

## ON THE EXTREMALITY OF QUASICONFORMAL MAPPINGS AND QUASICONFORMAL DEFORMATIONS

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(Communicated by Albert Baernstein II)

ABSTRACT. Given a family of quasiconformal deformations  $F(w, t)$  such that  $\bar{\partial}F$  has a uniform bound  $M$ , the solution  $f(z, t)(f(z, 0) = z)$  of the Löwner-type differential equation

$$\frac{dw}{dt} = F(w, t)$$

is an  $e^{2Mt}$ -quasiconformal mapping. An open question is to determine, for each fixed  $t > 0$ , whether the extremality of  $f(z, t)$  is equivalent to that of  $F(w, t)$ . The note gives this a negative approach in both directions.

### §1. INTRODUCTION

This paper deals with the quasiconformal solutions  $w = f(z, t)(f(z, 0) = 0)$  of the following Löwner-type differential equation

$$(1) \quad \frac{dw}{dt} = F(w, t)$$

in the unit disk. The maximal dilatation  $K[f]$  of  $f$  can be estimated in terms of the sup norm of  $\bar{\partial}F$ . It is of interest to find out whether minimizing the sup norm of  $\bar{\partial}F$  is equivalent to minimizing the maximal dilatation  $K[f]$ . To make this precise, we will need some definitions and notation.

In the terminology of Ahlfors [1], a complex-valued function  $F(w)$  defined in a Jordan domain  $\Omega$  is called a quasiconformal deformation if  $F(w)$  is continuous in  $\Omega \cup \partial\Omega$  and has locally  $L^2$ -generalized derivatives,  $\partial F, \bar{\partial}F$ , with  $\bar{\partial}F \in L^\infty(\Omega)$ . The deformation provides a way of generating a family of quasiconformal mappings, and is suitable for obtaining some distortion theorems (see Reich [6]).

Now for a homomorphism  $h$  of the unit circle  $\partial\Delta$  onto itself, denote by  $QC(h)$  the class of all quasiconformal mappings of the unit disk  $\Delta = \{z : |z| < 1\}$  with boundary values  $h$ . Then  $QC(h)$  is non-empty if and only if  $h$  is quasisymmetric in the sense of Beurling-Ahlfors [2]. A quasisymmetric function  $h$  then determines the extremal maximal dilatation  $K_h$ , defined as

$$K_h = \inf\{K[f] : f \in QC(h)\}.$$

$f_0 \in QC(h)$  is extremal if  $K[f_0] = K_h$ .

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Received by the editors December 23, 1997 and, in revised form, March 10, 1998.

1991 *Mathematics Subject Classification*. Primary 30C70, 30C62.

*Key words and phrases*. Quasiconformal mapping, quasiconformal deformation, extremality.

Project supported by the National Natural Science Foundation of China.

Similarly, for a continuous function  $H$  defined on  $\partial\Delta$ , denote by  $QD(H)$  the class of all quasiconformal deformations  $F$  on  $\Delta$  with boundary values  $H$ . Gardiner-Sullivan [3] and Reich-Chen [7] respectively proved that  $QD(H)$  is non-empty if and only if  $H$  is a Zygmund function, providing that  $F$  satisfies the following normalized conditions:

$$(2) \quad \operatorname{Re}[\bar{w}F(w)] = 0 \quad (|w| = 1), \quad F(0) = F(1) = 0.$$

Recall that a continuous function  $\phi(x)$  is called a Zygmund [9] function if it satisfies

$$|\phi(x+t) - 2\phi(x) + \phi(x-t)| = O(t)$$

for all real numbers  $x$  and  $t > 0$ . Define

$$B_H = \inf\{\|\bar{\partial}F\|_\infty : F \in QD(H)\}$$

for a Zygmund function  $H$ . Then  $F_0 \in QD(H)$  is extremal if  $\|\bar{\partial}F_0\|_\infty = B_H$ .

Now we can formulate our problem as follows. Given a family of quasiconformal deformations  $F(w, t)$  being continuous on  $\bar{\Delta} \times \{t \geq 0\}$  such that  $\bar{\partial}F$  is uniformly bounded, that is, there exists some constant  $M > 0$  such that  $|\bar{\partial}F(w, t)| \leq M$  for a.e.  $w \in \Delta$  and all  $t \geq 0$ , it is known (see [6]) that the differential equation (1) has a unique solution  $f(z, t)$  with the initial condition  $f(z, 0) = z$ , which is an  $e^{2Mt}$ -quasiconformal mapping on the unit disk  $\Delta$ . If, additionally,  $F(w, t)$  satisfy the normalized conditions (2), then  $f(z, t)$  map  $\Delta$  onto itself with  $f(0, t) = f(1, t) - 1 = 0$ . An open question is to determine, for each fixed  $t > 0$ , whether the extremality of  $f(z, t)$  is equivalent to that of  $F(w, t)$ . After giving some preliminary results, we will give a negative approach to the question in both directions. On the other hand, we will give a sufficient condition under which the answer to the question is affirmative.

## §2. PRELIMINARIES

**2.1.** Let  $A$  denote the Banach space of all functions  $\phi$  holomorphic in  $\Delta$  with the usual  $L^1$ -norm. The natural pairing:

$$(\mu, \phi) = \iint_{\Delta} \mu \phi, \quad \mu \in L^\infty, \phi \in A,$$

induces a linear map  $P$  from  $L^\infty$  onto  $A^*$ , the dual of  $A$ , defined by  $P\mu(\phi) = (\mu, \phi)$ .  $\mu \in L^\infty$  is said to satisfy the Hamilton-Krushkal ([4], [5]) condition if  $\|P\mu\| = \|\mu\|_\infty$ .

We will use the following standard result.

**Proposition 1** ([4], [5], [7], [8]). (1)  $f$  is an extremal quasiconformal mapping iff its complex dilatation  $\mu$  satisfies the Hamilton-Krushkal condition.

(2)  $F$  is an extremal quasiconformal deformation iff its  $\bar{\partial}$ -derivative  $\bar{\partial}F$  satisfies the Hamilton-Krushkal condition.

Note that  $\mu \in L^\infty$  satisfies the Hamilton-Krushkal condition iff there exists a sequence  $(\phi_n)$  in  $A$  with  $\|\phi_n\| = 1$  such that  $\iint_{\Delta} \mu \phi_n \rightarrow \|\mu\|_\infty$  as  $n \rightarrow \infty$ . Such a sequence  $(\phi_n)$  is called a Hamilton sequence for  $\mu$ . It is said to be degenerating if  $\phi_n \rightarrow 0$  locally uniformly in  $\Delta$ .

We will also need the following

**Proposition 2** ([8]).  $\mu$  satisfies the Hamilton-Krushkal condition iff  $\mu/(1 - |\mu|^2)$  does.

**2.2.** Let  $f(z, t)$  be the solution of the system (1). As done in Reich [6], differentiating both sides of the equation

$$\frac{df(z, t)}{dt} = F(f(z, t), t)$$

partially with respect to  $z$  and  $\bar{z}$  yields the relation

$$f_z(z, t)^2 \partial_t \mu(z, t) = (|f_z(z, t)|^2 - |f_{\bar{z}}(z, t)|^2) \bar{\partial} F(f(z, t), t),$$

or equivalently,

$$(3) \quad \bar{\partial} F(f(z, t), t) = \frac{\partial_t \mu(z, t)}{1 - |\mu(z, t)|^2} \cdot \frac{f_z(z, t)}{f_z(z, t)},$$

where  $\mu(z, t)$  is the complex dilatation of  $f(z, t)$ . Denote by  $\nu(w, t)$  the complex dilatation of the inverse mapping  $f^{-1}(w, t)$ ; then (3) is equivalent to

$$(4) \quad \bar{\partial} F(w, t) = -\frac{\partial_t \mu(z, t)}{\mu(z, t)} \cdot \frac{\nu(w, t)}{1 - |\nu(w, t)|^2} \quad (z = f^{-1}(w, t))$$

whenever  $\mu(z, t) \neq 0$ .

§3. COUNTEREXAMPLE THEOREMS

**Theorem 1.** *There exists a family of quasiconformal deformations  $F(w, t)$  on  $\bar{\Delta} \times [0, T]$  such that the solution  $f(z, t)$  of the system (1) and  $F(w, t)$  themselves satisfy the following conditions:*

- (1) For  $t \in (0, t_1]$ , both  $f(z, t)$  and  $F(w, t)$  are extremal.
- (2) For  $t \in (t_1, t_2]$ ,  $f(z, t)$  is extremal while  $F(w, t)$  is not.
- (3) For  $t \in (t_2, T]$ , neither  $f(z, t)$  nor  $F(w, t)$  is extremal.

**The example.** Let  $\mu$  be an extremal Beltrami differential in  $\Delta$  which has a degenerating Hamilton sequence and satisfies  $|\mu(z)| = \|\mu\|_\infty = k^2$  almost everywhere. Let  $G \subset \Delta$  be a compact positive-measure subset. Define

$$\mu(z, t) = t^2 \chi_G(z) \mu(z) + t \chi_{\Delta - G}(z) \mu(z)$$

for  $t \in [0, T]$ , where  $\chi$  denotes the characteristic function of a set, while  $1 < T < k^{-1}$  is a fixed number.

Let  $f(z, t)$  be the quasiconformal mapping of  $\Delta$  onto itself with complex dilatation  $\mu(z, t)$  and  $f(0, t) = f(1, t) - 1 = 0$ . Noting that  $\mu(z, t)$  satisfies the Hamilton-Krushkal condition iff  $t \in [0, 1]$ , by Proposition 1,  $f(z, t)$  is extremal iff  $t \in [0, 1]$ .

Define

$$F(w, t) = \partial_t f \circ f^{-1}(w, t).$$

Then  $f(z, t)$  automatically satisfy the equation (1). Now we investigate the extremality of  $F(w, t)$ .

By the relation (3), we get

$$|\bar{\partial} F(f(z, t), t)| = \frac{|\partial_t \mu(z, t)|}{1 - |\mu(z, t)|^2} = \begin{cases} \frac{k^2}{1 - t^2 k^4}, & z \in \Delta - G, \\ \frac{2tk^2}{1 - t^4 k^4}, & z \in G. \end{cases}$$

Thus, if

$$\frac{2tk^2}{1-t^4k^4} > \frac{k^2}{1-t^2k^4},$$

$\bar{\partial}F(w, t)$  cannot satisfy the Hamilton-Krushkal condition, and consequently  $F(w, t)$  cannot be extremal.

A direct computation will show that there exists a unique number  $t_0 \in (0, 1)$  such that

$$(5) \quad \frac{2tk^2}{1-t^4k^4} \leq \frac{k^2}{1-t^2k^4} \quad \text{as } t \in (0, t_0]$$

while

$$(6) \quad \frac{2tk^2}{1-t^4k^4} > \frac{k^2}{1-t^2k^4} \quad \text{as } t \in (t_0, T].$$

On the other hand, by the relation (4), we get

$$(7) \quad \bar{\partial}F(w, t) = \begin{cases} -\frac{2}{t} \cdot \frac{\nu(w, t)}{1-t^4k^4}, & w \in f(\cdot, t)(G), \\ -\frac{1}{t} \cdot \frac{\nu(w, t)}{1-t^2k^4}, & w \in \Delta - f(\cdot, t)(G). \end{cases}$$

Let  $t \in (0, t_0]$ . Then  $f(z, t)$  is extremal and so is  $f^{-1}(w, t)$ . By Proposition 1,  $\nu(w, t)$  satisfies the Hamilton-Krushkal condition. Now the relations (5) and (7) imply that  $\bar{\partial}F(w, t)$  also satisfies the Hamilton-Krushkal condition. Consequently,  $F(w, t)$  is extremal by Proposition 1 again.

Setting  $t_1 = t_0, t_2 = 1$ , we conclude that  $F(w, t)$  satisfy all the conditions in Theorem 1. This completes the proof of Theorem 1.

By considering

$$\mu(z, t) = t^2\chi_{\Delta-G}(z)\mu(z) + t\chi_G(z)\mu(z)$$

in the above example, we get

**Theorem 2.** *There exists a family of quasiconformal deformations  $F(w, t)$  on  $\bar{\Delta} \times [0, T]$  such that the solution  $f(z, t)$  of the system (1) and  $F(w, t)$  themselves satisfy the following conditions:*

- (1) For  $t \in (0, t_1)$ , neither  $f(z, t)$  nor  $F(w, t)$  is extremal.
- (2) For  $t \in [t_1, t_2)$ ,  $f(z, t)$  is not extremal while  $F(w, t)$  is.
- (3) For  $t \in [t_2, T]$ , both  $f(z, t)$  and  $F(w, t)$  are extremal.

#### §4. A SUFFICIENT CONDITION

**Theorem 3.** *Let  $f(z, t)$  be the solution of the system (1). If  $\mu(z, t)$ , the complex dilatation of  $f(z, t)$ , has the form  $\mu(z, t) = k(t)\mu(z)$  for some differentiable function  $k(t)$  with  $k(0) = 0, k'(t) > 0$ . Then, for each fixed  $t > 0$ ,  $f(z, t)$  is extremal iff  $F(w, t)$  is extremal.*

*Proof.* By the relation (4),

$$(8) \quad \bar{\partial}F(w, t) = -\frac{k'(t)}{k(t)} \cdot \frac{\nu(w, t)}{1-|\nu(w, t)|^2}.$$

Therefore,  $f(z, t)$  is extremal  $\Leftrightarrow f^{-1}(w, t)$  is extremal  $\Leftrightarrow \nu(w, t)$  satisfies the Hamilton-Krushkal condition (by Proposition 1)  $\Leftrightarrow \nu(w, t)/(1-|\nu(w, t)|^2)$  satisfies the Hamilton-Krushkal condition (by Proposition 2)  $\Leftrightarrow \bar{\partial}F(w, t)$  satisfies the

Hamilton-Krushkal condition (by relation (8))  $\Leftrightarrow F(w, t)$  is extremal (by Proposition 1).  $\square$

*Remark.* From the proof of Theorem 1, we find that the condition in Theorem 3 is not necessary.

#### ACKNOWLEDGEMENT

The author would like to thank the referee for valuable suggestions.

#### REFERENCES

1. L. V. Ahlfors, *Quasiconformal deformations and mappings in  $R^n$* , J. d'Analyse Math. **30** (1976), 74-97. MR **58**:11384
2. A. Beurling & L. V. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta. Math. **96** (1956), 125-142. MR **19**:258c
3. F. P. Gardiner & D. P. Sullivan, *Symmetric structures on a closed curve*, Amer. J. Math. **114** (1992), 683-736. MR **95h**:30020
4. R. S. Hamilton, *Extremal quasiconformal mappings with prescribed boundary values*, Tran. Amer. Math. Soc. **138** (1969), 399-406. MR **39**:7093
5. S. L. Krushkal, *Extremal quasiconformal mappings*, Siberian Math. J. **10** (1969), 411-418.
6. E. Reich, *A quasiconformal extension using the parametric representation*, J. d'Analyse Math. **54** (1990), 246-258. MR **91c**:30035
7. E. Reich & J. X. Chen, *Extensions with bounded  $\bar{\partial}$ -derivative*, Ann. Acad. Sci. Fenn. A I Math. **16** (1991), 377-389. MR **93b**:30018
8. E. Reich & K. Strebel, *Extremal quasiconformal mappings with prescribed boundary values*, Contributions to Analysis, A collection of papers dedicated to Lipman Bers. Academic Press, New York (1974), 375-391. MR **50**:13511
9. A. Zygmund, *Smooth functions*, Duke Math. J. **12** (1945), 47-76. MR **7**:60b

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