

## EXTENSIONS OF HEINZ-KATO-FURUTA INEQUALITY

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ABSTRACT. We give an extension of Lin's recent improvement of a generalized Schwarz inequality, which is based on the Heinz-Kato-Furuta inequality. As a consequence, we can sharpen the Heinz-Kato-Furuta inequality.

### 1. INTRODUCTION

First of all, we cite a generalized Schwarz inequality which is a base of Lin's recent paper [9]. For a (bounded linear) operator  $T$  acting on a Hilbert space  $H$ ,

$$(1) \quad |(Tx, y)|^2 \leq (|T|^{2\alpha}x, x)(|T^*|^{2(1-\alpha)}y, y)$$

for all  $\alpha \in [0, 1]$  and  $x, y \in H$ , where  $|X|$  is the square root of  $X^*X$  for an operator  $X$  on  $H$ . It implies the Heinz-Kato inequality via the Löwner-Heinz inequality, cf. [3],[10]. On the other hand, Furuta [7] extended the Heinz-Kato inequality, the so-called Heinz-Kato-Furuta inequality. Rephrasing it parallel to (1), we have

$$(2) \quad |(T|T|^{\alpha+\beta-1}x, y)|^2 \leq (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y)$$

for all  $\alpha, \beta \in [0, 1]$  with  $\alpha + \beta \geq 1$  and  $x, y \in H$ .

Very recently, Lin [9] sharpened (1) as follows:

**Theorem L.** *Let  $T$  be an operator on  $H$  and  $0 \neq y \in H$ . For  $z \in H$  satisfying  $Tz \neq 0$  and  $(Tz, y) = 0$ ,*

$$(3) \quad |(Tx, y)|^2 + \frac{(|T|^{2\alpha}x, z)|^2(|T^*|^{2(1-\alpha)}y, y)}{(|T|^{2\alpha}z, z)} \leq (|T|^{2\alpha}x, x)(|T^*|^{2(1-\alpha)}y, y)$$

for all  $\alpha \in [0, 1]$  and  $x, y \in H$ . The equality holds if and only if  $|T|^{2\alpha}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}z)$  and  $T^*y$  are proportional, or equivalently,  $Tx - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}Tz$  and  $|T^*|^{2(1-\alpha)}y$  are proportional.

In this note, we extend Theorem L, which is based on the Heinz-Kato-Furuta inequality (2). Our proof is quite simple, in which we clarify the meaning of the assumption in Theorem L that  $Tz \neq 0$  and  $(Tz, y) = 0$ . As a consequence, we can sharpen the Heinz-Kato-Furuta inequality, and Furuta's further generalization [6, Theorem 3] of the Heinz-Kato inequality via the Furuta inequality [4]. Incidentally we discuss Bernstein type inequality along the lines of our result.

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## 2. HEINZ-KATO-FURUTA INEQUALITY

For the sake of convenience, we first cite the Heinz-Kato-Furuta inequality [7]:

Let  $T$  be an operator on  $H$ . If  $A$  and  $B$  are positive operators on  $H$  such that  $T^*T \leq A^2$  and  $TT^* \leq B^2$ , then

$$(4) \quad |(T|T|^{\alpha+\beta-1}x, y)| \leq \|A^\alpha x\| \|B^\beta y\|$$

for all  $\alpha, \beta \in [0, 1]$  with  $\alpha + \beta \geq 1$  and  $x, y \in H$ .

We here remark that the Heinz-Kato inequality is just the case  $\alpha + \beta = 1$  in above and that it corresponds to (1). Thus we have the following extension of Theorem L. Throughout this paper, let  $T = U|T|$  be the polar decomposition of an operator  $T$  on  $H$ .

**Theorem 1.** *Let  $T$  be an operator on  $H$  and  $0 \neq y \in H$ . For  $z \in H$  satisfying  $T|T|^{\alpha+\beta-1}z \neq 0$  and  $(T|T|^{\alpha+\beta-1}z, y) = 0$ ,*

$$(5) \quad |(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2 (|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq (|T|^{2\alpha}x, x) (|T^*|^{2\beta}y, y)$$

for all  $\alpha, \beta \geq 0$  with  $\alpha + \beta \geq 1$  and  $x, y \in H$ . In the case  $\alpha, \beta > 0$ , the equality in (5) holds if and only if  $|T|^{\alpha+\beta-1}T^*y$  and  $|T|^{2\alpha}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}z)$  are proportional, or equivalently,  $|T^*|^{2\beta}y$  and  $T|T|^{\alpha+\beta-1}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}z)$  are proportional.

It is easily seen that Theorem L is the case  $\alpha + \beta = 1$  in Theorem 1. As a consequence, we have the following improvement of the Heinz-Kato-Furuta inequality via the Löwner-Heinz inequality, i.e.,  $A \geq B \geq 0$  implies  $A^\alpha \geq B^\alpha$  for  $\alpha \in [0, 1]$ :

**Theorem 2.** *Let  $T$  be an operator on  $H$ . If  $A$  and  $B$  are positive operators on  $H$  such that  $T^*T \leq A^2$  and  $TT^* \leq B^2$ , then*

$$(6) \quad |(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2 (|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq \|A^\alpha x\|^2 \|B^\beta y\|^2$$

for all  $\alpha, \beta \in [0, 1]$  with  $\alpha + \beta \geq 1$  and  $x, y, z \in H$  such that  $T|T|^{\alpha+\beta-1}z \neq 0$  and  $(T|T|^{\alpha+\beta-1}z, y) = 0$ . In the case  $\alpha, \beta > 0$ , the equality in (6) holds if and only if  $A^{2\alpha}x = |T|^{2\alpha}x$ ,  $B^{2\beta}y = |T^*|^{2\beta}y$  and  $|T|^{\alpha+\beta-1}T^*y$  and  $|T|^{2\alpha}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}z)$  are proportional; the third condition is equivalent to the condition that  $|T^*|^{2\beta}y$  and  $T|T|^{\alpha+\beta-1}(x - \frac{(|T|^{2\alpha}x, z)}{(|T|^{2\alpha}z, z)}z)$  are proportional.

*Proof of Theorem 1.* We only use the positivity of the Gram matrix

$$G = G(U|T|^\alpha x, |T^*|^\beta y, U|T|^\alpha z).$$

Noting that

$$(|T^*|^\beta y, U|T|^\alpha z) = (y, |T^*|^\beta U|T|^\alpha z) = (y, T|T|^{\alpha+\beta-1}z) = 0$$

by the assumption, we have

$$G = \begin{pmatrix} \| |T|^\alpha x \|^2 & (U|T|^\alpha x, |T^*|^\beta y) & (U|T|^\alpha x, U|T|^\alpha z) \\ (U|T|^\alpha x, |T^*|^\beta y)^* & \| |T^*|^\beta y \|^2 & 0 \\ (U|T|^\alpha x, U|T|^\alpha z)^* & 0 & \| |T|^\alpha z \|^2 \end{pmatrix}.$$

Since  $|T|^\alpha z \neq 0$ , we have

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2 (|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq (|T|^{2\alpha}x, x) (|T^*|^{2\beta}y, y).$$

To prove the equality condition, we set up the following lemma, which is applied to the vectors  $u = U|T|^\alpha x$ ,  $v = U|T|^\alpha z$  and  $w = |T^*|^\beta y$ .  $\square$

**Lemma.** (1) *If  $v \neq 0$  and  $(v, w) = 0$ , then  $\{u, v, w\}$  is linearly dependent if and only if  $w$  and  $u - \frac{(u,v)}{\|v\|^2}v$  are proportional.*

(2) *Let  $T = U|T|$  be the polar decomposition of an operator  $T$  on  $H$ , (namely  $\ker(U) = \ker(T)$ ). For  $\alpha, \beta > 0$  with  $\alpha + \beta \geq 1$  and  $y, w \in H$ , the following conditions are mutually equivalent:*

- (i)  $|T^*|^\beta y$  and  $U|T|^\alpha w$  are proportional.
- (ii)  $|T|^{\alpha+\beta-1}T^*y$  and  $|T|^{2\alpha}w$  are proportional.
- (iii)  $|T^*|^\beta y$  and  $T|T|^{\alpha-1}w$  are proportional.

*Proof.* (1) Suppose that  $au + bv + cw = 0$  for some  $(a, b, c) \neq 0$ . Then  $a(u, v) + b\|v\|^2 = 0$  and so  $b = -\frac{a(u,v)}{\|v\|^2}$ . Hence we have

$$0 = au + bv + cw = a\left(u - \frac{(u,v)}{\|v\|^2}v\right) + cw.$$

Since  $a = c = 0$  does not occur by  $v \neq 0$ , vectors  $u - \frac{(u,v)}{\|v\|^2}v$  and  $w$  are proportional. The converse is easily checked.

(2) (i) is equivalent to the statement that  $U|T|^\beta U^*y$  and  $U|T|^\alpha w$  are proportional. Noting that  $\alpha, \beta > 0$  and  $\ker(U) = \ker(T)$ , it is equivalent to (ii). Similarly we have the equivalence between (i) and (iii).  $\square$

### 3. FURUTA INEQUALITY

In [6], the Heinz-Kato-Furuta inequality is extended by the use of the Furuta inequality; Theorem 1 also gives us an improvement of the extension due to Furuta. For the sake of convenience, we cite the Furuta inequality [4]; see also [2],[5],[8].

**The Furuta inequality.** *If  $A \geq B \geq 0$ , then for each  $r \geq 0$ ,*

$$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

*holds for  $p \geq 0$  and  $q \geq 1$  with*

$$(*) \quad (1 + 2r)q \geq p + 2r.$$

The domain representing (\*) is drawn in Figure 1 and it is shown in [11] that this domain is the *best possible one* for the Furuta inequality.

**Theorem 3.** *Let  $T$  be an operator on  $H$ . If  $A$  and  $B$  are positive operators on  $H$  such that  $T^*T \leq A^2$  and  $TT^* \leq B^2$ . Then for each  $r, s \geq 0$*

$$(7) \quad \begin{aligned} & \left| (|T|^{(1+2r)\alpha+(1+2s)\beta-1} x, y) \right|^2 + \frac{\left| (|T|^{2(1+2r)\alpha} x, z) \right|^2 \left| (|T^*|^{2(1+2s)\beta} y, y) \right|^2}{\left| (|T|^{2(1+2r)\alpha} z, z) \right|^2} \\ & \leq \left( (|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x \right) \left( (|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y \right) \end{aligned}$$

*for all  $p, q \geq 1$ ,  $\alpha, \beta \in [0, 1]$  with  $(1+2r)\alpha + (1+2s)\beta \geq 1$  and  $x, y, z \in H$  such that  $T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z \neq 0$  and  $(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z, y) = 0$ . In the case  $\alpha, \beta > 0$ , the equality in (7) holds if and only if  $|T|^{2(1+2r)\alpha}x = (|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x$ ,  $|T^*|^{2(1+2s)\beta}y = (|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}y$  and  $|T|^{2(1+2r)\alpha}\left(x - \frac{(|T|^{2(1+2r)\alpha}x, z)}{(|T|^{2(1+2r)\alpha}z, z)}z\right)$  and*

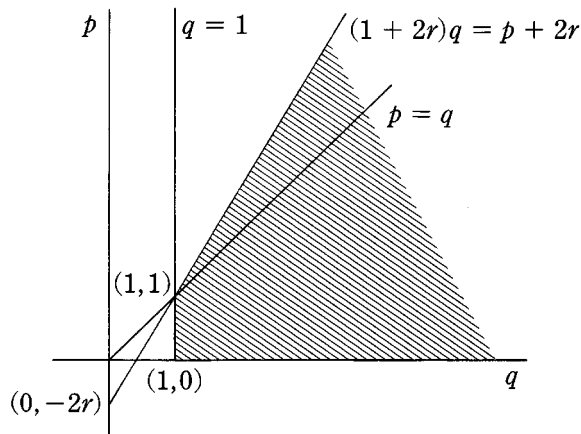


FIGURE 1

$|T|^{2(1+2r)\alpha+2(1+2s)\beta-1}T^*y$  are proportional; the latter is equivalent to the statement that  $T|T|^{(1+2r)\alpha+(1+2s)\beta-1}(x - \frac{(|T|^{2(1+2r)\alpha}x, z)}{(|T|^{2(1+2r)\alpha}z, z)}z)$  and  $|T^*|^{2(1+2s)\beta}y$  are proportional.

*Proof.* We use Theorem 1 by replacing  $\alpha$  (resp.  $\beta$ ) by  $\alpha_1 = (1+2r)\alpha$  (resp.  $\beta_1 = (1+2s)\beta$ ). Then we have

$$(8) \quad (|T|^{2\alpha_1+\beta_1-1}x, y)|^2 + \frac{(|T|^{2\alpha_1}x, z)|^2(|T^*|^{2\beta_1}y, y)}{(|T|^{2\alpha_1}z, z)} \leq (|T|^{2\alpha_1}x, x)(|T^*|^{2\beta_1}y, y).$$

Next we use the Furuta inequality for  $|T|^2 \leq A^2$  and  $|T^*|^2 \leq B^2$ ; namely (for the former) we replace  $A, B, q$  in the Furuta inequality by  $A^2, |T|^2, \frac{p+2r}{(1+2r)\alpha}$  respectively. Then we have

$$|T|^{2\alpha_1} = |T|^{2(1+2r)\alpha} \leq (|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}$$

and similarly

$$|T^*|^{2\beta_1} = |T^*|^{2(1+2s)\beta} \leq (|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}.$$

Combining with (8), we obtain the inequality (7).

The equality condition is shown similarly to Theorem 2.  $\square$

*Remark.* (1) We remark that the condition  $(1+2r)\alpha + (1+2s)\beta \geq 1$  in Theorem 3 is unnecessary if  $T$  is either positive or invertible.

(2) Though Theorem 3 follows from the Furuta inequality, they are equivalent actually, that is, Theorem 3 is an alternative representation of the Furuta inequality. As a matter of fact, we put  $T = B, \alpha = \beta, r = s$  and also  $x = y$  in Theorem 3. Thus it follows from the above remark (1) that if  $A^2 \geq B^2$ , then for  $B^{2(1+2r)\alpha}z \neq 0$  and  $(B^{2(1+2r)\alpha}z, x) = 0$

$$\begin{aligned} & |(B^{2(1+2r)\alpha}x, x)|^2 + \frac{|(B^{2(1+2r)\alpha}x, z)|^2(B^{2(1+2r)\alpha}x, x)}{(B^{2(1+2r)\alpha}z, z)} \\ & \leq ((B^{2r}A^{2p}B^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x, x)(B^{2(1+2r)\alpha}x, x), \end{aligned}$$

that is,  $A^2 \geq B^2$  ensures

$$(B^{2(1+2r)\alpha}x, x)^2 \leq ((B^{2r}A^{2p}B^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x, x)$$

for all  $p \geq 1$ ,  $r \geq 0$  and  $\alpha \in [0, 1]$ . This is nothing but the Furuta inequality.

#### 4. A CONCLUDING REMARK

Lin also discussed Bernstein type inequalities independently of Theorem L [9, Theorem 3], see [1]. As an application of Theorem 1, we have a generalization of it:

**Theorem 4.** *Let  $T$  be an operator on  $H$  having a nonzero normal eigenvalue  $\lambda$  with an eigenvector  $e$ . If  $y \in H$  satisfies  $(e, y) = 0$  and  $T^*y \neq 0$ , then*

$$|\lambda|^2|(x, e)|^2 \leq \frac{\|Tx\|^2\|T^*|^\beta T^*y\|^2 - |(T|T|^\beta x, T^*y)|^2}{\|T^*|^\beta T^*y\|^2}$$

for all  $x \in H$  and  $\beta \in [0, 1]$ .

*Proof.* We put  $\alpha = 1$ ,  $z = e$  and replace  $y$  by  $T^*y$  in Theorem 1. Since  $(|T|^\beta e, T^*y) = 0$  by  $(e, y) = 0$ , it follows from Theorem 1 that

$$|(T|T|^\beta x, T^*y)|^2 + \|T^*|^\beta T^*y\|^2|\lambda|^2|(x, e)|^2 \leq \|Tx\|^2\|T^*|^\beta T^*y\|^2,$$

so that we have the desired inequality.  $\square$

We obtain Lin's inequality [9, Theorem 3] by taking  $\beta = 0$  in Theorem 4.

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