

AN INFINITE FAMILY OF MANIFOLDS WITH BOUNDED TOTAL CURVATURE

A. N. DRANISHNIKOV

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ABSTRACT. The negative answer to the following problem of V. I. Arnold is given: Is the number of topologically different k -manifolds of bounded total curvature finite?

§1. INTRODUCTION

In [Ar] V. I. Arnold defined the total curvature $\alpha(g)$ of an immersion $g : M^k \rightarrow \mathbf{R}^m$ of a k -manifold in the m -dimensional euclidean space as the k -volume $\text{vol}_k(g\Delta\bar{g})(M^k)$ of a submanifold $(g\Delta\bar{g})(M^k) \subset \mathbf{R}^m \times G_{m,k}$ where $\bar{g} : M^k \rightarrow G_{m,k}$ is the induced map to the Grassmann manifold. He proved that for any two submanifolds $X, Y \subset \mathbf{R}^m$ and for every diffeomorphism $A : \mathbf{R}^m \rightarrow \mathbf{R}^m$ generically the total curvature $\alpha(A^n(X) \cap Y)$ is bounded by $Ce^{\lambda n}$ where $A^n = A \circ \dots \circ A$ and C and λ do not depend on n . Then the following question of Arnold appeared naturally: *Is the number of topologically different k -manifolds M lying in \mathbf{R}^m with the total curvature $\alpha(M)$ bounded from above finite?*

We note that Arnold's total curvature $\alpha(g)$ differs from the classic Chern-Lashof total curvature $\tau(g)$ [N-K]. Recall that $\tau(g) = \frac{1}{\text{vol}(S^{m-1})} \int_{SM^k} \nu^*(d\sigma)$ where SM^k is the unit sphere bundle of the normal bundle of an immersion $g : M^k \rightarrow \mathbf{R}^m$, $\nu : SM^k \rightarrow S^{m-1}$ is the Gauss map and $d\sigma$ is the volume element on S^{m-1} . Arnold's total curvature is always greater than Chern-Lashof's and the last can be estimated from below by the Morse number $m(M^k)$ = the minimal number of critical points of Morse functions.

It is easy to show that the answer to Arnold's question is affirmative for 2-manifolds. The purpose of this paper is to show that for 3-manifolds the answer is negative.

Theorem 1. *There exist a number $C > 0$ and an infinite family $\{M_i\}$ of pairwise nonhomeomorphic 3-dimensional compact submanifolds of \mathbf{R}^4 such that the total curvature $\alpha(M_i) < C$ for all i .*

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Theorem 2. *There exist a number $C > 0$ and an infinite family $\{\Sigma_i\}$ of pairwise nonhomeomorphic homology 3-spheres with embeddings $\{\eta_i : \Sigma_i \rightarrow \mathbf{R}^7\}$ such that $\alpha(\eta_i) < C$ for all i .*

The construction of both families of manifolds is basically due to the presence of the fundamental group. This situation suggests the following:

Conjecture. *The answer to Arnold's problem is positive for simply connected manifolds.*

§2. THE CONSTRUCTION OF THE FIRST FAMILY

Let $K(n, m) \subset S^3$ be an (n, m) -torus knot in S^3 and let $N(n, m)$ be its regular neighborhood. Let $\{p_i\}$ be a sequence of all odd numbers. For every i let M_i denote the result of doubling the compact 3-manifold $S^3 - \text{Int}N(2, p_i)$ along its boundary $\partial N(2, p_i)$.

Lemma 1. *For every $i \neq j$ manifolds M_i and M_j are not homeomorphic.*

Proof. An S^1 -action on $S^3 = S^1 * S^1$ defined by the formula

$$(\theta, (\phi, \psi)) \rightarrow (\phi + 2\theta, \psi + p_i\theta)$$

has two exceptional orbits with the orbit space homeomorphic to S^2 . We may assume that $N(2, p_i)$ is a preimage of a 2-disk under the projection to the orbit space. Then it is easy to see that each M_i is a Seifert manifold [Or] with the orbit space S^2 with 4 exceptional orbits of types $(2, 1), (2, 1), (p_i, 2), (p_i, 2)$. Since these orbit invariants are different for $i \neq j$, Seifert bundles on M_i and M_j are different. Note that M_i are large Seifert manifolds for all i . Hence M_i and M_j are not homeomorphic for $i \neq j$ [Or, Theorem 6]. \square

Lemma 2. *Suppose that $g_i : M \rightarrow N$ is a sequence of embeddings of a Riemannian manifold M into a Riemannian manifold N , converging in the C^1 -topology to an embedding $g : M \rightarrow N$. Then $\lim \text{vol}(g_i(M)) = \text{vol}(g(M))$, where $\text{vol}(L)$ for $L \subset N$ means the volume of submanifold L in N .*

The proof follows for instance from [D-N-F], v.1.

Let D^m denote the unit ball in \mathbf{R}^m and let D_r^m be a concentric ball of radius r . The normal regular neighborhood of radius r of immersed k -manifold $f : M^k \rightarrow \mathbf{R}^m$ is an immersion $F : M^k \times D_r^{m-k} \rightarrow \mathbf{R}^m$ such that 1) $F|_{M^k \times \{0\}} = f$ where 0 is the center of the disk D_r^{m-k} , 2) for every $t \in M^k$ the restriction $F|_{\{t\} \times D_r^{m-k}}$ is an isometrical embedding and the disk $F(\{t\} \times D_r^{m-k})$ is orthogonal to the manifold $f(M^k)$ at the point $f(t)$.

Every C^1 -manifold has locally a normal regular neighborhood for some r [H], and hence, every closed manifold with trivial normal bundle has a normal regular neighborhood. Therefore every smooth enough immersion of the circle in \mathbf{R}^3 has a normal regular neighborhood. Here is a parameterized version of that statement.

Lemma 3. *Suppose that $\{f_i : S^1 \rightarrow \mathbf{R}^3\}$ is a sequence of immersions of S^1 into \mathbf{R}^3 , converging in C^2 -metric to an immersion f_∞ . Then there exist $r > 0$ and normal regular neighborhoods $F_i : S^1 \times D_r \rightarrow \mathbf{R}^3$, $i = 1, 2, \dots, \infty$, such that $\{F_i\}$ converges in C^2 -metric to F_∞ .*

Let $C_k(\cdot, \cdot)$ be a space of mappings in C^k -topology. Denote by S^1 the boundary of the unit disk D^2 and denote by $w_r : S^1 \rightarrow \partial D_r$ the natural projection. In this notation the center of D^2 is ∂D_0 . Suppose that $f : S^1 \rightarrow \mathbf{R}^3$ is a C^2 -immersion with a normal regular neighborhood F of radius r . We define a map $R_f : [0, r] \rightarrow C_2(S^1 \times S^1, \mathbf{R}^3)$ by setting $R_f(x) = F \circ (id_{S^1} \times w_r) : S^1 \times S^1 \rightarrow \mathbf{R}^3$.

Lemma 4. *Suppose that a sequence $\{f_i : S^1 \rightarrow \mathbf{R}^3\}$ is converging to f_∞ in C^2 -metric and assume that r is given by Lemma 3. Then R_{f_i} uniformly converges to R_{f_∞} .*

We omit the proof since it is straightforward.

We define the ‘Gauss map’ $G_f : [0, r] \rightarrow C_1(S^1 \times S^1, G_{3,2})$ for R_f in the following way. For $x > 0$ the value $G_f(x)$ is the tangent bundle map $\nu : S^1 \times S^1 \rightarrow G_{3,2}$ for $S^1 \times \partial D_x$. For $x = 0$ for every $(t, \theta) \in S^1 \times S^1$ the value $G_f(0)(t, \theta)$ is the span of the derivative $f'(t)$ and the vector orthogonal to $F(t, (r, \theta)) - F(t, 0) = b(t, \theta)$. Here $(r, \theta) \in D_r$ means a point written in polar coordinates. Note that for all $x \in [0, r]$ the Gauss map is defined as $G_f(x)(t, \theta) = \text{span}\{\frac{\partial F(t, (x, \theta))}{\partial t}, b(t, \theta)^\perp\}$.

Lemma 5. *Suppose that a sequence $\{f_i : S^1 \rightarrow \mathbf{R}^3\}$ is converging to f_∞ in C^2 -metric and assume that the number r is provided by Lemma 3. Then G_{f_i} converges uniformly to G_{f_∞} .*

Proof. According to Lemma 3 a sequence F_i converges uniformly to F_∞ in C^2 -metric. It implies a uniform convergence of $\frac{\partial F_i}{\partial t}$ to $\frac{\partial F_\infty}{\partial t}$ in C^1 -metric. The second vector $b_i(t, \theta)^\perp$ does not depend on x , and therefore by virtue of Lemma 3 the sequence $b_i(t, \theta)^\perp$ converges to $b_\infty(t, \theta)^\perp$ in C^1 -metric as a sequence of functions in t and θ . This implies the lemma. \square

The proof of Theorem 1. We construct manifolds M_i in \mathbf{R}^4 symmetrically with respect to the hyperplane $\{0\} \times \mathbf{R}^3$. Let us consider two symmetric 3-dimensional spheres $S_+^3 \subset [1, \infty) \times \mathbf{R}^3$ and $S_-^3 \subset (-\infty, -1] \times \mathbf{R}^3$ such that the intersection $S_+^3 \cap (\{1\} \times \mathbf{R}^3)$ is a 3-disk D_+^3 and similarly $S_-^3 \cap (\{-1\} \times \mathbf{R}^3) = D_-^3$. Let $S^1 \subset D_+^3$ be a smoothly embedded circle and let $f : S^1 \rightarrow S^1 \subset D_+^3$ be a double winding around that circle ($f \in C^2$). For every i we consider the boundary of the normal regular neighborhood of S^1 of radius ϵ_i and realize a torus knot $K(2, p_i)$ on it. For small enough ϵ_i it is possible to realize the knot $K(2, p_i)$ by a map $f : S^1 \rightarrow \mathbf{R}^3$ which is $1/i$ -close to f in C^2 -metric. By virtue of Lemma 3 there exist a number r and the normal regular neighborhoods F_i of radius r . For every i there exists $r_i < r$ such that the restriction of F_i on $S^1 \times D_{r_i}^2$ is an embedding. We denote $N(2, p_i) = F_i(S^1 \times D_{r_i}^2)$. Let C_i be the cylinder over the boundary $\partial N(2, p_i)$ that connects S_+^3 with S_-^3 . We define $M_i = (S_+^3 - \text{Int}N(2, p_i)) \cup C_i \cup (S_-^3 - \text{Int}N(2, p_i))$.

We will write $\alpha(M)$ if the immersion of M is fixed. Since

$$\alpha(M_i) \leq \alpha(S_+^3) + \alpha(S_-^3) + \alpha(\partial N(2, p_i) \times [-1, 1]),$$

it suffices to find a common estimate for $\alpha(\partial N(2, p_i) \times [-1, 1])$. By an obvious version of the Kuiper theorem [N-K] for the Arnold total curvature it follows that $\alpha(\partial N(2, p_i) \times [-1, 1]) = 2\alpha(\partial N(2, p_i))$.

Lemmas 4, 5 imply that $\lim_{i \rightarrow \infty} R_{f_i}(r_i) = R_{f_\infty}(0)$ and $\lim_{i \rightarrow \infty} G_{f_i}(r_i) = G_{f_\infty}(0)$. Denote by A_i the diagonal product of R_i and G_i , $i = 1, 2, \dots, \infty$. By Lemma 2 it follows that $\lim_{i \rightarrow \infty} \text{vol}(A_i(r_i)(S^1 \times S^1)) = \text{vol}(A_\infty(0)(S^1 \times S^1))$. Since $\text{vol}(A_i(r_i)(S^1 \times S^1)) = \alpha(\partial N(2, p_i))$, it follows that the sequence $\alpha(\partial N(2, p_i))$

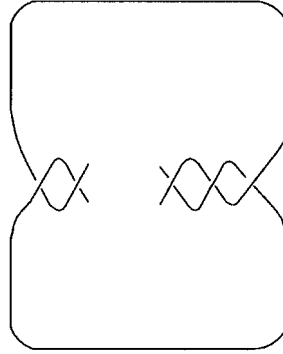


FIGURE 1

has an upper bound. So, it remains to note that by Lemma 1 the family $\{M_i\}$ is pairwise nonhomeomorphic. \square

Remark. Manifolds M_i are not smoothly embedded in \mathbf{R}^4 because at the points where spheres S^3_+ and S^3_- are attached to the cylinder C_i we have singularities. If we smooth these angles by a small perturbation near $\partial N(2, p_i)$, then $\alpha(M_i)$ will increase by approximately $vol(\partial N(2, p_i)) \times \pi/2$. Since $vol(\partial N(2, p_i)) \rightarrow 0$, then the sequence $\alpha(M_i)$ for resulting smooth manifolds M_i also will be bounded.

§3. THE CONSTRUCTION OF THE SECOND FAMILY

For any knot $K \subset S^3$ we denote by K_1 the 3-manifold obtained from S^3 by Dehn surgery on the knot K with the homology class $(1, 1)$ [R]. Let K^m be the following knot with $2m + 1$ -crossings shown in Figure 1:

Lemma 6. *The manifolds K_1^m and K_1^n are homeomorphic if and only if $m = n$.*

Proof. The direct computation of the Casson invariant $\lambda(K_1^k)$ shows that $\lambda(K_1^m)$ and $\lambda(K_1^n)$ are different for $m \neq n$ [A-M]. \square

By the definition a normal regular neighborhood induces a framing of the normal vector bundle. Let $\xi : S^1 \rightarrow \mathbf{R}^3$ be an immersion of the circle in \mathbf{R}^3 and let $F^k : S^1 \times D_r^2 \rightarrow \mathbf{R}^3$ be a normal regular neighborhood of ξ of radius r such that the induced framing has Hopf invariant equal to k . We consider the embedding of \mathbf{R}^3 in \mathbf{R}^7 as a factor $\mathbf{R}^3 \times 0 \times \dots \times 0 \subset \mathbf{R}^7$. A map $\phi : (M, N) \rightarrow (\mathbf{R}^7, \mathbf{R}^3)$ is said to be a relative embedding if the restriction of ϕ on N is an immersion into \mathbf{R}^3 and the restriction of ϕ on $M - N$ is an embedding in $\mathbf{R}^7 - \mathbf{R}^3$, transversal to \mathbf{R}^3 .

Lemma 7. *Let $f : (D^2, \partial D^2) \rightarrow (\mathbf{R}^7, \mathbf{R}^3)$ be a relative embedding of class C^2 . Then there exist $r > 0$ and a relative embedding of class C^2 , $F : (D^2 \times D_r^2, \partial D^2 \times D_r^2) \rightarrow (\mathbf{R}^7, \mathbf{R}^3)$, such that*

- 1) $F|_{D^2 \times \{0\}} = f$.
- 2) $f|_{\partial D^2 \times D_r^2} = F^1$ is a normal regular neighborhood in \mathbf{R}^3 , with Hopf invariant equal to one.
- 3) For every $x \in \text{Int}D^2$ the restriction $F|_{\{x\} \times D_r^2}$ imbeds the disk D_r^2 isometrically into the 5-plane, orthogonal to $f(D^2)$ at the point $f(x)$.

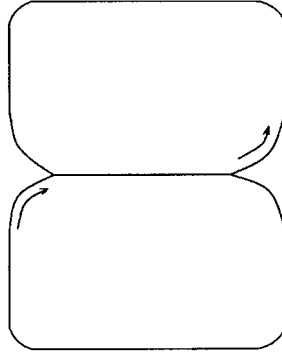


FIGURE 2

Proof. The map f induces a map $\nu(f) : D^2 \rightarrow G_{7,5}$ via the normal bundle. Let us consider a fibration $\eta : E \rightarrow G_{7,5}$ with the Stiefel manifold $V_{5,2}$ as a fiber. Here E is a subset of $G_{7,5} \times V_{7,2}$ defined in the following way: $L \times (a, b) \in E$ if and only if $a, b \in L$ and $\eta(L \times (a, b)) = L$. First we choose a normal regular neighborhood F^1 of $f|_{\partial D^2}$ of some radius r and with Hopf invariant equal to 1. The map F^1 induces a lifting $\beta : \partial D^2 \rightarrow E$ of a map $\nu(f)|_{\partial D^2} : \partial D^2 \rightarrow G_{7,5}$. Since the fundamental group $\pi_1(V_{5,2})$ is trivial, there is a lifting $\gamma : D^2 \rightarrow E$ extending β . The map γ defines a map $\Phi : D^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}^7$ such that (1) $\Phi|_{D^2 \times \{0\}} = f$, (2) $\Phi|_{\partial D^2 \times D_{r_0}^2} = F^1$ and (3) for every $x \in \text{Int}D^2$, the restriction $\Phi|_{\{x\} \times \mathbf{R}^2}$ is an isometric imbedding into a normal plane $\nu(f)(x)$ at the point $f(x) \in \mathbf{R}^7$. In order to complete the proof we choose $r < r_0$ such that the restriction $F = \Phi|_{D^2 \times D_r^2}$ is a relative embedding. \square

We note that the following parameterized version of Lemma 7 is valid.

Lemma 8. *Suppose that a sequence $f_i : (D^2, \partial D^2) \rightarrow (\mathbf{R}^7, \mathbf{R}^3)$ converges in C^2 -metric to f . Then there exists $r > 0$ and a sequence of maps $F_i : (D^2 \times D_r^2, \partial D^2 \times D_r^2) \rightarrow (\mathbf{R}^7, \mathbf{R}^3)$ with the properties (1)-(3) of Lemma 6 for f_i for every $i = 1, 2, \dots, \infty$, such that F_i converges in C^2 -metric to F_∞ .*

Proof of Theorem 2. Consider the 3-sphere S^3 in \mathbf{R}^7 with a flat part in $\mathbf{R}^3 \subset \mathbf{R}^7$. We define a sequence of embeddings $f_i : (D^2, \partial D) \rightarrow (\mathbf{R}^7, \mathbf{R}^3)$, C^2 -converging to a relative embedding f_∞ with $f_i(\partial D^2) \subset S^3$ and $f_i(\text{Int}D^2) \cap S^3 = \emptyset$ for $i = 1, 2, \dots, \infty$. Let $g : S^1 \rightarrow \mathbf{R}^3 \cap S^3$ be an immersion of the circle with the self-intersection along the interval as shown in Figure 2.

Let $f : D^2 \rightarrow \mathbf{R}^7$ be an extension of g to a relative C^2 -embedding with $f(\text{Int}D^2) \cap (S^3) = \emptyset$. For every i we realize the knot K^i by a map $g_i : S^1 \rightarrow \mathbf{R}^3 \cap S^3$ such that the distance between g and $g_i : S^1 \rightarrow \mathbf{R}^3$ in C^2 -metric is less than $1/i$. For every i there is an extension f_i of g_i which is $2/i$ -close to f and moreover f_i is an embedding. Apply Lemma 8 to the sequence f_i to obtain $F_i, i = 1, 2, \dots, \infty$. For every $i < \infty$ there exists a small number $\epsilon_i > 0$ such that the restriction $R_i = F_i|_{D^2 \times \partial D_{\epsilon_i}^2}$ is an embedding. We define $\Sigma_i = S^3 - (S^1 \times D_{\epsilon_i}^2) \cup \text{Im}R_i$ for all i . It easy to check that Σ_i is homeomorphic to K_1^i .

In order to complete the proof it is sufficient to find a common upper bound for $\alpha(\text{Im}R_i)$. For large i there is a rough estimate for $\alpha(\text{Im}R_i)$ as $2\pi\alpha(f(D^2))$. That implies an upper bound for the sequence $\{\alpha(\text{Im}R_i)\}$. Thus, Lemma 6 completes the proof. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, 358 LITTLE HALL, GAINESVILLE, FLORIDA 32611-8105

E-mail address: dranish@math.ufl.edu