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# SPECTRAL MULTIPLIER THEOREM FOR $H^1$ SPACES ASSOCIATED WITH SOME SCHRÖDINGER OPERATORS

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ABSTRACT. Let $T_t$ be the semigroup of linear operators generated by a Schrödinger operator $-A = \Delta - V$, where $V$ is a nonnegative polynomial. We say that $f$ is an element of $H^1_A$ if the maximal function $\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)|$ belongs to $L^1$. A criterion on functions $F$ which implies boundedness of the operators $F(A)$ on $H^1_A$ is given.

## 1. Introduction

Let $A = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, where $V(x) = \sum_{\beta\leq\alpha} a_\beta x^\beta$ is a nonnegative polynomial, $V \not\equiv 0$. Let $\{T_t\}_{t>0}$ be the semigroup of linear operators (e.g. on $L^2(\mathbb{R}^d)$) generated by $-A$. Since $V$ is nonnegative, the Feynman-Kac formula asserts that the integral kernels $T_t(x,y)$ of the operators $T_t$ satisfy

$$0 \leq T_t(x,y) \leq (4\pi t)^{-d/2} \exp(-|x-y|^2/(4t)). \tag{1.1}$$

We say that a function $f$ is an element of the space $H^1_A$ associated to the operator $A$ if the maximal function

$$\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)| = \sup_{t>0} \Big| \int_{\mathbb{R}^d} T_t(x,y) f(y)\, dy \Big| \tag{1.2}$$

belongs to $L^1(\mathbb{R}^d)$. We put

$$\|f\|_{H^1_A} = \|\mathcal{M}f\|_{L^1(\mathbb{R}^d)}.$$

The spaces $H^1_A$ were studied in [6]. It was proved there that the elements $H^1_A$ can be characterized in terms of special atoms. The definition of atoms for $H^1_A$ is as follows. We first define an auxiliary function $m(x,V)$ (see [13]) by

$$m(x,V) = \sum_{\beta\leq\alpha} |D^\beta V(x)|^{1/(|\beta|+2)}. \tag{1.3}$$

Since $V$ is a nonzero polynomial, there is $c > 0$ such that $c \leq m(x,V) < \infty$. We set $\mathcal{B}_0 = \{x \in \mathbb{R}^d : c \leq m(x,V) < 1\}$, $\mathcal{B}_n = \{x \in \mathbb{R}^d : 2^{(n-1)/2} \leq m(x,V) < 2^{n/2}\}$

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for $n = 1, 2, 3, \ldots$. We say that a function $a$ is an atom for $H^1_A$ associated to a ball $B(y_0, r)$ with the center at $y_0$ and radius $r$ if

$$\operatorname{supp} a \subset B(y_0, r), \tag{1.4}$$

$$\|a\|_{L^\infty} \leq |B(y_0, r)|^{-1}, \tag{1.5}$$

$$\text{if } y_0 \in \mathcal{B}_n, \quad \text{then} \quad r \leq 2^{1-n/2}, \tag{1.6}$$

$$\text{if } y_0 \in \mathcal{B}_n \quad \text{and} \quad r \leq 2^{-1-n/2}, \quad \text{then} \quad \int a(x)\, dx = 0. \tag{1.7}$$

The atomic norm in the space $H^1_A$ is defined by

$$\|f\|_{H^1_A\text{-atom}} = \inf\{\sum_j |c_j|\}, \tag{1.8}$$

where the infimum is taken over all decompositions $f = \sum_j c_j a_j$, with $a_j$ being $H^1_A$ atoms and $c_j$ being scalars.

The theorem below was proved in [6].

**Theorem 1.9.** *There exists a constant $C > 0$ such that*

$$\frac{1}{C}\|f\|_{H^1_A} \leq \|f\|_{H^1_A\text{-atom}} \leq C\|f\|_{H^1_A}. \tag{1.10}$$

Let $E_A$ be the spectral decomposition of the operator $A$. For a bounded function $F$ on $\mathbb{R}^+$ we define the operator $F(A)$ (bounded at least on $L^2(\mathbb{R}^d)$) by

$$F(A)f = \int_0^\infty F(\lambda)\, dE_A(\lambda)f.$$

We say that a function $F$ on $\mathbb{R}$ belongs to the space $C(s)$, $s \geq 0$, if

$$\|F\|_{C(s)} = \begin{cases} \sum_{k=0}^{s} \sup |F^{(k)}(\lambda)| & \text{if } \ s \in \mathbb{Z}, \\ \|F^{([s])}\|_{\text{Lip }(s-[s])} + \sum_{k=0}^{[s]} \sup |F^{(k)}(\lambda)| & \text{if } \ s \notin \mathbb{Z} \end{cases}$$

is finite.

Our aim is to prove

**Theorem 1.11.** *Let $F$ be a bounded continuous function on $(0, \infty)$. If for some $\varepsilon > 0$ and a nonzero function $\varphi \in C_c^\infty(0, \infty)$ there exists a constant $C > 0$ such that for every $t > 0$*

$$\|\varphi(\,\cdot\,)F(t\,\cdot\,)\|_{C(\frac{d}{2}+\varepsilon)} \leq C, \tag{1.12}$$

*then the operator $F(A)$ is bounded on $H^1_A$.*

Spectral multiplier theorems on $L^p$ spaces attracted attention of many authors (cf. e.g. [1], [2], [7], [9], [11], [12], and references there). A. Hulanicki and E. Stein (cf. [10]) observed that if $\mathcal{L}$ is a sublaplacian on a stratified Lie group, then the convolution kernel $F(-\mathcal{L})(x)$, which corresponds to a spectral multiplier $F$, satisfies Calderón-Zygmund type estimates. This combined with the fact that the atoms for Hardy spaces on homogeneous groups satisfy moment conditions implies boundedness of the operator $F(-\mathcal{L})$ on these spaces (see also [3]). Our $H^1_A$ spaces are of a different nature than the classical Hardy spaces or the Hardy spaces on homogeneous groups. The main difference is that for some $H^1_A$ atoms no mean-value zero is required. Therefore some additional decay of kernels associated with the multiplier $F$ is needed (see Proposition 2.2).

Our proof of the theorem is based on the following characterization of $H^1_A$; cf. [6]. Let $\psi \in C_c^\infty(1/2\,,2)$ be such that $|\psi(\lambda)| \geq c > 0$ for $\lambda \in [\frac{3}{4}, \frac{7}{4}]$. We define a Littlewood-Paley square function $Sf$ by

$$Sf(x) = (\sum_{\mu\in\mathbb{Z}} |\Psi_\mu f(x)|^2)^{1/2}, \tag{1.13}$$

where

$$\Psi_\mu f = \psi(2^{-\mu}A)f = \int_0^\infty \psi(2^{-\mu}\lambda)\, dE_A(\lambda)f. \tag{1.14}$$

We have

**Proposition 1.15** (cf. [6]). *There exists a constant $C > 0$ such that*

$$\frac{1}{C}\|f\|_{H^1_A} \leq \|Sf\|_{L^1} + \|f\|_{L^1} \leq C\|f\|_{H^1_A}. \tag{1.16}$$

## 2. Proof of Theorem 1.11

Let us note that under the assumption of Theorem 1.11 the inequality (1.12) holds for every $\varphi \in C_c^\infty(0,\infty)$.

We fix $\varphi \in C_c^\infty(\frac{1}{2}, 2)$ such that $|\varphi(\lambda)| > c > 0$ for $\lambda \in [\frac{3}{4}, \frac{7}{4}]$ and

$$\sum_{\mu\in\mathbb{Z}} \varphi(2^{-\mu}\lambda) = 1 \quad \text{for} \quad \lambda > 0. \tag{2.1}$$

Set $Q_\mu(\lambda) = \varphi(2^{-\mu}\lambda)F(\lambda)$. We shall denote by $Q_\mu(A)(x,y) = Q_\mu(x,y)$ the integral kernel of the operator

$$Q_\mu(A) = \int_0^\infty Q_\mu(\lambda)\, dE_A(\lambda),$$

that is,

$$Q_\mu(A)f(x) = \int_{\mathbb{R}^d} Q_\mu(x,y)f(y)\, dy.$$

The following two propositions are crucial in our proof of Theorem 1.11.

**Proposition 2.2.** *Assume that $F$ satisfies (1.12). Then there exist a constant $\delta > 0$ and kernels $K_\mu(x,y) \geq 0$ which satisfy*

$$\sup_x \int_{\mathbb{R}^d} K_\mu(x,y)\, dy \leq 1 \quad \textit{and} \quad \sup_y \int_{\mathbb{R}^d} K_\mu(x,y)\, dx \leq 1 \tag{2.3}$$

*such that for every $b > 0$ there is a constant $C_b > 0$ such that*

$$|Q_\mu(x,y)| \leq C_b K_\mu(x,y)(1 + 2^{\mu/2}|x-y|)^{-\delta}(1 + 2^{-\mu/2}m(x,V))^{-b}. \tag{2.4}$$

**Proposition 2.5.** *There exists a constant $C > 0$ such that for every $\mu \in \mathbb{Z}$ we have*

$$\int_{\mathbb{R}^d} |Q_\mu(x,y) - Q_\mu(x,y_0)|\, dx \leq C2^{\mu/2}|y-y_0|. \tag{2.6}$$

Proofs of Propositions 2.2 and 2.5 will be presented in Section 3.

**Lemma 2.7.** *Let $a$ be an $H^1_A$ atom associated to a ball $B(y_0, r)$. Then*

$$\int_{B(y_0,4r)} |F(A)a(x)|\,dx \le C, \tag{2.8}$$

*where $C$ is independent of $a$.*

*Proof.* Obviously the operator $F(A)$ is bounded on $L^2(\mathbb{R}^d)$. Therefore, by the Schwarz inequality,

$$\int_{B(y_0,4r)} |F(A)a(x)|\,dx \le \|F(A)a\|_{L^2(\mathbb{R}^d)} |B(y_0, 4r)|^{1/2}$$
$$\le Cr^{d/2}\|a\|_{L^2(\mathbb{R}^d)} \le C.$$

□

Since $\|Sf\|_{L^2(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}$, the same argument as above leads to

**Lemma 2.9.** *There exists a constant $C > 0$ such that if $a$ is an $H^1_A$ atom associated to a ball $B(y_0, r)$, then*

$$\int_{B(y_0,4r)} S(F(A)a)(x)\,dx \le C. \tag{2.10}$$

**Lemma 2.11.** *Assume that $a$ is an $H^1_A$ atom associated to a ball $B(y_0, r)$. Let $l$ be a nonnegative integer such that $2^{-(l+1)/2} < r \le 2^{-l/2}$. Then*

$$\int_{|x-y_0|>4r} \sum_{\mu=l}^{\infty} |Q_\mu a(x)|\,dx \le C,$$

*with a constant $C$ independent of $a$.*

*Proof.* Applying Proposition 2.2, we get

$$\sum_{\mu=l}^{\infty} \int_{|x-y_0|>4r} |Q_\mu a(x)|\,dx \le \sum_{\mu=l}^{\infty} \sup_{y\in B(y_0,r)} \int_{|x-y_0|>4r} |Q_\mu(x, y)|\,dx$$
$$\le C_0 \sum_{\mu=l}^{\infty} \sup_{y\in B(y_0,r)} \int_{|x-y_0|>4r} K_\mu(x, y)(1 + 2^{\mu/2}|x - y|)^{-\delta}\,dx$$
$$\le C_0 \sum_{\mu=l}^{\infty} 2^{-\delta(\mu-l)/2} \le C.$$

□

Let us denote by $\sigma$ the largest integer such that spectrum of the operator $A \ge 2^\sigma$. Since the bottom of the spectrum of $A$ is strictly positive, the number $\sigma$ is well defined.

**Proposition 2.12.** *There exists a constant $C > 0$ such that if $a$ is an $H^1_A$ atom associated to a ball $B(y_0, r)$, where $y_0 \in \mathcal{B}_n$, $2^{-(n+2)/2} < r \le 2^{1-n/2}$, then*

$$\int_{\mathbb{R}^d} \sum_{\mu=\sigma}^{n+2} |Q_\mu a(x)|\,dx \le C. \tag{2.13}$$

*Proof.* By Proposition 2.2, we obtain

$$|Q_\mu(x,y)| \le \sum_{j=2}^{\infty} b_j K_\mu(x,y)(1+2^{\mu/2}|x-y|)^{-\delta}\chi_{[0,j]}(1+2^{-\mu/2}m(x,V))$$

$$= \sum_{j=2}^{\infty} b_j P_\mu^j(x,y),$$

where $b_j$ is a sequence of rapidly decaying positive numbers, that is,

$$\sum_{j=2}^{\infty} j^N b_j = C_N < \infty \quad \text{for every} \quad N > 0. \tag{2.14}$$

We have

$$\begin{aligned}P_\mu^j(x,y) &= K_\mu(x,y)(1+2^{\mu/2}|x-y|)^{-\delta}\chi_{[0,j]}(1+2^{-\mu/2}m(x,V))\\ &= K_\mu(x,y)(1+2^{\mu/2}|x-y|)^{-\delta}\chi_{[0,j]}(1+2^{-\mu/2}m(x,V))\chi_{[0,1]}(2^{\mu/2}|x-y|)\\ &+\sum_{k=1}^{\infty} K_\mu(x,y)(1+2^{\mu/2}|x-y|)^{-\delta}\chi_{[0,j]}(1+2^{-\mu/2}m(x,V))\chi_{(2^{(k-1)/2},2^{k/2}]}(2^{\mu/2}|x-y|)\\ &= P_\mu^{j,0}(x,y) + \sum_{k=1}^{\infty} P_\mu^{j,k}(x,y).\end{aligned}$$

Obviously,

$$\sum_{\mu=\sigma}^{n+2} |Q_\mu a(x)| \le \sum_{\mu=\sigma}^{n+2}\sum_{j=2}^{\infty}\sum_{k=0}^{\infty} b_j P_\mu^{j,k}|a|(x),$$

where

$$P_\mu^{j,k}|a|(x) = \int P_\mu^{j,k}(x,y)|a(y)|\,dy.$$

Let $a$ be an $H^1_A$ atom associated to a ball $B(y_0,r)$, where $y_0 \in \mathcal{B}_n$ and $2^{-(n+2)/2} < r \le 2^{1-n/2}$. Let us note that in this case no moment condition on $a$ is required.

If $P_\mu^{j,0}|a| \not\equiv 0$, then there exist $y \in B(y_0,r)$ and $x \in \mathbb{R}^d$ such that $P_\mu^{j,0}(x,y) \ne 0$. This implies $2^{\mu/2}|x-y| \le 1$, $|y-y_0| \le 2^{1-n/2}$, and $m(x,V) \le 2^{\mu/2}j$. Since $D^\beta V(y) = \sum_{\gamma\le\alpha} \frac{1}{\gamma!} D^{\gamma+\beta}V(x)(y-x)^\gamma$, we have $|D^\beta V(y)| \le C 2^{(|\beta|+2)\mu/2} j^{2|\alpha|+2}$ for every $\beta \le \alpha$. On the other hand there exists $\beta \le \alpha$ such that $|D^\beta V(y)| \ge \frac{1}{C} 2^{(|\beta|+2)n/2}$. Thus $n \le \mu + C\log_2 j$. This gives

$$\int \sum_{\mu=\sigma}^{n+2}\sum_{j=2}^{\infty} b_j P_\mu^{j,0}|a|(x)\,dx \le C\sum_{j=2}^{\infty} b_j \log_2 j \le C. \tag{2.15}$$

We now consider $P_\mu^{j,k}|a|$ for $k \ge 1$. If $P_\mu^{j,k}|a| \not\equiv 0$, then there exist $x \in \mathbb{R}^d$ and $y \in B(y_0,r)$ such that $P_\mu^{j,k}(x,y) \ne 0$. Therefore $|y-y_0| \le 2^{1-n/2}$, $m(x,V) \le 2^{\mu/2}j$, $2^{(k-1)/2} \le 2^{\mu/2}|x-y| \le 2^{k/2}$. This leads to $n \le \mu + C\log_2 j + k|\alpha|$. Let us note that $\int P_\mu^{j,k}|a|(x)\,dx \le C2^{-\delta(k-1)/2}$. Thus, by (2.14),

$$\sum_{\mu=\sigma}^{n+2}\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\int b_j P_\mu^{j,k}|a|(x)\,dx$$

$$\le C\sum_{j=2}^{\infty} b_j \sum_{\mu=\sigma}^{n+2}\sum_{k\ge(n-\mu-C\log_2 j)/|\alpha|} 2^{-\delta(k-1)/2} \le C<\infty.$$

□

**Proposition 2.16.** *There exists a constant $C>0$ such that if $a$ is an $H^1_A$ atom associated to a ball $B(y_0,r)$, $y\in\mathcal{B}_n$, $2^{-(l+1)/2}<r\le 2^{-l/2}\le 2^{-(n+2)/2}$, then*

$$\sum_{\mu=\sigma}^{l}\int_{\mathbb{R}^d}|Q_\mu(a)|\,dx\le C.$$

*Proof.* Let $a$ be an $H^1_A$ atom associated to a ball $B(y_0,r)$, such that $y\in\mathcal{B}_n$, $2^{-(l+1)/2}<r\le 2^{-l/2}\le 2^{-(n+2)/2}$. Then, by the definition of $H^1_A$ atoms, $a$ has mean-value 0. Therefore, applying Proposition 2.5, we obtain

$$\sum_{\mu=\sigma}^{l}\int_{\mathbb{R}^d}|Q_\mu a(x)|\,dx = \sum_{\mu=\sigma}^{l}\int_{\mathbb{R}^d}|\int_{B(y_0,r)}[Q_\mu(x,y)-Q_\mu(x,y_0)]a(y)\,dy|\,dx$$

$$\le \sum_{\mu=\sigma}^{l} C2^{\mu/2}\int_{B(y_0,r)}|y-y_0|\,|a(y)|\,dy \le C\sum_{\mu=\sigma}^{l}2^{\mu/2}r\le C.$$

□

*Proof of Theorem 1.11.* We first note that $\varphi(2^{-\mu}A)=0$ for every $\mu\le\sigma-1$. By Theorem 1.9 and Proposition 1.15 it suffices to show that there exists a constant $C>0$ such that

$$\|F(A)a\|_{L^1}\le C \tag{2.17}$$

and

$$\|(\sum_{\mu=\sigma}^{\infty}|\varphi(2^{-\mu}A)F(A)a(x)|^2)^{1/2}\|_{L^1(dx)}\le C \tag{2.18}$$

for every $H^1_A$ atom $a$. Let $a$ be an $H^1_A$ atom associated to a ball $B(y_0,r)$. In order to prove (2.17) it is sufficient, by Lemma 2.7, to show that

$$\int_{B(y_0,4r)^c}|F(A)a(x)|\,dx\le C, \tag{2.19}$$

with a constant $C$ independent of $a$. But this is a consequence of the equality $F(A)a(x)=\sum_{\mu=\sigma}^{\infty}Q_\mu(A)a(x)$, Lemma 2.11 and Propositions 2.12 and 2.16.

Now we turn to our proof of (2.18). By Lemma 2.9 we only need to show that

$$\int_{B(y_0,4r)^c}S(F(A)a)(x)\,dx\le C,$$

with $C$ independent of $a$.

By virtue of Lemma 2.11 and Propositions 2.12 and 2.16, we get

$$\int_{B(y_0,4r)^c} S(F(A)a)(x)\,dx = \int_{B(y_0,4r)^c} (\sum_{\mu=\sigma}^{\infty} |\varphi(2^{-\mu}A)F(A)a(x)|^2)^{1/2}\,dx$$

$$\leq \int_{B(y_0,4r)^c} \sum_{\mu=\sigma}^{\infty} |Q_\mu(A)a(x)|\,dx \leq C.$$

□

## 3. Proofs of Propositions 2.2 and 2.5

For an integral kernel $K(x,y)$ and a number $\delta > 0$ we write

$$\|K\|_{\omega(\delta)} = \max\{\sup_y \int |K(x,y)|(1+|x-y|)^\delta\,dx, \sup_x \int |K(x,y)|(1+|x-y|)^\delta\,dy\}.$$

For $t > 0$ we set $A^{[t]} = -\Delta + V^{[t]}$, where $V^{[t]}(x) = tV(t^{1/2}x)$. If $F$ is a bounded function on $[0,\infty)$, we denote by $F(A^{[t]}) = F(-\Delta + V^{[t]})$ the operator $\int_0^\infty F(\lambda)\,dE_{A^{[t]}}(\lambda)$, where $E_{A^{[t]}}$ is the spectral resolution for $A^{[t]}$.

The following two lemmas follow from [7], [8].

**Lemma 3.1.** *For every $\varepsilon > 0$ and $0 < a < b < \infty$ there exist $\delta > 0$ and $C > 0$ such that if $F \in C(\frac{d}{2}+\varepsilon)$, supp $F \subset (a,b)$, then the kernel $F(A^{[t]})(x,y)$ of the operator $F(A^{[t]})$ satisfies*

$$\|F(A^{[t]})\|_{\omega(\delta)} \leq C\|F\|_{C(\frac{d}{2}+\varepsilon)}. \tag{3.2}$$

*The constant $C$ in (3.2) depends on $d$, $\varepsilon$, $\delta$, $a$, $b$, but it is independent of $F$ and $A^{[t]}$.*

**Lemma 3.3.** *If $F$ is a continuous function on $\mathbb{R}^+$ supported on $(a,b)$, $0 < a < b < \infty$, then*

$$F(tA)(x,y) = t^{-d/2}F(A^{[t]})(\frac{x}{t^{1/2}}, \frac{y}{t^{1/2}}).$$

It is shown in [5] that for every multi-index $\alpha \in \mathbb{Z}^d_+$ there exist a homogeneous Lie group $G$ and a regular symmetric kernel $P$ of order 2 such that for every nonnegative polynomial $W(x) = \sum_{\beta\leq\alpha} b_\beta x^\beta$ there is a unitary representation $\Pi^W$ that acts on $L^2(\mathbb{R}^d)$ such that

$$\Pi^W_P = -\Delta + W.$$

The Lie algebra of the group $G$ is generated, as a Lie algebra, by $X_1, X_2, ..., X_d, Y$; cf. [4], [5]. The elements $X_1, ..., X_d$ are homogeneous of degree 1, whereas $Y$ is homogeneous of degree 2. Moreover,

$$\Pi^W_{X_j} = \frac{\partial}{\partial x_j}, \quad \Pi^W_Y = iW(x). \tag{3.4}$$

The kernel $P$ satisfies maximal subelliptic estimates, that is, for every left-invariant homogeneous differential operator $\partial$ on $G$ there exists a constant $C > 0$ such that

$$\|\partial f\|_{L^2(G)} \leq C\|P^{s/2}f\|_{L^2(G)} = C\|\int_0^\infty \lambda^{s/2}\,dE_P(\lambda)f\|_{L^2(G)}, \tag{3.5}$$

where $E_P$ is the spectral resolution of the operator $P : f \mapsto f * P$ and $s$ is the degree of homogeneity of $\partial$ (cf. [5]).

**Lemma 3.6.** *For every left-invariant differential operator $\partial$ on $G$ there is a constant $C>0$ such that for every function $F \in C_c(\frac{1}{2},2)$, and every unitary representation $\Pi^W$, where $W$ is a nonnegative polynomial which has the form $W(x)=\sum_{\beta\leq\alpha} b_\beta x^\beta$, we have*

$$\|\Pi^W_\partial F(-\Delta+W)f\|_{L^2(\mathbb{R}^d)} \leq C\|F\|_{C_c(1/2\,,2)}\|f\|_{L^2(\mathbb{R}^d)}. \tag{3.7}$$

*Proof.* The required estimate (3.7) is a consequence of (3.5) and a transference method. □

We are in a position to prove

**Lemma 3.8.** *For every $N>0$ there exists a constant $C>0$ such that for every $F\in C_c(\frac{1}{2},2)$ and every nonnegative polynomial $W$ of the form $W(x)=\sum_{\beta\leq\alpha} b_\beta x^\beta$ the kernel $F(-\Delta+W)(x,y)$ of the operator $F(-\Delta+W)$ satisfies*

$$|F(-\Delta+W)(x,y)| \leq C\|F\|_{C_c(1/2,2)}(1+m(x,W))^{-N}. \tag{3.9}$$

*Proof.* It follows from (3.4) and Lemma 3.6 that for every $q=0,1,2,...$ and $x\in\mathbb{R}^d$ the functional $f\mapsto W^q(x)F(-\Delta+W)f(x)$ is bounded on $L^2(\mathbb{R}^d)$ and

$$|W^q(x)F(-\Delta+W)f(x)| \leq C_q\|F\|_{C_c(1/2,2)}\|f\|_{L^2(\mathbb{R}^d)} \tag{3.10}$$

with a constant $C_q>0$ independent of $x\in\mathbb{R}^d$ and $W$. Therefore,

$$\int_{\mathbb{R}^d} |W^q(x)F(-\Delta+W)(x,y)|^2\,dy \leq C_q\|F\|^2_{C_c(1/2\,,2)}, \tag{3.11}$$

where $C_q$ is independent of $W$ and $x$. Since $F(-\Delta+W)=F(-\Delta+W)\phi(-\Delta+W)^*$, where $\phi\in C_c^\infty(\frac{1}{4},4)$, $\phi\equiv 1$ on $[\frac{1}{2},2]$, we have

$$|W^q(x)F(-\Delta+W)(x,y)| \leq C_q\|F\|_{C_c(1/2\,,2)}. \tag{3.12}$$

Similarly, we can prove using (3.4) and Lemma (3.6) that

$$|(D^\beta W(x))^q(x)F(-\Delta+W)(x,y)| \leq C_{q,\alpha}\|F\|_{C_c(1/2\,,2)}. \tag{3.13}$$

By (3.12) and (3.13) we obtain (3.9). □

*Proof of Proposition 2.2.* It follows from Lemma 3.3 that

$$\begin{aligned} Q_\mu(x,y) &= 2^{\mu d/2}\tilde{Q}_\mu(-\Delta+V^{[2^{-\mu}]})(2^{\mu/2}x,2^{\mu/2}y) \\ &= 2^{\mu d/2}\tilde{Q}_\mu(A^{[2^{-\mu}]})(2^{\mu/2}x,2^{\mu/2}y), \end{aligned} \tag{3.14}$$

where $\tilde{Q}_\mu(\lambda)=Q_\mu(2^\mu\lambda)=\varphi(\lambda)F(2^\mu\lambda)$. Applying Lemma 3.1 we obtain that there exist constants $C>0$ and $\delta'>0$ such that

$$\|\tilde{Q}_\mu(A^{[2^{-\mu}]})\|_{\omega(\delta')} \leq C\|\tilde{Q}_\mu\|_{C(\frac{d}{2}+\varepsilon)} \leq C. \tag{3.15}$$

Setting $\tilde{K}_\mu(x,y)=\tilde{Q}_\mu(A^{[2^{-\mu}]})(x,y)(1+|x-y|)^{\delta'/2}$, we have

$$\|\tilde{K}_\mu(x,y)\|_{\omega(\frac{\delta'}{2})} \leq C.$$

This implies that there exist constants $C>0$ and $0<\varepsilon'<1$ such that

$$\sup_y \int |\tilde{K}_\mu(x,y)|^{1-\varepsilon'}\,dx \leq C, \quad \sup_x \int |\tilde{K}_\mu(x,y)|^{1-\varepsilon'}\,dy \leq C. \tag{3.16}$$

Moreover, by Lemma 3.8, for every $N>0$ there exists a constant $C_N$ such that

$$|\tilde{Q}_\mu(A^{[2^{-\mu}]})(x,y)| \leq C_N\|\tilde{Q}_\mu\|_{C_c(1/2\,,2)}(1+m(x,V^{[2^{-\mu}]}))^{-N}. \tag{3.17}$$

Therefore, by (3.17), we get

$$|\tilde{Q}_\mu(A^{[2^{-\mu}]})(x,y)| \le C_N |\tilde{K}_\mu(x,y)|^{1-\varepsilon'}(1+|x-y|)^{-\delta'(1-\varepsilon')/2}(1+m(x,V^{[2^{-\mu}]}))^{-N\varepsilon'}.$$

Since $m(\frac{x}{t^{1/2}}, V^{[t]}) = t^{1/2}m(x,V)$, we obtain, by (3.14) and (3.16), the required estimates (2.3) and (2.4). □

*Proof of Proposition 2.5.* It follows from (3.2) of [5] that the kernels $T_t^{[2^{-\mu}]}(x,y)$ of the semigroup $T_t^{[2^{-\mu}]} = e^{-tA^{[2^{-\mu}]}}$ satisfy

$$|\nabla_y T_t^{[2^{-\mu}]}(x,y)| \le Ct^{-(d+1)/2}(1+\frac{|x-y|}{t^{1/2}})^{-d-3}, \tag{3.18}$$

with a constant $C>0$ independent of $\mu$.

Let $\tilde{Q}_\mu(\lambda)$ be as in the proof of Proposition 2.2. Since $\tilde{Q}_\mu(\lambda) = \tilde{Q}_\mu(\lambda)e^{\lambda}e^{-\lambda} = \tilde{R}_\mu(\lambda)e^{-\lambda}$, we have $\tilde{Q}_\mu(A^{[2^{-\mu}]}) = \tilde{R}_\mu(A^{[2^{-\mu}]})T_1^{[2^{-\mu}]}$. By virtue of Lemma 3.1 we obtain $\int_{\mathbb{R}^d} |\tilde{R}_\mu(A^{[2^{-\mu}]})(x,z)|\,dx \le C$, with $C>0$ independent of $\mu$. Therefore, applying (3.18), we get

$$\int_{\mathbb{R}^d} |\tilde{Q}_\mu(A^{[2^{-\mu}]})(x,y) - \tilde{Q}_\mu(A^{[2^{-\mu}]})(x,y_0)|\,dx$$

$$= \int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} \tilde{R}_\mu(A^{[2^{-\mu}]})(x,z)\left(T_1^{[2^{-\mu}]}(z,y) - T_1^{[2^{-\mu}]}(z,y_0)\right)\,dz|\,dx \le C|y-y_0|.$$

The estimate (2.6) is now a consequence of (3.14). □

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