

## NO SUBMAXIMAL TOPOLOGY ON A COUNTABLE SET IS $T_1$ -COMPLEMENTARY

MIKHAIL G. TKAČENKO, VLADIMIR V. TKACHUK, RICHARD G. WILSON,  
AND IVAN V. YASCHENKO

(Communicated by Alan Dow)

ABSTRACT. Two  $T_1$ -topologies  $\tau$  and  $\mu$  given on the same set  $X$ , are called *transversal* if their union generates the discrete topology on  $X$ . The topologies  $\tau$  and  $\mu$  are  $T_1$ -complementary if they are transversal and their intersection is the cofinite topology on  $X$ . We establish that for any connected Tychonoff topology there exists a connected Tychonoff transversal one. Another result is that no  $T_1$ -complementary topology exists for the maximal topology constructed by van Douwen on the rational numbers. This gives a negative answer to Problem 162 from *Open Problems in Topology* (1990).

### 0. INTRODUCTION

The lattice  $\mathcal{L}_1(X)$  of all  $T_1$ -topologies on a given set  $X$  has been under intensive study since 1966 when A.K. Steiner [St] showed that for any infinite set  $X$  there exist  $T_1$ -topologies on  $X$  which do not have a complement in the lattice  $\mathcal{L}_1(X)$ . Recall that a topology  $\mu$  on  $X$  is a complement of  $\tau$  in  $\mathcal{L}_1(X)$  if  $\tau \cup \mu$  is a subbase of the discrete topology and  $\tau \cap \mu$  coincides with the cofinite topology on  $X$ .

The papers [An], [AnSt], [StSt1] and [StSt2] contain positive results on the existence of complements in the lattice  $\mathcal{L}_1(X)$  which are also called  $T_1$ -complements. It was proved, in particular, that every Hausdorff locally compact or Frechet-Urysohn topology has a  $T_1$ -complement [An]. In the same paper, Anderson constructed an example of an irresolvable ( $\equiv$  not representable as a union of two disjoint dense subsets) dense in itself space which has a  $T_1$ -complement and asked whether every MI-space has one [An, Question 1]. Recall that  $(X, \tau)$  is a MI-space if it is dense in itself and every dense subset of  $X$  is open. It is easy to see that every MI-space is irresolvable.

Later, S. Watson asked whether each Hausdorff space has a  $T_1$ -complement. This question is published as Problem 162 (Problem 94 in the internal enumeration of Watson's paper) in *Open Problems in Topology* [Wa1]. The second part of Problem

---

Received by the editors January 15, 1998 and, in revised form, March 19, 1998.

1991 *Mathematics Subject Classification*. Primary 54H11, 54C10, 22A05, 54D06; Secondary 54D25, 54C25.

*Key words and phrases*. Transversal topology,  $T_1$ -complement, connected space, strongly  $\sigma$ -discrete space.

This research was supported by Consejo Nacional de Ciencia y Tecnología (CONACYT) de México, grant 400200-5-3012PE..

162 of [Wa1] is an inquiry whether every completely regular  $T_1$ -topology has a  $T_1$ -complement. The same question is repeated in [Wa2] (Problem 6.6).

In this paper we work only with  $T_1$ -spaces. We use the modern term “submaximal space” instead of “MI-space”. Our main result is Theorem 3.6 which implies that no submaximal Hausdorff topology on a countable set is  $T_1$ -complementary. A narrower class than submaximal topologies is formed by the maximal ones. A topology  $\tau$  is *maximal* if it is dense in itself but any strictly stronger one is not. As there exists in ZFC a Tychonoff countable maximal space [vD], Theorem 3.6 and Corollary 3.8 provide the negative answer to the respective questions from [An], [Wa1] and [Wa2].

Given a set  $X$  and  $\tau, \mu \in \mathcal{L}_1(X)$  we say that  $\tau$  and  $\mu$  are *transversal* if  $\tau \cup \mu$  is a subbase of the discrete topology on  $X$ . It is immediate that if  $\tau$  and  $\mu$  are  $T_1$ -complementary, then they are transversal.

We prove in particular, that every connected Tychonoff topology has a connected Tychonoff transversal topology. Examples of the non-existence of transversal connected topologies are given.

## 1. NOTATIONS AND TERMINOLOGY

All spaces are assumed to be  $T_1$ . If  $X$  is a space, then  $\tau(X)$  is its topology and  $\tau^*(X) = \tau(X) \setminus \{\emptyset\}$ . Analogously, if  $\tau$  is a topology, then  $\tau^* = \tau \setminus \{\emptyset\}$ . A topology  $\tau$  on a set  $X$  is called *cofinite* (and is denoted by  $\mathcal{CF}(X)$ ) if every non-empty element of  $\tau$  is a complement of a finite set. If  $\tau$  is a topology on a set  $X$  and  $x \in X$ , then  $\tau(x) = \{U \in \tau : x \in U\}$ . We will write  $\tau(x, X)$  instead of  $\tau(X)(x)$ . Given a space  $(X, \tau)$  and a subset  $A \subset X$  we denote by  $\text{cl}_\tau(A)$  and  $\text{Int}_\tau(A)$  the  $\tau$ -closure and  $\tau$ -interior of  $A$  respectively. A space  $X$  is called *submaximal* if it is dense in itself and every dense subset of  $X$  is open. A topology  $\tau$  on a set  $X$  is *maximal* if  $(X, \tau)$  has no isolated points, but  $(X, \mu)$  has an isolated point if  $\mu$  is a topology strictly stronger than  $\tau$ .

All other notions are standard and can be found in [En].

## 2. TRANSVERSAL TOPOLOGIES

Let us start with the main definitions and some auxiliary results.

**2.1. Definition.** Two topologies  $\tau$  and  $\mu$  on the same set  $X$  are called *transversal* if their join  $\tau \vee \mu$  ( $\equiv$  the smallest topology that contains  $\tau \cup \mu$ ) is discrete. The topology  $\mu$  will be referred to as a  $\tau$ -transversal one.

**2.2. Proposition.** *If  $\tau$  and  $\mu$  are topologies on the same set  $X$ , then the following conditions are equivalent:*

- (1)  $\tau$  and  $\mu$  are transversal;
- (2) for each point  $x \in X$  there exist  $U \in \tau$  and  $V \in \mu$  such that  $U \cap V = \{x\}$ ;
- (3) there is a  $\tau$ -open cover  $\gamma$  of  $X$  such that  $X$  is a union of  $\mu$ -isolated points of elements of  $\gamma$ .

*Proof.* It is clear that the family  $\mathcal{B}(\tau, \mu) = \{U \cap V : U \in \tau, V \in \mu\}$  is a base of  $\tau \vee \mu$ . As each singleton belongs to  $\tau \vee \mu$ , we have  $\{x\} \in \mathcal{B}(\tau, \mu)$  for every  $x \in X$ . This proves (1)  $\implies$  (2).

For each  $x \in X$  take  $U_x \in \tau(x)$  and  $V_x \in \mu(x)$  such that  $\{x\} = U_x \cap V_x$ . Let  $\gamma = \{U_x : x \in X\}$ . It is clear that  $\gamma$  is a  $\tau$ -open cover of  $X$  and  $x$  is  $\mu$ -isolated in  $U_x$  for each  $x \in X$ . Thus (2)  $\implies$  (3).

To prove (3)  $\implies$  (1) consider any  $\gamma$  as in (3). For every  $x \in X$  there is a  $U_x \in \gamma$  such that  $x$  is  $\mu$ -isolated in  $U_x$ . This means that  $U_x \cap V_x = \{x\}$  for some  $V_x \in \mu$  and therefore  $\{x\}$  is open in  $\tau \vee \mu$ .  $\square$

**2.3. Theorem.** *Let  $(X, \tau)$  be a space of weight  $\kappa$ . If  $\mu$  is a  $\tau$ -transversal topology on  $X$ , then  $(X, \mu)$  is a union of  $\leq \kappa$  discrete subspaces.*

*Proof.* Take a base  $\mathcal{B}$  for  $(X, \tau)$  of cardinality  $\kappa$ . For every  $x \in X$  fix a  $U_x \in \mathcal{B}$  and  $V_x \in \mu$  with  $U_x \cap V_x = \{x\}$ . For every  $U \in \mathcal{B}$  put  $A_U = \{x \in X : U = U_x\}$ . Since for each  $x \in A_U$  we have  $V_x \cap A_U = \{x\}$ , the set  $A_U$  is discrete in  $(X, \mu)$ . Now  $X = \bigcup\{A_U : U \in \mathcal{B}\}$  is a union of  $\kappa$  many sets discrete in  $(X, \mu)$ .  $\square$

**2.4. Corollary.** *If  $(X, \tau)$  is a second countable space and  $\mu$  is a  $\tau$ -transversal topology on  $X$ , then  $(X, \mu)$  is  $\sigma$ -discrete.*

**2.5. Corollary.** *Let  $(X, \tau)$  be an infinite connected second countable space. Then there is no dense-in-itself compact (or even Baire)  $\tau$ -transversal topology  $\mu$  on  $X$ .*

*Proof.* Suppose that  $\mu$  is transversal for  $\tau$  and  $(X, \mu)$  is a Baire space. According to Corollary 2.4 we have  $X = \bigcup\{X_n : n \in \omega\}$ , where  $X_n$  is a  $\mu$ -discrete subset of  $X$ . The Baire property of  $(X, \mu)$  implies  $U = \text{Int}_\mu(\text{cl}_\mu(X_n)) \neq \emptyset$  for some  $n \in \omega$ . Then  $X_n \cap U \neq \emptyset$  and any  $x \in X_n \cap U$  is an isolated point of  $(X, \mu)$  which contradicts the fact that  $(X, \mu)$  is dense-in-itself.  $\square$

**2.6. Theorem.** *For every cardinal  $\kappa \geq \omega$  there exists a space  $D_\kappa$  with the following properties:*

- (1)  $D_\kappa$  is homeomorphic to a dense subspace of  $I^\kappa$  and hence  $D_\kappa$  is a Tychonoff space without isolated points;
- (2)  $D_\kappa = \bigcup\{D_\kappa^n : n \in \omega\}$ , where each  $D_\kappa^n$  is closed, discrete in  $D_\kappa$  and  $|D_\kappa^n| = \kappa$ ;
- (3) if  $A \subset D_\kappa$  and  $|A| < \kappa$ , then  $A$  is closed and discrete in  $D_\kappa$ ;
- (4) if  $\kappa \geq \mathfrak{c}$ , then the space  $D_\kappa$  is connected.

*Proof.* It is possible to represent the set  $\kappa$  in the following form:  $\kappa = \bigcup\{K_\alpha : \alpha < \kappa\}$ , where  $|K_\alpha| = \kappa$  for all  $\alpha < \kappa$  and  $K_\alpha \cap K_\beta = \emptyset$  if  $\alpha \neq \beta$ .

If  $\kappa < \mathfrak{c}$ , then for every finite  $A \subset \kappa$  pick a countable dense subspace  $C_A$  of the space  $I^A$  and let  $\mathcal{F} = \bigcup\{C_A : A \text{ is a finite subset of } \kappa\}$ .

If  $\kappa \geq \mathfrak{c}$ , then for every finite  $A \subset \kappa$  put  $C_A = I^A$  and let  $\mathcal{F} = \bigcup\{C_A : A \text{ is a finite subset of } \kappa\}$ .

It is clear that in both cases  $|\mathcal{F}| = \kappa$ , so it is possible to enumerate the elements of  $\mathcal{F}$  as  $\{x_\alpha : \alpha < \kappa\}$  in such a way that for each  $x \in \mathcal{F}$  the set  $\{\alpha : x_\alpha = x\}$  has cardinality  $\kappa$ . Given an  $x_\alpha \in \mathcal{F}$  we denote by  $S_\alpha$  the finite set of coordinates corresponding to the face to which the point  $x_\alpha$  belongs.

For each  $\alpha < \kappa$  let

$$d_\alpha(t) = \begin{cases} 0, & \text{if } t \notin K_\alpha \cup S_\alpha; \\ x_\alpha(t), & \text{if } t \in S_\alpha; \\ 1, & \text{if } t \in K_\alpha \setminus S_\alpha, \end{cases}$$

and denote by  $D_\kappa$  the subspace  $\{d_\alpha : \alpha < \kappa\}$  of  $I^\kappa$ .

Let us prove that the space  $D_\kappa$  has the properties we promised. Observe first that  $D_\kappa$  is dense in  $I^\kappa$  for all  $\kappa \geq \omega$ .

Indeed, for a finite  $S \subset \kappa$  every  $x \in C_S$  occurs  $\kappa$  times in the enumeration of  $\mathcal{F}$ . Thus, there is an  $\alpha(x) < \kappa$  such that  $x_{\alpha(x)} = x$  and hence  $S_{\alpha(x)} = S$ . It is clear

that  $\pi_S(d_{\alpha(x)}) = x$ . Therefore,  $\pi_S(D_\kappa) \supset C_S$  for every finite  $S \subset \kappa$ . Since the set  $C_S$  is dense in  $I^S$ , we can conclude that  $\pi_S(D_\kappa)$  is dense in  $I^S$  for every finite  $S \subset \kappa$ . Thus  $D_\kappa$  is dense in  $I^\kappa$ . This shows that (1) is true for  $D_\kappa$ .

Now let  $D_\kappa^n = \{d_\alpha \in D_\kappa : |S_\alpha| = n\}$  for every  $n \in \omega$ . It is evident that  $|D_\kappa^n| = \kappa$  for each  $n \in \omega$  and  $\bigcup\{D_\kappa^n : n \in \omega\} = D_\kappa$ . Given an  $\alpha < \kappa$  take any distinct  $t_1, \dots, t_{n+1} \in K_\alpha \setminus S_\alpha$ . Then  $d_\alpha(t_i) = 1$  for every  $i \leq n + 1$ . Thus,  $W_\alpha = \{d \in D_\kappa : d(t_i) > \frac{1}{2} \text{ for all } i \leq n + 1\}$  is an open neighbourhood of  $d_\alpha$ . If  $\beta \neq \alpha$  and  $|S_\beta| = n$ , then

$$\{t_1, \dots, t_{n+1}\} \setminus (K_\beta \cup S_\beta) \neq \emptyset,$$

which implies  $d_\beta(t_i) = 0$  for some  $i \leq n + 1$ . Therefore,  $d_\beta \notin W_\alpha$ . This proves that  $D_\kappa^n$  is closed and discrete in  $D_\kappa$ . Hence we established (2) for  $D_\kappa$ .

Take a subset  $A$  of  $D_\kappa$  with  $|A| < \kappa$ . The sets  $B = \{\alpha < \kappa : d_\alpha \in A\}$  and  $H = \bigcup\{S_\alpha : \alpha \in B\}$  have cardinality less than  $\kappa$ . Take any  $\alpha_0 < \kappa$ . It is clear that  $K_{\alpha_0} \setminus H \neq \emptyset$ . Pick any  $t \in K_{\alpha_0} \setminus H$  and let  $V_{\alpha_0} = \{d \in D_\kappa : d(t) > \frac{1}{2}\}$ . Then  $V_{\alpha_0}$  is an open neighbourhood of  $d_{\alpha_0}$  which does not contain any point of  $A$ , distinct from  $d_{\alpha_0}$ . Thus,  $A$  is closed and discrete in  $D_\kappa$ . This proves (3).

Finally, suppose that  $\kappa \geq \mathfrak{c}$ . If  $D_\kappa$  is disconnected, then there is a continuous surjective function  $\varphi : D_\kappa \rightarrow \{0, 1\}$ . As  $D_\kappa$  is dense in  $I^K$ , there is a countable set  $T \subset \kappa$  and a continuous function  $\varphi_1 : \pi_T(D_\kappa) \rightarrow \{0, 1\}$  such that  $\varphi_1 \circ \pi_T = \varphi$  [Ar]. In particular,  $\varphi_1$  is surjective. Now, if  $S$  is a finite subset of  $T$  and  $x \in I^S$ , then the set  $P(S) = \{\alpha < \kappa : x_\alpha = x\}$  has cardinality  $\kappa > \omega$ . Therefore,  $K_\alpha \cap T = \emptyset$  for some  $\alpha \in P(S)$  and hence  $d_\alpha|_S = x$  and  $d_\alpha|_{T \setminus S} \equiv 0$ . This proves that  $\pi_T(D_\kappa)$  contains the set  $\sigma_T = \{z \in I^T : |\{t \in T : z(t) \neq 0\}| < \omega\}$  which is connected and dense in  $I^T$ . Since  $\pi_T(D_\kappa)$  contains a dense connected subspace, it is connected. This gives a contradiction with the continuity and surjectivity of  $\varphi_1$ . Consequently,  $D_\kappa$  is connected.  $\square$

**2.7. Corollary.** *For every cardinal  $\kappa \geq \omega$  there exists a Tychonoff space  $E_\kappa$  with the following properties:*

- (1)  $E_\kappa = A_\kappa \cup B_\kappa$ , where each one of the subspaces  $A_\kappa, B_\kappa$  is closed in  $E_\kappa$  and homeomorphic to the space  $D_\kappa$  from Theorem 2.6. In particular, if  $A \subset E_\kappa$  and  $|A| < \kappa$ , then  $A$  is closed and discrete in  $E_\kappa$ ;
- (2) there is a point  $p \in E_\kappa$  such that  $A_\kappa \cap B_\kappa = \{p\}$ ;
- (3)  $A_\kappa \setminus \{p\} = \bigcup\{A_\kappa^n : n \in \omega\}$  and  $B_\kappa \setminus \{p\} = \bigcup\{B_\kappa^n : n \in \omega\}$ , where  $A_\kappa^n$  and  $B_\kappa^n$  are closed and discrete in  $E_\kappa$  for all  $n \in \omega$ ;
- (4)  $|A_\kappa^n| = |B_\kappa^n| = \kappa$  for all  $n \in \omega$ ;
- (5)  $A_\kappa^n \cap A_\kappa^m = \emptyset$  and  $B_\kappa^n \cap B_\kappa^m = \emptyset$  whenever  $m \neq n$ ;
- (6) if  $\kappa \geq \mathfrak{c}$ , then the space  $E_\kappa$  is connected.

*Proof.* Let  $P_\kappa$  and  $Q_\kappa$  be two disjoint copies of the space  $D_\kappa$  with  $P_\kappa^n \subset P_\kappa$  and  $Q_\kappa^n \subset Q_\kappa$  as respective copies of  $D_\kappa^n$ . Pick points  $a \in P_\kappa$ ,  $b \in Q_\kappa$  and identify  $a$  and  $b$  in the space  $P_\kappa \oplus Q_\kappa$ . Denote by  $E_\kappa$  the resulting quotient space and by  $\varphi_\kappa : P_\kappa \oplus Q_\kappa \rightarrow E_\kappa$  the relevant quotient map. Define the point  $p$  by the equality  $\{p\} = \varphi_\kappa(\{a, b\})$  and let  $A_\kappa = \varphi_\kappa(P_\kappa)$ ,  $A_\kappa^n = \varphi_\kappa(P_\kappa^n \setminus \{a\})$  and  $B_\kappa = \varphi_\kappa(Q_\kappa)$ ,  $B_\kappa^n = \varphi_\kappa(Q_\kappa^n \setminus \{b\})$ . It is straightforward that the space  $E_\kappa$  satisfies (1)-(6).  $\square$

The following example shows that not every Tychonoff space has a transversal connected or even dense-in-itself topology.

**2.8. Example.** The one-point compactification of any infinite discrete space has no transversal connected (or even dense-in-itself) topology.

*Proof.* Let  $X$  be the one-point compactification of an infinite discrete space. If  $a$  is the only non-isolated point of  $X$ , then for every  $U \in \tau(a, X)$  the set  $X \setminus U$  is finite. Suppose that  $\mu$  is a dense-in-itself topology, transversal to  $\tau(X)$ . Then there is a  $V \in \mu$  such that  $U \cap V = \{a\}$  for some  $U \in \tau(a, X)$ . Hence  $V \subset (X \setminus U) \cup \{a\}$  is a non-empty finite set. Since  $(X, \mu)$  is a  $T_1$ -space, any point of  $V$  is isolated in  $(X, \mu)$ , which is a contradiction.  $\square$

**2.9. Example.** There exists a dense-in-itself Tychonoff space  $X$  of cardinality  $\mathfrak{c}$  such that  $\tau(X)$  has no transversal connected Tychonoff topology.

*Proof.* Consider the discrete union  $Y = \bigoplus\{Q_\alpha : \alpha < \mathfrak{c}\}$  of  $\mathfrak{c}$  copies of rationals. Take any point  $z \notin Y$  and let  $X = \{z\} \cup Y$ . Declare the family  $\tau(y, Y)$  to be the local base in  $X$  at any  $y \in Y$ . Let  $\mathcal{B} = \{U_A = \{z\} \cup \bigcup\{Q_\beta : \beta \in \mathfrak{c} \setminus A\} \mid A \text{ is a countable subset of } \mathfrak{c}\}$  be the neighbourhood base of  $z$  in  $X$ .

Suppose that  $X$  has a connected transversal Tychonoff topology  $\mu$ . By Proposition 2.2 there exist  $U \in \tau(X)$  and  $V \in \mu$  such that  $\{z\} = U \cap V$ . But the complement of  $U$  is countable, which implies that  $V \subset (X \setminus U) \cup \{z\}$  is countable. Since in a connected Tychonoff space no non-empty proper open set can be countable, we have a contradiction.  $\square$

**2.10. Lemma.** Let  $(X, \tau)$  be a space of cardinality  $\kappa \geq \omega$ . Suppose that there exist families  $\{U_n : n \in \omega\}$  and  $\{V_n : n \in \omega\}$  of open subsets of  $X$  with the following properties:

- (1)  $\overline{U}_{n+1} \subset U_n$  and  $\overline{V}_{n+1} \subset V_n$  for any  $n \in \omega$ ;
- (2)  $\overline{U}_0 \cap \overline{V}_0 = \emptyset$ ;
- (3)  $|P| = \kappa$ , where  $P = X \setminus (U_0 \cup V_0)$ ;
- (4)  $|U_n \setminus U_{n+1}| = |V_n \setminus V_{n+1}| = \kappa$  for all  $n \in \omega$ .

Then there exists a  $\tau$ -transversal topology  $\mu$  on the set  $X$  such that  $(X, \mu)$  is homeomorphic to  $E_\kappa$  from Corollary 2.7. In particular,  $\tau$  has a transversal dense-in-itself Tychonoff topology and if  $\kappa \geq \mathfrak{c}$ , then  $\tau$  has a transversal connected Tychonoff topology.

*Proof.* Let  $F = \bigcap\{U_n : n \in \omega\}$  and  $G = \bigcap\{V_n : n \in \omega\}$  (the sets  $F$  and  $G$  can be empty). Take a point  $p' \in P$ . Using evident decompositions of  $E_\kappa$  and  $X$  into countably many pieces of cardinality  $\kappa$ , we conclude that there exists a bijection  $\varphi : E_\kappa \rightarrow X$  such that

- (i)  $\varphi(p) = p'$ ;
- (ii)  $\varphi(A_\kappa^0) = F \cup (P \setminus \{p'\})$ ;
- (iii)  $\varphi(B_\kappa^0) = G \cup (U_0 \setminus U_1)$ ;
- (iv)  $\varphi(B_\kappa^i) = U_i \setminus U_{i+1}$  for all  $i \geq 1$ ;
- (v)  $\varphi(A_\kappa^{i+1}) = V_i \setminus V_{i+1}$  for every  $i \in \omega$ .

Let  $\mu = \{\varphi(W) : W \text{ is open in } E_\kappa\}$ . It is clear that  $(X, \mu)$  is homeomorphic to  $E_\kappa$ . We are going to prove that the topology  $\mu$  is  $\tau$ -transversal.

Given an  $x \in X \setminus (F \cup G)$ , there exists an  $n \in \omega$  such that  $x \notin \overline{U}_n \cup \overline{V}_n$ . Let  $U = X \setminus (\overline{U}_n \cup \overline{V}_n)$ . Then  $x \in U$  and  $\varphi^{-1}(U)$  is contained in the closed and discrete subset

$$A = \{p\} \cup A_\kappa^0 \cup \dots \cup A_\kappa^n \cup B_\kappa^0 \cup \dots \cup B_\kappa^n$$

of the space  $E_\kappa$ . Therefore, there is a  $W \in \tau(E_\kappa)$  such that  $\{\varphi^{-1}(x)\} = W \cap \varphi^{-1}(U)$ . Then  $V = \varphi(W) \in \mu$  and  $U \cap V = \{x\}$ .

Now if  $x \in F$ , then let  $U = U_1$ . Observe that  $\varphi^{-1}(U) = (B_\kappa \setminus B_\kappa^0) \cup \varphi^{-1}(F) \subset B_\kappa \cup A_\kappa^0$  and  $\varphi^{-1}(x) \in \varphi^{-1}(F) \subset A_\kappa^0$ . Thus  $A_\kappa \setminus \{p\}$  is an open neighbourhood of  $\varphi^{-1}(x)$ , whose intersection with  $\varphi^{-1}(U)$  is contained in  $A_\kappa^0$ . Therefore,  $\varphi^{-1}(x)$  is isolated in  $\varphi^{-1}(U)$ . Take any  $W \in \tau(E_\kappa)$  such that  $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$ . Then  $V = \varphi(W) \in \mu$  and  $V \cap U = \{x\}$ .

Finally, assume that  $x \in G$ . For  $U = V_0$  we have  $\varphi^{-1}(U) = (A_\kappa \setminus A_\kappa^0) \cup \varphi^{-1}(G) \subset A_\kappa \cup B_\kappa^0$ . Thus  $B_\kappa \setminus \{p\}$  is an open neighbourhood of  $\varphi^{-1}(x)$ , whose intersection with  $\varphi^{-1}(U)$  is contained in  $B_\kappa^0$ . Therefore,  $\varphi^{-1}(x)$  is isolated in  $\varphi^{-1}(U)$ . Take any  $W \in \tau(E_\kappa)$  such that  $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$ . Then  $V = \varphi(W) \in \mu$  and  $V \cap U = \{x\}$ . We have established that for any  $x \in X$  property (2) of Proposition 2.2 is fulfilled and hence  $\mu$  is  $\tau$ -transversal.  $\square$

**2.11. Lemma.** *Suppose that in a space  $(X, \tau)$  of cardinality  $\kappa$  there exist  $a, b \in X$  and  $U_0, U_1, V_0, V_1 \in \tau$  such that*

- (1)  $a \in U_1 \subset \overline{U_1} \subset U_0$ ;
- (2)  $b \in V_1 \subset \overline{V_1} \subset V_0$ ;
- (3)  $|U_1| = |V_1| = \kappa$ ;
- (4)  $U_0 \cap V_0 = \emptyset$  and  $X \setminus (U_0 \cup V_0) \neq \emptyset$ ;
- (5) for any  $x \in (U_0 \setminus \{a\}) \cup (V_0 \setminus \{b\})$  there exists a  $U \in \tau(x, X)$  such that  $|U| < \kappa$ .

*Then there exists a  $\tau$ -transversal topology  $\mu$  on the set  $X$  such that  $(X, \mu)$  is homeomorphic to the space  $E_\kappa$  from Corollary 2.7. In particular,  $\tau$  has a transversal dense-in-itself Tychonoff topology and if  $\kappa \geq \mathfrak{c}$ , then  $\tau$  has a transversal connected Tychonoff topology.*

*Proof.* Pick any  $p' \in X \setminus (U_0 \cup V_0)$  and denote by  $P$  the set  $X \setminus (U_0 \cup V_1 \cup \{p'\})$ . Using evident decompositions of  $X$  and  $E_\kappa$  into finitely many pieces we can construct a bijection  $\varphi : E_\kappa \rightarrow X$  such that

- (i)  $\varphi(p) = p'$ ;
- (ii)  $\varphi(A_\kappa^0) \supset P \cup \{b\}$  and  $\varphi(A_\kappa \setminus \{p\}) = (U_1 \setminus \{a\}) \cup P \cup \{b\}$ ;
- (iii)  $\varphi(B_\kappa^0) \supset U_0 \setminus U_1$  and  $\varphi(B_\kappa \setminus \{p\}) = (V_1 \setminus \{b\}) \cup (U_0 \setminus U_1) \cup \{a\}$ .

Let  $\mu = \{\varphi(W) : W \text{ is open in } E_\kappa\}$ . It is clear that  $(X, \mu)$  is homeomorphic to  $E_\kappa$ . We are going to prove that the topology  $\mu$  is  $\tau$ -transversal.

Given an  $x \in X \setminus (U_0 \cup V_0)$  let  $U = X \setminus (\overline{U_1} \cup \overline{V_1})$ . The set  $U$  is a  $\tau$ -open neighbourhood of the point  $x$  and  $\varphi^{-1}(U) \subset \varphi^{-1}(P \cup \{p'\} \cup (U_0 \setminus U_1)) \subset A_\kappa^0 \cup \{p\} \cup B_\kappa^0$ . Since the set  $A_\kappa^0 \cup \{p\} \cup B_\kappa^0$  is closed and discrete in  $E_\kappa$ , there is a  $W \in \tau(E_\kappa)$  such that  $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$ . Then  $V = \varphi(W) \in \mu$  and  $U \cap V = \{x\}$ .

Suppose that  $x \in (U_0 \setminus \{a\}) \cup (V_0 \setminus \{b\})$ . Apply (5) to find a  $U \in \tau$  such that  $x \in U$  and  $|U| < \kappa$ . Then  $|\varphi^{-1}(U)| < \kappa$  and therefore  $\varphi^{-1}(U)$  is closed and discrete in  $E_\kappa$  by condition (1) of Proposition 2.7. Pick a  $W \in \tau(E_\kappa)$  such that  $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$ . Then  $V = \varphi(W) \in \mu$  and  $U \cap V = \{x\}$ .

If  $x = a$ , let  $U = U_1$ . Then  $\varphi^{-1}(U) \cap (B_\kappa \setminus \{p\}) = \{\varphi^{-1}(a)\}$ . Thus for  $W = B_\kappa \setminus \{p\}$  we have  $V = \varphi(W) \in \mu$  and  $U \cap V = \{a\}$ .

If  $x = b$ , let  $U = V_1$ . Then  $\varphi^{-1}(U) \cap (A_\kappa \setminus \{p\}) = \{\varphi^{-1}(b)\}$ . Thus for  $W = A_\kappa \setminus \{p\}$  we have  $V = \varphi(W) \in \mu$  and  $U \cap V = \{b\}$ .

Condition (2) of Proposition 2.2 having been checked for every  $x \in X$  we conclude that  $\tau$  and  $\mu$  are transversal.  $\square$

**2.12. Theorem.** *Given a cardinal number  $\lambda \geq \omega$  suppose that  $(X, \tau)$  is a regular space such that  $|U| \geq \lambda$  for any  $U \in \tau^*$ . Then  $\tau$  has a dense-in-itself transversal Tychonoff topology  $\mu$ . Moreover, if  $\lambda \geq \mathfrak{c}$ , then  $\mu$  can be chosen to be Tychonoff and connected.*

*Proof.* Let

$$M = \{x \in X : |U| = |X| = \kappa \text{ for any } U \in \tau(x)\}.$$

There are three cases to consider.

*Case 1.* The set  $M$  has at least two cluster points, say  $a$  and  $b$ . It is clear that  $M$  has to be infinite and  $a, b \in M$ .

Pick two distinct points  $x_0, y_0 \in M \setminus \{a, b\}$ . There exist  $U_0 \in \tau(a)$ ,  $V_0 \in \tau(b)$  such that  $\overline{U}_0 \cap \overline{V}_0 = \emptyset$  and  $\{x_0, y_0\} \subset X \setminus (\overline{U}_0 \cup \overline{V}_0)$ . In particular  $|P| = \kappa$ , where  $P = X \setminus (U_0 \cup V_0)$ .

Suppose that we have constructed open sets  $U_i \in \tau(a)$ ,  $V_i \in \tau(b)$  and points  $x_i, y_i \in M$  for all  $i \leq n$  in such a way that

- (i)  $\overline{U}_{i+1} \subset U_i$  and  $\overline{V}_{i+1} \subset V_i$  for all  $i < n$ ;
- (ii)  $x_{i+1} \in U_i \setminus \overline{U}_{i+1}$  and  $y_{i+1} \in V_i \setminus \overline{V}_{i+1}$  for all  $i < n$ .

Since  $a$  and  $b$  are cluster points of  $M$ , there exist  $x_{n+1} \in (U_n \setminus \{a\}) \cap M$  and  $y_{n+1} \in (V_n \setminus \{b\}) \cap M$ . Since the space  $X$  is regular, we can find  $U_{n+1} \in \tau(a)$  and  $V_{n+1} \in \tau(b)$  such that  $\overline{U}_{n+1} \subset (U_n \setminus \{x_{n+1}\})$  and  $\overline{V}_{n+1} \subset (V_n \setminus \{y_{n+1}\})$ . It is evident that the properties (i) and (ii) are fulfilled for all  $i \leq n$ .

Observe that the families  $\mathcal{U} = \{U_i : i \in \omega\}$  and  $\mathcal{V} = \{V_i : i \in \omega\}$  satisfy the conditions (1)-(3) of Lemma 2.10. The condition (4) is also fulfilled because each of the sets  $U_n \setminus \overline{U}_{n+1}$  and  $V_n \setminus \overline{V}_{n+1}$  is open and meets  $M$  for any  $n \in \omega$ . Therefore, we can apply Lemma 2.10 and conclude that  $(X, \tau)$  has a dense-in-itself transversal topology  $\mu$  which will be connected if  $\kappa \geq \mathfrak{c}$ .

*Case 2.* The set  $M$  has at least two isolated points, say  $a$  and  $b$ .

Pick any  $p' \in X \setminus \{a, b\}$ . Since  $X$  is Hausdorff, there exist  $U_0 \in \tau(a)$ ,  $V_0 \in \tau(b)$  such that  $U_0 \cap V_0 = \emptyset$ ,  $p' \in X \setminus (U_0 \cup V_0)$  and  $U_0 \cap M = \{a\}$ ,  $V_0 \cap M = \{b\}$ . Applying the regularity of  $X$  find  $U_1 \in \tau(a)$  and  $V_1 \in \tau(b)$  such that  $\overline{U}_1 \subset U_0$  and  $\overline{V}_1 \subset V_0$ . It is clear that the conditions (1), (2) and (4) of Lemma 2.11 are satisfied for  $a, b, U_0, U_1, V_0, V_1$ . The condition (3) is fulfilled because  $a \in M$  and  $b \in M$ . The condition (5) holds due to the fact that there are no points of  $M$  in  $U_0 \cup V_0$  distinct from  $a$  and  $b$ .

Thus we can apply Lemma 2.11 and conclude that  $(X, \tau)$  has a dense-in-itself transversal Tychonoff topology  $\mu$  which will be connected if  $\kappa \geq \mathfrak{c}$ .

*Case 3.* The set  $M$  has at most one point.

If  $M = \emptyset$ , then let  $\varphi : E_\kappa \rightarrow X$  be any bijection. The topology  $\mu = \{\varphi(W) : W \in \tau(E_\kappa)\}$  is as promised and to prove it we must only establish  $\tau$ -transversality of  $\mu$ . Let  $x \in X$ . As  $M = \emptyset$ , there is a  $U \in \tau(x)$  with  $|U| < \kappa$ . Therefore, the set  $\varphi^{-1}(U)$  is closed and discrete in  $E_\kappa$  by (1) of Corollary 2.7. Take any  $W \in \tau(E_\kappa)$  such that  $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$ . Then  $V = \varphi(W) \in \mu$  and  $U \cap V = \{x\}$  so that  $\mu$  is  $\tau$ -transversal.

Assume that  $M = \{a\}$ . Let  $\nu = \min\{|Z| : Z \in \tau^*\}$ . Since the space  $X$  is regular, there exists an  $H \in \tau^*$  such that  $|H| = \nu$  and  $a \notin \overline{H}$ . Since  $|Z| = \nu \geq \lambda$  for any  $Z \in \tau^*(H)$ , the conclusion of Case 1 is applicable to the space  $H$ . This makes it possible to find a bijection  $\xi : E_\nu \rightarrow H$  such that the topology  $\{\xi(W) : W \in \tau(E_\nu)\}$  is  $\tau(H)$ -transversal.

Since  $\nu < \kappa$  we have  $|X \setminus (\{a\} \cup H)| = \kappa$ . Let  $\psi : E_\kappa \rightarrow X \setminus (H \cup \{a\})$  be any bijection. The spaces  $E_\nu$  and  $E_\kappa$  are not compact by Corollary 2.7, so it is possible to choose points  $z \in \beta E_\nu \setminus E_\nu$  and  $t \in \beta E_\kappa \setminus E_\kappa$ . Take any  $q \in H$  and in the space  $E^* = (E_\kappa \cup \{t\}) \oplus (E_\nu \cup \{z\})$  identify the points  $q' = \xi^{-1}(q)$  and  $t$ . Denote the resulting quotient space by  $E$ , the point  $\{q', t\}$  by  $w$ , and let  $f : E^* \rightarrow E$  be the relevant quotient map. It is clear that  $E$  is a dense-in-itself Tychonoff space which is connected if  $\lambda \geq \mathfrak{c}$ . Identifying  $E_\kappa \cup (E_\nu \setminus \{q'\})$  with  $f(E_\kappa \cup (E_\nu \setminus \{q'\}))$  we have

$$E = E_\kappa \cup (E_\nu \setminus \{q'\}) \cup \{w\} \cup \{z\}.$$

Now let

$$\varphi(y) = \begin{cases} \xi(y), & \text{if } y \in E_\nu \setminus \{q'\}; \\ \psi(y), & \text{if } y \in E_\kappa; \\ a, & \text{if } y = z, \\ q, & \text{if } y = w. \end{cases}$$

Then  $\varphi : E \rightarrow X$  is a bijection. To conclude our proof it suffices to establish that  $\mu = \{\varphi(W) : W \in \tau(E)\}$  is a  $\tau$ -transversal topology.

Take any  $x \in X \setminus \{a\}$ . If  $x \in H$ , then there is a  $U \in \tau(H)$  and  $W' \in \tau(E_\nu)$  such that  $W' \cap \xi^{-1}(U) = \{\xi^{-1}(x)\}$ . If  $x \neq q$ , then  $\varphi^{-1}(x) = \xi^{-1}(x)$ . Otherwise  $\varphi^{-1}(x) = w$ . But in both cases  $\varphi^{-1}(U) \subset \xi^{-1}(U) \cup \{w\}$  and the set  $W = f(W') \cup E_\kappa$  is open in  $E$ . It is immediate that  $U \cap V = \{x\}$ , where  $V = \varphi(W) \in \mu$ . This shows that the condition (2) of Proposition 2.2 holds for  $x$ .

Assume that  $x \in X \setminus (H \cup \{a\})$ . There exists a  $U \in \tau(x)$  with  $|U| < \kappa$ . Therefore,  $\psi^{-1}(U \setminus H)$  is closed and discrete in  $E_\kappa$ . Take any  $W \in \tau(E_\kappa)$  such that  $W \cap \psi^{-1}(U \setminus H) = \{\psi^{-1}(x)\}$ . Then  $W \in \tau(E)$  and  $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$ . Therefore,  $V = \varphi(W) \in \mu$  and  $U \cap V = \{x\}$  which proves the property (2) of Proposition 2.2 for  $x$ .

Finally, if  $x = a$ , then  $W = \{z\} \cup (E_\nu \setminus \{q'\})$  is an open neighbourhood of  $z$  in  $E$ . If  $U = X \setminus \overline{H}$ , then  $\varphi^{-1}(U) \cap W = \{z\} = \{\varphi^{-1}(a)\}$ . Therefore,  $U \in \tau$ ,  $V = \varphi(W) \in \mu$  and  $U \cap V = \{a\}$  which shows that the condition (2) of Proposition 2.2 is fulfilled for  $x = a$ . Since this condition holds for every  $x \in X$  we conclude that  $\mu$  is a  $\tau$ -transversal topology.  $\square$

**2.13. Corollary.** *Let  $X$  be a regular space without isolated points. Then  $X$  has a transversal dense-in-itself Tychonoff topology.*

**2.14. Corollary.** *Let  $X$  be a connected Tychonoff space. Then  $X$  has a connected Tychonoff transversal topology.*

*Proof.* Any open subset of a Tychonoff connected space has cardinality  $\geq \mathfrak{c}$ . Now apply Theorem 2.12.  $\square$

### 3. COMPLEMENTARY TOPOLOGIES

Recall that topologies  $\tau$  and  $\mu$  on the same set  $X$  are called  $T_1$ -complementary [Wa2], if their join is discrete and  $\tau \cap \mu = \mathcal{CF}(X)$ .

**3.1. Definition.** A closed subset  $F$  of a space  $X$  is called well-placed if  $F$  and  $X \setminus F$  are infinite.

**3.2. Proposition.** *A space  $(X, \tau)$  has a well-placed subset if and only if*

$$\tau|_{X \setminus A} = \{U \cap (X \setminus A) : U \in \tau\} \neq \mathcal{CF}(X \setminus A)$$

for each finite  $A \subset X$ .

*Proof.* If there is a finite  $A \subset X$  such that the topology of  $X \setminus A$  is cofinite, then for any infinite closed  $F \subset X$  the set  $F \cap (X \setminus A)$  is infinite and closed in  $X \setminus A$ . Therefore  $(X \setminus A) \setminus F$  is finite whence  $X \setminus F$  is finite.

Suppose that there is no finite  $A \subset X$  with  $\tau|_{X \setminus A} = \mathcal{CF}(X \setminus A)$ . If  $D$  is an infinite discrete subspace of  $X$ , then splitting it into two disjoint infinite parts  $D_0, D_1$  we see that  $\overline{D_0}$  is closed, infinite and does not intersect  $D_1$  which implies  $F = \overline{D_0}$  is well-placed.

Thus, if the set  $A$  of isolated points of  $X$  is infinite, our proof is complete. If not, then  $\tau|_{X \setminus A}$  is not cofinite and hence  $X \setminus A$  has a proper infinite closed subset  $F$ . If  $B = (X \setminus A) \setminus F$  is finite, then every point of  $B$  is isolated in  $X \setminus A$  and hence in  $X$  which is a contradiction because  $B \cap A = \emptyset$ . Thus  $F$  is well-placed.  $\square$

**3.3. Lemma.** *Let  $\tau$  and  $\mu$  be transversal topologies on a set  $X$ . Suppose that  $\mu|_A = \mathcal{CF}(A)$  for some  $A \subset X$ . Then  $A$  is a discrete subspace of  $(X, \tau)$ .*

*Proof.* By Proposition 2.2 there exist  $U \in \tau$  and  $V \in \mu$  such that  $U \cap V = \{a\}$  for every  $a \in A$ . But  $F = A \setminus V$  is finite, so that  $U \cap A \subset \{a\} \cup F$  is also finite. Hence  $a$  is a  $\tau$ -isolated point of  $A$ .  $\square$

**3.4. Corollary.** *Let  $(X, \tau)$  be a space in which every discrete subset is closed. If  $\mu$  is a  $T_1$ -complementary topology for  $\tau$ , then for every well-placed set  $F$  of  $(X, \mu)$  there exists a well-placed set  $G$  of  $(X, \mu)$  such that  $G \subset F$  and  $G \neq F$ .*

*Proof.* Indeed, if  $\mu|_F = \mathcal{CF}(F)$ , then by Lemma 3.3 the set  $F$  is  $\tau$ -discrete and hence closed in  $(X, \tau)$ , which is a contradiction with  $\tau \cap \mu = \mathcal{CF}(X)$ . Thus there exists an infinite closed proper subset  $G$  of  $F$ . It is clear that  $G$  is as required.  $\square$

**3.5. Lemma.** *Let  $(X, \tau)$  be a Hausdorff space in which every discrete subset is closed. Suppose that  $\mu$  is a  $T_1$ -complementary topology for  $\tau$  and  $\mathcal{F} = \{F_n : n \in \omega\}$  is a family of well-placed subsets of  $(X, \mu)$  with  $F_{n+1} \subset F_n$  for each  $n \in \omega$ . Then  $F = \bigcap \mathcal{F}$  is well-placed in  $(X, \mu)$ .*

*Proof.* It suffices to prove that  $F$  is infinite. Suppose not. Since every  $F_n$  is infinite, we can assume that  $F_n \setminus F_{n+1} \neq \emptyset$  for each  $n \in \omega$ . Let  $x_n \in F_n \setminus F_{n+1}$ . It is straightforward that  $F \cup A$  is  $\mu$ -closed for every  $A \subset Y = \{x_n : n \in \omega\}$ . Any infinite subspace of a Hausdorff space has an infinite discrete subspace, so there is an infinite  $A \subset Y$  such that  $A$  is  $\tau$ -discrete and hence closed in  $(X, \tau)$ . Then  $F \cup A$  is also  $\tau$ -closed and hence it is well-placed in  $(X, \mu)$  and  $(X, \tau)$ , which is a contradiction with  $\tau \cap \mu = \mathcal{CF}(X)$ .  $\square$

**3.6. Theorem.** *Let  $(X, \tau)$  be a dense-in-itself Hausdorff countable space in which every discrete subset is closed. Then  $\tau$  does not have a  $T_1$ -complement.*

*Proof.* Assume that  $\mu$  is a  $T_1$ -complementary topology for  $\tau$ . If  $A$  is a finite subset of  $X$  and  $\mu|_{X \setminus A} = \mathcal{CF}(A)$ , then by Lemma 3.3 the set  $X \setminus A$  is closed and discrete in  $(X, \tau)$ . Then  $X = (X \setminus A) \cup A$  is discrete, which is a contradiction. Consequently, we can apply Proposition 3.2 to conclude that  $(X, \mu)$  has a well-placed subset  $F$ . Let  $F_0 = F$ .

Suppose that for some  $\alpha < \omega_1$  we have constructed  $\mu$ -well-placed subsets  $\{F_\beta : \beta < \alpha\}$  such that  $F_\delta$  is a proper subset of  $F_\beta$  if  $\beta < \delta < \alpha$ . If  $\alpha$  is a limit ordinal, let  $F_\alpha = \bigcap \{F_\beta : \beta < \alpha\}$ . Lemma 3.5 makes it possible to assert that  $F_\alpha$  is  $\mu$ -well-placed. If  $\alpha = \beta + 1$  use Corollary 3.4 to find a  $\mu$ -well-placed  $G$  which is a proper subset of  $F_\beta$ . Putting  $F_\alpha = G$  we finish our transfinite construction.

As a result we get a family  $\mathcal{F} = \{F_\alpha : \alpha < \omega_1\}$  of subsets of a countable set  $X$  such that  $F_\beta \subset F_\alpha$  and  $F_\beta \neq F_\alpha$  if  $\alpha < \beta$ . It is evident that such an  $\mathcal{F}$  cannot exist so the theorem is proved.  $\square$

**3.7. Corollary.** *If  $(X, \tau)$  is a submaximal Hausdorff countable space, then  $\tau$  has no  $T_1$ -complement.*

*Proof.* It is well-known (see e.g. [ArCo]) that in a submaximal Hausdorff space any discrete subspace is closed.  $\square$

Corollary 3.7 gives a negative answer to Question 1 from [An].

**3.8. Corollary.** *There exists a Tychonoff countable dense-in-itself space  $(X, \tau)$  which has no  $T_1$ -complement.*

*Proof.* Van Douwen constructed in [vD] an example of a Tychonoff maximal (and hence submaximal) countable space  $(X, \tau)$ . Now apply 3.6 to see that  $(X, \tau)$  is as promised.  $\square$

Corollary 3.8 gives a negative answer to Problem 162 (Problem 94 in its internal enumeration) of [Wa1] as well as to Problem 6.6 of [Wa2].

**3.9. Question.** Let  $X$  be a Hausdorff dense-in-itself space. Does  $\tau(X)$  have a transversal dense-in-itself Hausdorff (or Tychonoff) topology?

**3.10. Question.** Let  $X$  be a Hausdorff connected space. Does  $\tau(X)$  have a transversal connected Hausdorff (or Tychonoff) topology? What is the answer if  $X$  is regular?

#### REFERENCES

- [An] B.A. Anderson, A class of topologies with  $T_1$ -complements, *Fundamenta Mathematicae*, 1970, vol. 69, 267-277. MR **43**:6859
- [AnSt] B.A. Anderson and D.G. Stewart,  $T_1$ -complements of  $T_1$  topologies, *Proceedings of the Amer. Math. Soc.*, 1969, vol. 23, 77-81. MR **39**:6240
- [Ar] A.V. Arhangel'skii, Continuous mappings, factorization theorems and function spaces (in Russian), *Trudy Mosk. Mat. Obsch.*, 1984, vol. 47, 3-21. MR **86i**:54002
- [ArCo] A.V. Arhangel'skii and P.J. Collins, On submaximal spaces, *Topology and Its Applications*, 1995, vol. 64, no. 3, 219-241. MR **96m**:54002
- [vD] E.K. van Douwen, Applications of maximal topologies, *Topology and Its Applications*, 1993, vol. 51, 125-139.
- [En] R. Engelking, *General Topology*, PWN, Warszawa, 1977. MR **58**:18316b
- [St] A.K. Steiner, Complementation in the lattice of  $T_1$ -topologies, *Proceedings of the Amer. Math. Soc.*, 1966, vol. 17, 884-885. MR **33**:1255
- [StSt1] E.F. Steiner and A.K. Steiner, Topologies with  $T_1$ -complements, *Fundamenta Mathematicae*, 1967, vol. 61, 23-28. MR **37**:5840
- [StSt2] E. F. Steiner and A. K. Steiner, A  $T_1$ -complement for the reals, *Proceedings of the Amer. Math. Soc.*, 1968, vol. 19, 177-179. MR **37**:6897

- [Wa1] S. Watson, Problems I wish I could solve, Open Problems in Topology, ed. by J. van Mill and G.M. Reed, Elsevier Science Publishers B.V. (North Holland), 1990, 37-76. CMP 91:03
- [Wa2] S. Watson, The number of complements in the lattice of topologies on a fixed set, Topology and Its Applications, 1994, vol. 55, 101-125. MR **94k**:54002

(M. G. Tkačenko, V. V. Tkachuk and R. G. Wilson) DEPARTAMENTO DE MATEMATICAS, UNIVERSIDAD AUTÓNOMA METROPOLITANA, AV. MICHOACAN Y LA PURÍSIMA, IZTAPALAPA, A.P. 55-532, C.P. 09340, MÉXICO D.F.

*E-mail address:* `mich@xanum.uam.mx`

*E-mail address:* `vova@xanum.uam.mx`

*Current address,* R. G. Wilson: Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad Universitaria, México 20, D.F.

*E-mail address:* `rgw@xanum.uam.mx`

(I. V. Yaschenko) MOSCOW CENTER FOR CONTINUOUS MATHEMATICAL EDUCATION, B. VLAS'-EVSKIJ, 11, 121002, MOSCOW, RUSSIA

*E-mail address:* `ivan@mccme.ru`