Clearly, it suffices to consider the complex case (in the case of real space, we need to consider its complexification). Then, there exist eigenvectors $e_1, e_2, \dots$ such that $Te_n = a_n e_n$. We will show that if $T_1, \dots, T_k \in B(H)$ are nonnegative contractions (H is a Hilbert space) and, for some nonzero vector $e \in H$, $T_k \dots T_1 e = a_n e$, then $k \geq \sqrt{n^2-1}$. This will imply that T cannot be represented as a product of nonnegative contractions.

We will assume, w.l.o.g., that $||e|| = 1$. Let $f_0 = e$, $f_i = T_i \dots T_1 e$ ($1 \leq i \leq k$) and $g_i = f_{i+1} - f_i$ ($0 \leq i \leq k-1$). Then $\langle g_i, f_i \rangle = -\langle (I - T_{i+1}) f_i, f_i \rangle \leq 0$. We know that $\sum_{i=0}^{k-1} g_i = (a_n - 1)e$. Hence

$$\sum_{i=0}^{k-1} \Im \langle g_i, e \rangle = \Im (a_n - 1) = \frac{\sqrt{n^2-1}}{n^2}$$

(here $\Im z$ stands for the imaginary part of z). However, $e = f_i - \sum_{j=0}^{i-1} g_j$. Thus, $\langle g_i, e \rangle = \langle g_i, f_i \rangle - \sum_{j=0}^{i-1} \langle g_i, g_j \rangle$. Hence, $\Im \langle g_i, e \rangle \leq ||g_i|| \sum_{j=0}^{k-1} ||g_j||$. Therefore, $(\sum_{i=0}^{k-1} ||g_i||)^2 \geq \sqrt{n^2-1}/n^2$.

On the other hand, it was shown in [KW2] (see the proof of Proposition 2.1) that $||g_i||^2 \leq ||f_i||^2 - ||f_{i+1}||^2$. Thus, $\sum_{i=0}^{k-1} ||g_i||^2 \leq 1 - |a_n|^2 = 1/n^2$, which implies $(\sum_{i=0}^{k-1} ||g_i||)^2 \leq k/n^2$. Therefore, $k \geq \sqrt{n^2-1}$. This shows that T defined above cannot be represented as a product of finitely many nonnegative contractions. □

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