

## ON A THEOREM BY SERRE

ARNE LEDET

(Communicated by Ronald M. Solomon)

ABSTRACT. We present a short proof of a theorem by Serre on the trace form of a finite separable field extension.

Let  $M/K$  be a finite Galois extension in characteristic  $\neq 2$ , and assume that  $M$  is the splitting field over  $K$  of an irreducible polynomial  $f(X) \in K[X]$  of degree  $n$ . We embed the Galois group  $G = \text{Gal}(M/K)$  transitively into  $S_n$  by considering the elements of  $G$  as permutations of the roots of  $f(X)$ . From the ‘positive’ double cover

$$1 \rightarrow \mu_2 \rightarrow \tilde{S}_n^+ \rightarrow S_n \rightarrow 1$$

of  $S_n$  (i.e., the double cover in which transpositions lift to elements of order 2) we then get an extension

$$(*) \quad 1 \rightarrow \mu_2 \rightarrow \tilde{G}^+ \rightarrow G \rightarrow 1$$

of  $G$  with the cyclic group  $\mu_2 = \{\pm 1\}$ . Let  $\gamma^+ \in H^2(G, \mu_2)$  be the characteristic class of  $(*)$ .

We embed  $S_n$  into the orthogonal group  $O_n(\bar{K}_{\text{sep}})$  as permutations of the standard basis vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n \in \bar{K}_{\text{sep}}^n$ . ( $\bar{K}_{\text{sep}}$  being the separable closure of  $K$ .) As the pre-image in the Clifford group  $C_n^*(\bar{K}_{\text{sep}})$  of a transposition  $(ij)$ ,  $i < j$ , we can then take the element  $x_{ij} = (\mathbf{e}_i - \mathbf{e}_j)/\sqrt{2}$ . The subgroup of  $C_n^*(\bar{K}_{\text{sep}})$  generated by these is exactly the double cover  $\tilde{S}_n^+$  of  $S_n$ , and we get a diagram

$$\begin{array}{ccccccc}
 & & & & \text{Gal}(K) & & \\
 & & & & \downarrow \text{res} & & \\
 1 & \longrightarrow & \mu_2 & \longrightarrow & \tilde{G}^+ & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \mu_2 & \longrightarrow & \tilde{S}_n^+ & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \bar{K}_{\text{sep}}^* & \longrightarrow & C_n^*(\bar{K}_{\text{sep}}) & \longrightarrow & O_n(\bar{K}_{\text{sep}}) & \longrightarrow & 1,
 \end{array}$$

Received by the editors March 7, 1998.

1991 *Mathematics Subject Classification*. Primary 12G05.

This work was supported by a Queen’s University Advisory Research Committee Postdoctoral Fellowship.

where  $\text{Gal}(K) = \text{Gal}(\bar{K}_{\text{sep}}/K)$  is the absolute Galois group of  $K$ . The last row of this diagram is a short-exact sequence of  $G$ -groups, inducing a connecting map  $\delta : H^1(\text{Gal}(K), O_n(\bar{K}_{\text{sep}})) \rightarrow H^2(\text{Gal}(K), \bar{K}_{\text{sep}}^*)$ , cf. [Se1], and we know from [Sp] that the image of the crossed homomorphism  $e : \text{Gal}(K) \rightarrow O_n(\bar{K}_{\text{sep}})$  in the last column of the diagram is the Hasse-Witt invariant of the quadratic form obtained from  $\langle 1, \dots, 1 \rangle$  by Galois twist with  $e$ . The *Hasse-Witt invariant* of a regular quadratic form  $q \sim \langle a_1, \dots, a_n \rangle$  is

$$\text{hw}(q) = \prod_{i < j} (a_i, a_j) \in H^2(\text{Gal}(K), \bar{K}_{\text{sep}}^*),$$

where the elements  $(a_i, a_j) \in H^2(\text{Gal}(K), \bar{K}_{\text{sep}}^*)$  are *quaternion symbols*: For  $a, b \in K^*$  the quaternion symbol  $(a, b)$  is represented by the factor system  $(\sigma, \tau) \mapsto (-1)^{\chi_a(\sigma)\chi_b(\tau)}$ , where  $\chi_a, \chi_b : \text{Gal}(K) \rightarrow \mathbb{F}_2$  are the homomorphisms with kernels  $\text{Gal}(K(\sqrt{a}))$  and  $\text{Gal}(K(\sqrt{b}))$ , resp.

Now, let  $L = K(\theta)$ , where  $\theta$  is a root of  $f(X)$ , and let  $\theta_1 = \theta, \theta_2, \dots, \theta_n \in M$  be the conjugates. This numbering fixes our embedding of  $G$  into  $S_n$ . The Galois twist corresponding to  $e$  above is obtained by restricting  $\langle 1, \dots, 1 \rangle$  from  $\bar{K}_{\text{sep}}^n$  to the space of fixed points under the  $G$ -action  $\sigma \mathbf{x} = e_\sigma(\sigma \mathbf{x})$ . It is easy to see that the fixed points are exactly the points

$$(g(\theta_1), \dots, g(\theta_n)), \quad g(X) \in K[X],$$

meaning that the twisted quadratic space is  $L$  equipped with the *trace form*  $q_L : x \mapsto \text{Tr}_{L/K}(x^2)$ .

We compute  $\delta(e)$  directly as follows: Let  $s_\sigma \in \tilde{G}^+$  be a pre-image of  $\sigma \in G$ . Then  $s_{\text{res } \sigma} \in C_n^*(\bar{K}_{\text{sep}})$  is a pre-image of  $e_\sigma \in O_n(\bar{K}_{\text{sep}})$  for  $\sigma \in \text{Gal}(K)$ , and  $\delta(e)$  is given by the factor system

$$(\sigma, \tau) \mapsto s_{\text{res } \sigma} \sigma s_{\text{res } \tau} s_{\text{res } \sigma \tau}^{-1} = (-1)^{\chi_2(\sigma)\chi_d(\tau)} s_{\text{res } \sigma} s_{\text{res } \tau} s_{\text{res } \sigma \tau}^{-1}, \quad \sigma, \tau \in \text{Gal}(K),$$

where  $d = d_{L/K}$  is the discriminant of  $L/K$ , since  $\sigma$  operates on  $s_{\text{res } \tau}$  through the factor  $1/\sqrt{2}$  contributed by each transposition. Here,  $(\sigma, \tau) \mapsto s_{\text{res } \sigma} s_{\text{res } \tau} s_{\text{res } \sigma \tau}^{-1}$  is the inflation of  $\gamma^+$  to  $H^2(\text{Gal}(K), \bar{K}_{\text{sep}}^*)$ , and  $(\sigma, \tau) \mapsto (-1)^{\chi_2(\sigma)\chi_d(\tau)}$  is the quaternion symbol  $(2, d)$ . Hence, we have

**Theorem** (Serre, [Se2]). *With notation as above,*

$$\inf_{G \rightarrow \text{Gal}(K)} (\gamma^+) = \text{hw}(q_L) \cdot (2, d_{L/K}) \in H^2(\text{Gal}(K), \bar{K}_{\text{sep}}^*).$$

If we look instead at the ‘negative’ double cover

$$1 \rightarrow \mu_2 \rightarrow \tilde{S}_n^- \rightarrow S_n \rightarrow 1$$

of  $S_n$ , where transpositions lift to elements of order 4, we get an extension

$$(**) \quad 1 \rightarrow \mu_2 \rightarrow \tilde{G}^- \rightarrow G \rightarrow 1$$

of  $G$  with  $\mu_2$ . Let  $\gamma^- \in H^2(G, \mu_2)$  be the characteristic class of (\*\*). The elements  $y_{ij} = (\mathbf{e}_i - \mathbf{e}_j)/\sqrt{-2} \in C_n(\bar{K}_{\text{sep}}^*)$ ,  $i < j$ , generate a copy of  $\tilde{S}_n^-$  mapping onto  $S_n \subseteq O_n(\bar{K}_{\text{sep}})$ , and we can repeat the entire argument above with  $-2$  instead of  $2$ ,<sup>1</sup> getting

<sup>1</sup>The author would like to thank the referee for suggesting this modified argument.

**Theorem.** *With notation as above,*

$$\inf_{G \rightarrow \text{Gal}(K)}(\gamma^-) = \text{hw}(q_L) \cdot (-2, d_{L/K}) \in H^2(\text{Gal}(K), \bar{K}_{\text{sep}}^*).$$

## REFERENCES

- [Se1] J.-P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Mathematics 5, Springer-Verlag, 1973. MR **53**:8030
- [Se2] ———, *L'invariant de Witt de la forme  $\text{Tr}(x^2)$* , *Comm. Math. Helv.* **59** (1984), 651–676. MR **86k**:11067
- [Sp] T. A. Springer, *On the Equivalence of Quadratic Forms*, *Proc. Neder. Acad. Sci.* **62** (1959), 241–253. MR **21**:7184

DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO,  
CANADA K7L 3N6

*E-mail address:* ledet@mast.queensu.ca